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M. K. Singh, A. K. Singh (Inst. Sci., Banaras Hindu Univ., Varanasi, India)

STUDY OF FROZEN-TYPE NEWTON-LIKE METHOD IN A BANACH SPACE WITH DYNAMICS

ВИВЧЕННЯ МЕТОДУ НЬЮТОНА ЗАМОРОЖЕНОГО ТИПУ У БАНАХОВОМУ ПРОСТОРІ З ДИНАМІКОЮ

The main objective of this work is investigation of positives and negatives of the three steps iterative frozen-type Newton-like method for solving nonlinear equations in a Banach space. We perform a local convergence analysis by Taylor expansion and semilocal convergence by recurrence relations technique under the conditions of Kantorovich theorem for the Newton's method. The convergence results are examined by comparing the proposed method with the Newton method and the fourth order Jarratt method using some test functions. We discuss the corresponding conjugacy maps for quadratic polynomials along with the extraneous fixed points. Additionally, the theoretical and numerical results are examined by using the dynamical analysis of a selected test function. It not only confirms the theoretical and numerical results, but also reveals some drawbacks of the frozen-type Newton-like method.

Мета цієї роботи — вивчення плюсів та мінусів трикрокового ітераційного методу Ньютона замороженого типу для розв'язання нелінійних рівнянь у банаховому просторі. Проведено аналіз локальної збіжності за допомогою рядів Тейлора та напівлокальної збіжності за допомогою рекурентних співвідношень за умов теореми Канторовича для методу Ньютона. Отримані результати збіжності перевірено шляхом порівняння запропонованого методу з методом Ньютона та методом Джарратта четвертого порядку з використанням деяких тестових функцій. Обговорено відповідні спряжені відображення для квадратичних поліномів, а також додаткові нерухомі точки. Крім того, отримані теоретичні та числові результати перевірено за допомогою методів динамічного аналізу певної тестової функції. Тим самим не тільки підтверджено теоретичні та числові результати, але й виявлено деякі недоліки запропонованого методу Ньютона замороженого типу.

1. Introduction. Convergence analysis results are the central part of a paper related to the solution of the nonlinear equations of the form

$$F(x) = 0. \tag{1}$$

Generally, three type of convergence analysis are used for the numerical solution of nonlinear equations. First one is local convergence analysis in which we start with the assumption of the existence of the particular solution, around this solution, there exists a neighborhood starting with any vector in this neighborhood leads to a sequence which converges to the solutions under some suitable conditions [13, 15]. Second one is global convergence analysis, it also start with the assumption of existence of the solution but it does not requires any local neighborhood for any initial vector to converge to the solution. Third and last is semilocal convergence analysis, it does not requires the knowledge of the existence of a solution, rather than demands that some conditions around the initial vector.

Newton method is commonly used and basic method for solving the nonlinear equation (1). It is only of order two under some conditions [1 – 24]. It is defined as follows:

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n = 0, 1, 2, \ldots. \tag{2}$$

The semilocal convergence of Newton method in Banach spaces was established by Kantorovich [16], where it was supposed that the second Fréchet derivative should be bounded in some domain. The convergence of the sequence obtained by the iterative expression was derived from the convergence of majorizing sequences. This technique has been used by many authors in order to establish the order of convergence of the variants of Newton methods (see, for example, [7, 23]). In [11], Rall introduced a different technique for the semilocal convergence of these methods, which is based on recurrence relations. Parida and Gupta in [10], Madhu in [17], Ezquerro et al. in [8] and Chun et al. in [6] have used this idea to prove the semilocal convergence for several Newton-like method of different orders.

The earlier study of the multistep Newton-like methods for the solution of (1) in real space and Banach space uses the higher order derivatives, while the Newton-like methods, e.g., the proposed method (3) involves only the first-order derivative, hence, it limits the applications of the method (3):

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - F'(x_n)^{-1}F(y_n), \\ x_{n+1} &= z_n - F'(x_n)^{-1}F(z_n), \quad n = 0, 1, 2, \ldots . \end{aligned} \tag{3}$$

In this paper first we have performed the local convergence analysis of the method (3) around the local root x^*, which requires fourth-order Fréchet derivative of operator F. Then we analyze the semilocal convergence analysis using the recurrence relations technique which requires only the first-order Fréchet derivative. In this technique, we generate a sequence of positive real numbers that guarantees the convergence of the iterative scheme in Banach spaces, providing a suitable convergence domain. The main advantage of this technique is that we get a result of semilocal convergence under the same conditions of Kantorovich theorem for Newton method, which has quadratic convergence. This allows us to apply the fourth-order convergence method for solving nonlinear equations $F(x) = 0$ under the same conditions that assure us the convergence of Newton method. Moreover, the semilocal convergence analysis by recurrence relations technique is important in other aspects, especially if F has no third order Fréchet derivative. As a motivational example, let us define function M on $X = Y = R,\ D = [-1, 1]$ by

$$M(t) = \begin{cases} t^5 \sin\dfrac{1}{t} + t^5 - t^4, & \text{if} \quad t \neq 0, \\ 0, & \text{if} \quad t = 0. \end{cases}$$

Then we have $t^* \approx 0.5169$ and

$$M'(t) = 5t^4 \sin\frac{1}{t} - t^3 \cos\frac{1}{t} + 5t^4 - 4t^3,$$

$$M''(t) = 20t^3 \sin\frac{1}{t} - 8t^2 \cos\frac{1}{t} - t \sin\frac{1}{t} + 20t^3 - 12t^2,$$

$$M'''(t) = 60t^2 \sin\frac{1}{t} - 36t \cos\frac{1}{t} - 9 \sin\frac{1}{t} + \frac{1}{t} \cos\frac{1}{t} + 60t^2 - 24t.$$

It is clear that $M'''(t)$ is not bounded on D.

Numerical results consist of the comparative study of the efficiency of the proposed scheme with the Newton method and Jarratt method (see [9]) by using the classical efficiency index defined by Ostrowski in [19]. In addition, we include the comparative study of the Newton method and Jarratt method for the solutions of a nonlinear equation and a system of nonlinear equations. One important aspect of this study is the discussion of the extraneous fixed points and the corresponding conjugacy maps for quadratic polynomials as well as the comparative study of the dynamical analysis of the proposed frozen-type Newton-like method along with fourth-order Jarratt method. The basins of attraction reveal some other defects of these types of modified frozen-type Newton-like method.

2. Local convergence analysis.

Theorem 1. *Let D be a convex subset of $\Re^n$ and $F: D \to \Re^n$ be a function such that*

(1) *it has simple zero $x^* \in I$,*

(2) *Jacobian matrix $F'(x^*)$ is non singular at the zero x^*,*

(3) *F is a fourth-order Fréchet differential in the convex set D at some neighborhood S of the zero x^*.*

Then the iterative method (3) *has convergence of fourth-order to the zero x^*.*

Proof. Let $x^* \in I$ be a simple zero of a function F, $e_n = x_n - x^*$ and

$$A_k = \left(\frac{1}{k!}\right) F'(x^*)^{-1} F^{(k)}(x^*).$$

Using Taylor expansion of F around x^* and taking into account $F(x^*) = 0$, we get

$$F(x_n) = F'(x^*)\big[e_n + A_2 e_n^2 + A_3 e_n^3 + O(e_n^4)\big], \tag{4}$$

$$F'(x_n) = F'(x^*)\big[1 + 2A_2 e_n + 3A_3 e_n^2 + 4A_4 e_n^3 + O(e_n^4)\big]. \tag{5}$$

Now from (4) and (5), we get

$$F'(x_n)^{-1} F(x_n) = e_n - A_2 e_n^2 + \big(2A_2^2 - 2A_3\big) e_n^3 + O(e_n^4).$$

Since $y_n = x_n - F'(x_n)^{-1} F(x_n)$, we obtain

$$y_n = x^* + A_2 e_n^2 + \big(2A_3 - 2A_2^2\big) e_n^3 + \big(4A_2^3 - 7A_2A_3 + 3A_4\big) e_n^4 + O(e_n^5),$$

$$F(y_n) = F'(x^*)\Big[A_2 e_n^2 - 2(A_2^2 - A_3) e_n^3 + \big(5A_2^3 - 7A_2A_3 + 3A_4\big) e_n^4 -$$

$$-2\big(6A_2^4 - 12A_2^2A_3 + 3A_3^2 + 5A_2A_4 - 2A_5\big) e_n^5 + O(e_n^6)\Big].$$

Next

$$z_n = 2A_2^2 e_n^3 + (-9A_2^3 + 7A_2A_3) e_n^4 + (30A_2^4 - 44A_2^2A_3 + 6A_3^2 + 10A_2A_4) e_n^5 + O(e_n)^6,$$

$$F(z_n) = F'(x^*)\Big[2A_2^2 e_n^3 + (-9A_2^3 + 7A_2A_3) e_n^4 +$$

$$+2\big(15A_2^4 - 22A_2^2A_3 + 3A_3^2 + 5A_2A_4\big) e_n^5 + O(e_n^6)\Big].$$

Hence, using the proposed method (3), we have

$$x_{n+1} = 4A_2^3 e_n^4 + \left(-26A_2^4 + 20A_2^2 A_3\right) e_n^5 + O(e_n^6).$$

Therefore,

$$e_{n+1} = 4A_2^3 e_n^4 + \left(-26A_2^4 + 20A_2^2 A_3\right) e_n^5 + O(e_n^6). \tag{6}$$

Equation (6) confirms that the proposed method (3) converges with fourth-order to the root of (1) locally, if there exist a fourth-order Fréchet differentiable operator in an open convex domain D.

3. Recurrence relations. Let X, Y be Banach spaces and $F : D \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator in an open convex domain D. Now we study the semilocal convergence analysis for the fourth-order method (3):

$$\begin{aligned} y_n &= x_n - \tau_n F(x_n), \\ z_n &= y_n - \tau_n F(y_n), \\ x_{n+1} &= z_n - \tau_n F(z_n), \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{7}$$

where $\tau_n = [F'(x_n)]^{-1}$ for $n \in N$. We assume that the inverse of F' at x_0, $[F'(x_0)]^{-1} = \tau_0 \in$ $\in L(Y, X)$ exists at some $x_0 \in D$, where $L(Y, X)$ is the set of bounded linear operators from Y into X. In the following we assume that $y_0, z_0 \in D$ and

1) $\|\tau_0\| \leq \beta$,
2) $\|\tau_0 F(x_0)\| \leq \eta$,
3) $\|F'(x) - F'(y)\| \leq k\|x - y\|$

in order to obtain the recurrence relations which satisfy the steps that appear in the iterative process (7). Notice that these are the classical Kantorovich conditions [16] for the semilocal convergence of Newton method. Let us also denote by $a_0 = \beta\eta k$ and define the sequence $a_{n+1} = a_n f(a_n)^2 g(a_n)$, where

$$f(x) = \frac{1}{1 - x(h(x) + 1)}, \tag{8}$$

$$g(x) = \frac{x}{2} + (x + 1)h(x) + \frac{x}{2}h(x)^2, \tag{9}$$

and

$$h(x) = \frac{x}{2} + \frac{x^2}{2} + \frac{x^3}{8}. \tag{10}$$

To study the convergence of x_n defined by (7) to a solution of $F(x) = 0$ in a Banach space, we have to prove that x_n is a Cauchy sequence. To do this, we need to analyze some properties of sequence a_n and, previously, of the real functions described in (8)–(10), respectively.

Lemma 1. *Let $f(x)$, $g(x)$ and $h(x)$ be the real functions described in* (8)–(10). *Then*:

1) *f is increasing and $f(x) > 1$ for $x \in (0, 0.6)$,*
2) *h and g are increasing for $x \in (0, 0.6)$.*

Lemma 2. *Let $f(x)$ and $g(x)$ as before and $a_0 \in (0, b)$ for $b = 0.2990377177778545$. Then*:

1) $f(a_0)^2 g(a_0) < 1$,
2) $f(a_0)g(a_0) < 1$,
3) *the sequence a_n is decreasing and $a_n < b$ for $n \geq 0$.*

Proof. From the definition of functions f and g (i) follows trivially. From (i) and $f(a_0) > 1$, we obtain (ii). Now we prove (iii) by induction method for $n \geq 0$. Firstly, from (i) and the definition of a_1, we have $a_1 < a_0$. Now, we suppose that $a_k < a_{k-1}$ for $k \leq n$. Then

$$a_{n+1} = a_n f(a_n)^2 g(a_n) < a_{n-1} f(a_n)^2 g(a_n) < a_{n-1} f(a_{n-1})^2 g(a_{n-1}) = a_n.$$

As f and g are increasing, $f(x) > 1$. Finally, for all $n \geq 0$, $a_n < b$, since a_n is a decreasing sequence and $a_0 < b$.

Note that $a_0 = b$, is the value of the solution of equation $f(a_0)^2 g(a_0) - 1 = 0$. By using Taylor expansion of $F(y_0)$ around x_0, we get

$$z_0 - x_0 = y_0 - x_0 - \tau_0 F(y_0) = y_0 - x_0 - \tau_0 \int\limits_0^1 \big(F'(x_0 + t(y_0 - x_0)) - F'(x_0)\big)(y_0 - x_0)\, dt,$$

$$\|z_0 - x_0\| \leq \|y_0 - x_0\| + \frac{1}{2} K\beta \|y_0 - x_0\|^2.$$

In a similar way, $\|z_0 - y_0\| \leq \frac{1}{2}\|y_0 - x_0\|$. Now, by using Taylor expansion of $F(z_0)$ and (1), we have

$$\|x_1 - x_0\| = \| - \tau_0 \big(F(x_0) + F(y_0) + F(z_0)\big)\| =$$

$$= \left\| y_0 - x_0 - \tau_0 \int\limits_{x_0}^{y_0} (F'(x) - F'(x_0))dx - \tau_0 \int\limits_{x_0}^{z_0} (F'(x) - F'(x_0))dx \right\| \leq$$

$$\leq \|y_0 - x_0\| + \frac{1}{2} K\beta \|y_0 - x_0\|^2 + \frac{1}{2} K\beta \|z_0 - x_0\|^2 \leq$$

$$\leq \left(1 + \frac{a_0}{2} + \frac{1}{2} a_0^2 + \frac{1}{8} a_0^3\right)\eta = \big(1 + h(a_0)\big)\eta.$$

Assuming that $a_0 < 0.6$ and applying assumptions (i) – (iii), we have

$$\|I - \tau_0 F'(x_1)\| \leq \|\tau_0\|\, \|F'(x_1) - F'(x_0)\| \leq$$

$$\leq \beta K \|x_1 - x_0\| \leq \beta K \eta (1 + h(a_0)) \leq$$

$$\leq a_0(1 + h(a_0)) < 1.$$

Next, by the Banach lemma, τ_1 exists and

$$\|\tau_1\| \leq \frac{\tau_0}{1 - \tau_0 K \|x_1 - x_0\|} \leq \frac{1}{1 - a_0(1 + h(a_0))} \tau_0 = f(a_0)\|\tau_0\|.$$

Note that we need $a_0 < 0.6$ in order to guaranty $a_0(1 + h(a_0)) < 1$. We also note that $K\tau_0\|y_0 - x_0\| \leq a_0$, so it can be deduced that x_1 is well defined and

$$\|x_1 - x_0\| \leq \|\tau_0\|\, \|F(x_0) + F(y_0) + F(z_0)\| \leq (h(a_0) + 1)\|\tau_0 F(x_0)\|. \tag{11}$$

Again we assume that $x_n, y_n, z_n \in D$ and $a_n < 0.6$ for $n \geq 1$. Then the following estimations can be proved by induction for $n \geq 1$:

(I_n) $\|\tau_n\| \le f(a_{n-1})\|\tau_{n-1}\|$,
(II_n) $\|y_n - x_n\| = \|\tau_n F(x_n)\| \le f(a_{n-1})g(a_{n-1})\|y_{n-1} - x_{n-1}\|$,
(III_n) $\|z_n - y_n\| \le \dfrac{1}{2}K\beta(f(a_0))^n\|y_n - x_n\|^2$,
(IV_n) $K\|\tau_n\|\,\|y_n - x_n\| \le a_n$,
(V_n) $\|x_n - x_{n-1}\| \le (1 + h(a_{n-1}))\|y_{n-1} - x_{n-1}\|$.
Let us consider $n = 1$. So, (I_1) has been proved before.
(II_1) By using Taylor formula

$$F(x_1) = F(y_0) + F'(y_0)(x_1 - y_0) + \int\limits_{y_0}^{x_1} (F'(x) - F'(y_0))dx =$$

$$= \int\limits_0^1 \big(F'(x_0 + t(y_0 - x_0)) - F'(x_0)\big)(y_0 - x_0)\, dt - (F'(y_0) -$$

$$-F'(x_0) + F'(x_0))\tau_0\big(F(y_0) + F(z_0)\big) -$$

$$-\tau_0(F(y_0) + F(z_0)) \int\limits_0^1 \big(F'(y_0 + t(x_1 - y_0)) - F'(y_0)\big)\, dt.$$

On the other hand,

$$F(y_0) + F(z_0) \le \eta \frac{ha_0}{\beta}.$$

Then we have

$$\|F(x_1)\| \le \frac{1}{2}K\eta^2 + K\eta^2 h(a_0) + \eta\frac{h(a_0)}{\beta} + \frac{k}{2}\eta^2 h(a_0)^2,$$

$$\|y_1 - x_1\| = \|\tau_1\|\,\|F(x_1)\| \le \|f(a_0)\|\,\|\tau_0\|\,\|F(x_1)\| \le f(a_0)g(a_0)\|y_0 - x_0\|.$$

(III_1) It is clear that

$$\|z_1 - y_1\| = \tau_1\|F(y_1)\| \le$$

$$\le \beta f(a_0)\left\| \int\limits_0^1 \big(F'(x_1 + t(y_1 - x_1)) - F'(x_1)\big)(y_1 - x_1)\, dt \right\| \le$$

$$\le \frac{1}{2}\beta K f(a_0)\|(y_1 - x_1)\|^2.$$

(IV_1) By using (I_1) and (II_1), we get

$$K\|\tau_1\|\,\|y_1 - x_1\| \le Kf(a_0)\|\tau_0\| f(a_0)g(a_0)\|y_0 - x_0\| = a_1.$$

(V_1) It has been shown in (11) that

$$\|x_1 - x_0\| \le (1 + h(a_0))\|y_0 - x_0\|.$$

By considering that the induction hypothesis of items (I_n) to (V_n) are true for a fixed $n \geq 1$, it can be proved (I_{n+1}) to (V_{n+1}) in a similar way. Note that condition $a_n < 0.6$, for $n \geq 1$, is necessary for the existence of operators τ_n, $n \geq 1$. The above recurrence relations for the proposed method given in (3) allow us to establish a new semilocal convergence result for this method under mild conditions.

4. Semilocal convergence analysis. From the Lemmas 1 and 2 and the recurrence relations proved in the previous section, we are in position to prove the semilocal convergence result for method (3) under mild conditions. In the previous results we have used different conditions for parameter a_0. In the following, we consider the most restrictive one in order to prove the semilocal convergence.

Theorem 2. *Let X and Y be Banach spaces and $F\colon D \subseteq X \to Y$ be a twice Fréchet differentiable nonlinear operator in an open convex domain D. Let $\tau_0 = F'(x_0)^{-1} \in B(Y, X)$ exists at some $x_0 \in D$ and*

(i) $\|\tau_0\| \leq \beta$,

(ii) $\|\tau_0 F(x_0)\| \leq \eta$,

(iii) $\|F'(x) - F'(y)\| \leq k\|x - y\|$, $x, y \in D$,

are satisfied. Let $a_0 = \beta\eta k$, $a_0 < 0.29903\ldots$ and $B(x_0, R\eta) = \{x \in X : \|x - x_0\| < R\eta\} \subset D$, where $R = \dfrac{1}{2}a_0 + \dfrac{1 + h(a_0)}{1 - f(a_0)g(a_0)}$. Then the following conditions hold:

(a) *the solution x^* and the iterates x_n, y_n and z_n belong to $\overline{B(x_0, R\eta)}$*,

(b) *the sequence $\{x_n\}$ generated by* (7) *is well defined furthermore with initial point x_0 converges to a solution x^* of operator $F(x) = 0$*,

(c) *the solution x^* of $F(x) = 0$ is unique and belong to $B\left(x_0, \dfrac{2}{K\beta} - R\eta\right) \cap D$.*

Proof. Firstly, let us recall that τ_n exists for $n \geq 1$, since $a_0 < 0.29903\ldots$. Moreover, we are going to prove that y_n and z_n belong to $B(x_0, R\eta) \subset D$. By recurrence relation (V_n), it is easy to observe that

$$\|x_n - x_0\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \ldots + \|x_1 - x_0\| \leq$$

$$\leq (1 + h(a_0))\|y_0 - x_0\| \sum_{k=0}^{n-1} \big(f(a_0)g(a_0)\big)^k.$$

Hence

$$\|y_n - x_0\| \leq \|y_n - x_n\| + \|x_n - x_0\| \leq$$

$$\leq (1 + h(a_0))\big(f(a_0)g(a_0)\big)^n \|y_0 - x_0\| + \ (1 + h(a_0))\|y_0 - x_0\| \sum_{k=0}^{n-1} \big(f(a_0)g(a_0)\big)^k <$$

$$< (1 + h(a_0)) \frac{1 - \big(f(a_0)g(a_0)\big)^{n+1}}{1 - f(a_0)g(a_0)} \eta < R\eta.$$

Now, by applying recurrence relations (I_n) and (II_n), we have

$$\|z_n - y_n\| \leq \tau_n F(y_n) \leq \frac{1}{2} K\beta (f(a_0))^n \|y_n - x_n\|^2 \leq$$

$$\leq \frac{1}{2}a_0\big((f(a_0))^3(g(a_0))^2\big)^n\|y_0 - x_0\|.$$

Therefore,

$$\|z_n - x_0\| \leq \|z_n - y_n\| + \|y_n - x_0\| \leq$$

$$\leq \frac{1}{2}a_0\big((f(a_0))^3(g(a_0))^2\big)^n\|y_0 - x_0\| + \ (1 + h(a_0))\frac{1 - \big(f(a_0)g(a_0)\big)^{n+1}}{1 - f(a_0)g(a_0)}\|y_0 - x_0\| <$$

$$< \left(\frac{1}{2}a_0 + (1 + h(a_0))\frac{1 - \big(f(a_0)g(a_0)\big)^{n+1}}{1 - f(a_0)g(a_0)}\right)\eta < R\eta.$$

In order to prove the convergence of the sequence $x_n,$ let us state that

$$\|x_{n+1} - x_n\| \leq (1 + h(a_n))\|y_n - x_n\| \leq$$

$$\leq (1 + h(a_n))f(a_{n-1})g(a_{n-1})\|y_{n-1} - x_{n-1}\| \leq \ldots$$

$$\ldots \leq (1 + h(a_n))\left[\prod_{j=0}^{n-1} f(a_j)g(a_j)\right]\|y_0 - x_0\| \tag{12}$$

by (V_n) and (II_n).

Then, from (12), we obtain

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \ldots + \|x_{n+1} - x_n\| \leq$$

$$\leq \big(1 + h(a_{n+m-1})\big)\eta \prod_{j=0}^{n+m-2} f(a_j)g(a_j) + \big(1 + h(a_{n+m-2})\big)\eta \prod_{j=0}^{n+m-3} f(a_j)g(a_j) + \ldots$$

$$\ldots + \big(1 + h(a_n)\big)\eta \prod_{j=0}^{n-1} f(a_j)g(a_j),$$

and, as h is increasing and a_n is decreasing by Lemmas 1 and 2,

$$\|x_{n+m} - x_n\| \leq \big(1 + h(a_0)\big)\eta \sum_{l=0}^{m-1}\left[\prod_{j=0}^{n+l-1} f(a_j)g(a_j)\right] \leq$$

$$\leq \big(1 + h(a_0)\big)\eta \sum_{l=0}^{m-1}\big(f(a_0)g(a_0)\big)^{l+n}.$$

Since f and g are also increasing. Therefore, by applying the partial sum of a geometric sequence, we have

$$\|x_{n+m} - x_n\| \leq \big(1 + h(a_0)\big)\frac{1 - \big(f(a_0)g(a_0)\big)^m}{1 - f(a_0)g(a_0)}\big(f(a_0)g(a_0)\big)^n\eta.$$

Then we conclude that $\{x_n\}$ is a Cauchy sequence if $f(a_0)g(a_0) < 1.$

In order to prove that x^* is a solution of $F(x)=0$, we start with the bound of $\|F'(x_n)\|$,

$$\|F'(x_n)\| \leq \|F'(x_0)\| + \|F'(x_n) - F'(x_0)\| \leq$$

$$\leq \|F'(x_0)\| + K\|x_n - x_0\| \leq \|F'(x_0)\| + KR\eta, \tag{13}$$

by applying hypothesis (ii) and Lemmas 1 and 2. Then, by (12), we get

$$\|F(x_n)\| \leq \|F'(x_n)\| \, \|\tau_n F(x_n)\| \leq$$

$$\leq \|F'(x_n)\| \leq f(a_{n-1})g(a_{n-1})\|y_{n-1} - x_{n-1}\| \leq$$

$$\leq \|F'(x_n)\| \left[\prod_{j=0}^{n-1} f(a_j)g(a_j)\eta\right],$$

and, as f and g are increasing and a_n is decreasing,

$$\|F(x_n)\| \leq \|F'(x_n)\| \big(f(a_0)g(a_0)\big)^n \eta.$$

Since $\|F'(x_n)\|$ is bounded (see (13)) and $\big(f(a_0)g(a_0)\big)^n$ tends to zero when $n \to \infty$, we conclude that $\|F(x_n)\| \to 0$. By continuity of F in D, $F(x^*) = 0$.

Let us observe that, if $a_0 \in (0, 0.29903\ldots)$, $\dfrac{2}{K\beta} - R\eta > 0$. So, we are going to prove the uniqueness of x^* in $B\left(x_0, \dfrac{2}{K\beta} - R\eta\right) \cap D$. Let us assume that y^* is the another solution of $F(x) = 0$ in $B\left(x_0, \dfrac{2}{K\beta} - R\eta\right) \cap D$. Then, in order to prove that $y^* = x^*$, and taking into account the Taylor expansion

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\, dt (y^* - x^*),$$

it is necessary to show that the operator $P = \displaystyle\int_0^1 F'(x^* + t(y^* - x^*))\, dt$ is invertible. So, by applying hypothesis (iii), we have

$$\|I - \tau_0 P\| = \|\tau_0 (F'(x_0) - P)\| =$$

$$= \left\| \tau_0 \int_0^1 \big(F'(x^* + t(y^* - x^*)) - F'(x_0)\big)\, dt \right\| \leq$$

$$\leq K\beta \int_0^1 \big\|(x^* + t(y^* - x^*)) - x_0\big\| dt \leq$$

$$\leq K\beta \int_0^1 \big((1-t)\|x^* - x_0\| + t\|y^* - x_0\|\big)\, dt <$$

Table 1. Efficiency indices for Newton, Jarratt and proposed methods

m	Newton method (2)	Jarratt method (14)	Proposed method (3)
2	1.1220	1.1487	1.1487
3	1.0595	1.0682	1.0801
4	1.0353	1.0393	1.0508
5	1.0234	1.0255	1.0353
6	1.0166	1.0179	1.0260
7	1.0125	1.0133	1.0200
8	1.0097	1.0102	1.0159

$$< \frac{1}{2}K\beta\left(R\eta + \frac{2}{K\beta} - R\eta\right) = 1.$$

Therefore, by the Banach lemma, the integral operator P is invertible and, hence, $y^* = x^*$.

5. Numerical results. In this section, we consider the situation $X = Y = R^m$ to study the efficiency of iterative method (3). Notice that we have proved in Theorem 2 of previous section, that the method (3) has a fourth-order of convergence. Nevertheless, it is not the only advantage of the scheme: the number of evaluations of the nonlinear function F and its associated Jacobian matrix are also lower than the respective one of known methods. The most used tool to compare the efficiency of different iterative methods is the efficiency index, defined by Ostrowski as $\mathrm{EI} = p^{1/d}$, where p is the order of convergence and d is the total number of functional evaluations per iteration. The efficiency index of proposed frozen-type Newton-like method (3) is $\mathrm{EIPM} = 4^{1/(m^2+3m)}$. We compare it in Table 1 with not only the index of classical Newton method, $\mathrm{EIN} = 2^{1/(m^2+m)}$, but also with fourth-order Jarratt method, $\mathrm{EIJ} = 4^{1/(2m^2+m)}$, whose iterative expression is given by

$$\begin{aligned} y_n &= x_n - 2/3F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - \left[6F'(y_n) - 2F'(x_n)\right]^{-1}\left[3F'(y_n) + F'(x_n)\right]F'(x_n)^{-1}F(x_n), \end{aligned} \tag{14}$$

where $n = 0, 1, 2, \ldots$.

In Table 1, the efficiency indices for systems of size $m \leq 8$ can be observed. We remark that the best efficiency index is the one of method (3). In a similar way, the same conclusion can be reached for higher sizes of the system.

***Example* 1.** Let $X = R,\ D = (-1, 1)$ and $F : D \to R$ be an operator defined by

$$F(x) = e^x - 1 \quad \forall x \in D.$$

Then its Fréchet derivative $F^{'}(x)$ at any point $x \in D$ is given by

$$F'(x) = e^x.$$

We have computed the numerical results with the help of MATLAB 2007 and the stopping criterion used for the computation is $|x_{n+1} - x^*| + |f(x_{n+1})| < 10^{-14}$. The initial approximation is 0.1 and approximate solution is 0. The numerical solution of Example 1 by 2nd order Newton method (2), 4th order Jarratt method (14) and proposed frozen-type Newton-like method (3) are given in Table 2. Numerical results in Table 2 reveals that the proposed method (3) is converging to the root 0 in much better way in comparison to the other methods.

Table 2. Comparison of the different methods for Example 1

Method	N	x	$f(x)$
Newton method (2)	1	0.1000000	0.10517091807565
	2	0.00483741803596	0.00484913723185
	3	1.168146299657721e-005	1.168153122521609e-005
	4	6.822793885920731e-011	6.822786779991930e-011
	5	7.106394304750309e-017	0.00000000
Jarratt method (14)	1	0.1000000	0.10517091807565
	2	4.469864996595185e-006	4.469874986368083e-006
	3	8.853103661407222e-017	0.00000000
Proposed method (3)	1	0.1000000	0.10517091807565
	2	4.270663384022157e-005	4.270754578139524e-005
	3	5.636437603011429e-017	0.00000000

***Example* 2.** Let $X = R,\ D = (-2, 2)$ and $F: D \to R$ be an operator defined by

$$F(x) = x^3 - 1 \quad \forall x \in D.$$

Then F is Fréchet differentiable and its Fréchet derivative $F^{'}(x)$ at any point $x \in D$ is given by

$$F'(x) = 3x^2.$$

We have computed the numerical results with the help of MATLAB 2007 and the stopping criterion used for the computation is $|x_{n+1} - x^*| + |f(x_{n+1})| < 10^{-14}$. The initial approximation is -2.0 and the approximate solution is 1.0. The numerical solution of Example 2 by 2nd order Newton method (2), 4th order Jarratt method (14) and proposed method (3) are given in Table 3. Numerical results in Table 3 reveals that the proposed method (3) is converging to the root 1.0 in much better way in comparison to the others starting with the point -2.0.

***Example* 3.** Let $D = X = Y = R^2$. Consider an operator $F: R^2 \to R^2$ defined by

$$F(x, y) = \left(-x^2 + 1/3, -y^2 + 1/3\right) \quad \forall (x, y) \in R^2.$$

The starting vector is $[0.1, 0.1]$ and approximate solution is $[0.57735, 0.57735]$. The numerical solution of Example 3 by 2nd order Newton method (2), 4th order Jarratt method (14) and proposed method (3) are shown in Table 4. Numerical results show that the proposed method (3) is converging to the root much faster in comparison to the other method.

***Example* 4.** Consider the boundary problem

$$x'' + 3xx' = 0, \quad x(0) = 0, \quad x(2) = 1.$$

We take $t_0 = 0 < t_1 < t_2 < t_3 < \ldots < t_{n-1} < t_n = 2,\ t_{i+1} = t_i + h,\ h = \dfrac{2}{n}$. Here $x_0 = x(t_0) = 0$, $x_1 = x(t_1),\ x_2 = x(t_2),\ x_3 = x(t_3), \ldots, x_{n-1} = x(t_{n-1})$ and $x_n = x(t_n) = 1$.

Table 3. Comparison of the different methods for Example 2

Method	N	x	$f(x)$
Newton method (2)	1	–2.0000000	–9.0000000
	2	–1.25000000000000	–2.95312500000000
	3	–0.62000000000000	–1.23832800000000
	4	0.45381893860562	–0.90653525030627
	5	1.92104897791877	6.08949519579644
	6	1.37102304993910	1.57711779001700
	7	1.09134823246687	0.29984045141206
	8	1.00743271644716	0.02246429578528
	9	1.00005470281893	1.641174341520113e–004
	10	1.00000000299218	8.976540177840775e–009
	11	1.0000000000	0.00000000
Jarratt method (14)	1	–2.0000000	–9.0000000
	2	–0.53409090909091	–1.15235108705860
	3	3.58636394131284	45.12783567736838
	4	1.60490771004722	3.13380694233384
	5	1.02464061498428	0.07575828546674
	6	1.00000023207008	6.962104164287553e–007
	7	1.0000000000	0.00000000
Proposed method (3)	1	–2.0000000	–9.0000000
	2	–0.83625920116901	–1.58482068849867
	3	0.84307872254916	–0.40075504502418
	4	0.84307872254916	0.01634336199084
	5	1.00000000335886	1.007659089502511e–008
	6	1.0000000000	0.00000000

We discretize the above problem by using the central difference schemes for the first and second order derivatives, i.e.,

$$x_i'' = \frac{x_{i-1} - 2x_i + x_{i+1}}{h^2}, \quad i = 1, 2, 3, \ldots, n-1,$$

$$x_i' = \frac{x_{i+1} - x_{i-1}}{2h}, \quad i = 1, 2, 3, \ldots, n-1,$$

$$x_i = \frac{x_{i+1} - x_{i-1}}{2}, \quad i = 1, 2, 3, \ldots, n-1.$$

Thus we get an $(n-1) \times (n-1)$ nonlinear system

$$4(x_{i-1} - 2x_i + x_{i+1}) + 3h(x_{i+1}^2 - x_{i-1}^2) = 0, \quad i = 1, 2, 3, \ldots, n-1. \tag{15}$$

Table 4. Comparison of the different methods for Example 3

Method	N	x	y	$f(x,y)$	$g(x,y)$
Newton method (2)	1	0.1000000	0.1000000	0.323333	0.323333
	2	1.71667	1.71667	–2.61361	–2.61361
	3	0.955421	0.955421	–0.579495	–0.579495
	4	0.652154	0.652154	–0.091971	–0.091971
	5	0.58164	0.58164	–0.00497212	–0.00497212
	6	0.577366	0.577366	-0.000018269	-0.000018269
	7	0.57735	0.57735	-2.50303×10^{-10}	-2.50303×10^{-10}
	8	0.57735	0.57735	0.00000000	0.00000000
Jarratt method (14)	1	0.1000000	0.1000000	0.323333	0.323333
	2	0.955421	0.955421	–0.579495	–0.579495
	3	0.58164	0.58164	–0.00497212	–0.00497212
	4	0.57735	0.57735	–2.50303$\times 10^{-10}$	-2.50303×10^{-10}
	5	0.57735	0.57735	0.00000000	0.00000000
Proposed method (3)	1	0.1000000	0.1000000	0.323333	0.323333
	2	0.348886	0.348886	0.211612	0.211612
	3	0.577366	0.577366	–0.000018269	–0.000018269
	4	0.57735	0.57735	0.00000000	0.00000000

Table 5. Solution of Example 4 by proposed method

N	x_1	x_2
1	0.7642513878376436	0.9813462685344896
2	0.7321437776456221	0.9820633448941537
3	0.7321436796857499	0.9820632479169275
N	$f(x_1,x_2)$	$g(x_1,x_2)$
1	–0.2625450310300572	0.038075035451129
2	–1.481897515809294$\times 10^{-8}$	–6.708611455241709$\times 10^{-7}$
3	–2.220446049250313$\times 10^{-16}$	0.000000

Next, we solve the above problem for $n = 3$ by the proposed method using the initial approximations $x_0 = [0.1, 0.1]$. The solution of the problem is shown in Table 5 with $x = [x_1, x_2]$ and $F = [f, g]$. We use the numerical iterations up to 3 and solution comes out to be $[0.7321436796857499, 0.9820632479169275]$.

6. Corresponding conjugacy maps for quadratic polynomials. In this section, we have discussed the rational map $R(z)$ arising from various methods applied to a generic polynomial with simple roots.

Theorem 3 (Newton method). *For a rational map $R(z)$ arising from Newton method applied to $P(z) = (z - a)(z - b),\ a \neq b,\ R(z)$ is conjugate via the Mobius transformation given by $M(z) = (z - a)/(z - b)$ to*

$$S(z) = MoRoM^{-1}(z) = M\left(R\left(\frac{zb - a}{z - 1}\right)\right),$$

$$S(z) = z^2.$$

Theorem 4 (Jarratt method [9]). *For a rational map $R(z)$ arising from Jarratt method* (14) *applied to $P(z) = (z - a)(z - b),\ a \neq b,\ R(z)$ is conjugate via the Mobius transformation given by $M(z) = (z - a)/(z - b)$ to*

$$S(z) = z^4 M(z),$$

where $M(z) = 1$.

Theorem 5 (proposed Newton-like method). *For a rational map $R(z)$ arising from proposed Newton-like method* (3) *applied to $P(z) = (z - a)(z - b),\ a \neq b,\ R(z)$ is conjugate via the Mobius transformation given by $M(z) = (z - a)/(z - b)$ to*

$$S(z) = z^4 M(z),$$

where $M(z) = (4 + 14z + 14z^2 + 6z^3 + z^4)/(1 + 6z + 14z^2 + 14z^3 + 4z^4)$.

Theorem 6 (Newton-like method). *For a rational map $R(z)$ arising from any Newton-like method of order p applied to $P(z) = (z - a)(z - b),\ a \neq b,\ R(z)$ is conjugate via the Mobius transformation given by $M(z) = (z - a)/(z - b)$ to*

$$S(z) = z^p M(z),$$

where $M(z)$ is either unity or a rational function and p is the order of the Newton-like method.

7. Extraneous fixed points. The Newton-like iterative methods discussed in earlier sections can be written in the fixed-point iteration form as

$$x_{n+1} = x_n - E_f(x_n)\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots.$$

Clearly, the root x^* of $f(x) = 0$ is a fixed point of the method. However, the points $\xi \neq x^*$ at which $E_f(\xi) = 0$ are also fixed points of the method as, with $E_f(\xi) = 0$, second term on right-hand side of (15) vanishes. These points are called extraneous fixed points (see [24]). In this section, we have discussed the extraneous fixed points of some Newton-like method for the polynomial $z^3 - 1$.

Theorem 7. *Newton method given by $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)},\ n = 0, 1, 2, \ldots,$ has no extraneous fixed points.*

Proof. For Newton method, we have $E_f(x_n) = 1$. Hence, it has no extraneous fixed point.

Theorem 8. *Jarratt method* [9] *given by equation* (14) *has* 6 *extraneous fixed points.*

Proof. For Jarratt method (14), $E_f(x_n)$ given by the equation $(1 + 7z^3 + 19z^6)/(2 + 14z^3 + + 11z^6)$. In this equation numerator is of degree 6 and hence, Jarratt method has 6 extraneous fixed points

$$z = -0.59835788680381586464044506025 61 - 0.12932136746874947783068606644 55i,$$

$$z = -0.5983578868038158646404450602561 + 0.1293213674687494778306860664455i,$$

$$z = 0.1871833539218283973810430511561 + 0.5828538142612528060331642836771i,$$

$$z = 0.1871833539218283973810430511561 - 0.5828538142612528060331642836771i,$$

$$z = 0.4111745328819874672594020091000 - 0.4535324467925033282024782172316i,$$

$$z = 0.4111745328819874672594020091000 + 0.4535324467925033282024782172316i.$$

These fixed points are repelling (the magnitude of the derivative at these points are greater than 1).

Theorem 9. *There are* 27 *extraneous fixed points for the proposed frozen-type Newton-like method* (3).

Proof. For the proposed Newton-like method (3) we have $E_f(x_n)$ given by the equation

$$-(1/(1594323z^{26}))(-1 - 18z^3 + 18z^6 + 1434z^9 + 900z^{12} -$$
$$-38376z^{15} - 21495z^{18} + 87246z^{21} + 561861z^{24} + 1002754z^{27}).$$

In this equation numerator is of degree 27 and hence, proposed Newton-like method (3) has 27 extraneous fixed points:

$$z = -0.817521886913956291382533799779,$$

$$z = -0.7098160218218003705197801214 31 - 0.3039738768290894876751530986 73i,$$

$$z = -0.709816021821800370519780121431 + 0.303973876829089487675153098673i,$$

$$z = -0.489097660021366552881238963 30,$$

$$z = -0.455370545479753046629856907 11 - 0.0169465585223846117720177893 8i,$$

$$z = -0.45537054547975304662985690711 + 0.0169465585223846117720177893 8i,$$

$$z = -0.327728143334710450688745469971 - 0.567641795325934006419800381417i,$$

$$z = -0.327728143334710450688745469971 + 0.567641795325934006419800381417i,$$

$$z = -0.316119935047690763754218947392 - 0.456037024774720125076116948043i,$$

$$z = -0.316119935047690763754218947392 + 0.456037024774720125076116948043i,$$

$$z = -0.236879680997335665242581064393 - 0.501786406781346974501530338165i,$$

$$z = -0.236879680997335665242581064393 + 0.501786406781346974501530338165i,$$

$$z = 0.0916589114900667429816184411 88 - 0.76670564532543332351162092724 0i,$$

$$z = 0.091658911490066742981618441188 + 0.766705645325433323511620927240i,$$

$$z = 0.213009122552771769949466657842 - 0.402835739781835520906304451 68i,$$

$$z = 0.213009122552771769949466657842 + 0.402835739781835520906304451 68i,$$

$$z = 0.242361422926981276680390328 69 - 0.385889181259450909134286662 29i,$$

$$z = 0.242361422926981276680390328 69 + 0.385889181259450909134286662 29i,$$

$$z = 0.244548830010683276440619481 65 - 0.423570998510028064132160109 87i,$$

$$z = 0.244548830010683276440619481 65 + 0.423570998510028064132160109 87i,$$

$$z = 0.408760943456978145691266899889 - 0.707994722217275186239453660464i,$$

$$z = 0.408760943456978145691266899889 + 0.707994722217275186239453660464i,$$

$$z = 0.552999616045026428996800011785 - 0.045749382006626849425413390123i,$$

$$z = 0.552999616045026428996800011785 + 0.045749382006626849425413390123i,$$

$$z = 0.618157110331733627538161680243 - 0.462731768496343835836467828567i,$$

$$z = 0.618157110331733627538161680243 + 0.462731768496343835836467828567i,$$

$$z = 0.655456286669420901377490939942.$$

These fixed points are repelling (the magnitude of the derivative at these points are greater than 1).

***Remark*.** Similarly we may calculate the extraneous fixed points for other Newton-like method. These fixed points are repelling (the derivative at these points has its magnitude > 1). These fixed points can be seen in the basin of attractions plot for Example 2 $(z^3 - 1)$, Fig. 2 (see dynamics of methods in Subsection 8.2).

8. Dynamics of methods. In numerical and theoretical sections, we have seen the advantages of the fourth-order frozen-type Newton-like method. Now we have disclosed some defects of the fourth order frozen-type Newton-like method by the study of the dynamical analysis of the functions $F(z) = (e^z - 1)$ and $z^3 - 1$. For these purpose we have plotted the basins of attraction of above two examples by using different iterative methods. The dynamics of the function by iterative methods usually help us to study the important information about the convergence, divergence and stability of the methods. The basic definitions and dynamical concepts of function can be found in [1, 4].

***8.1. For Example* 1.** We have taken a square $R \times R = [-5.0, 5.0] \times [-5.0, 5.0]$ of $500 \times \times 500$ points to study the dynamics of function $F(z) = (e^z - 1)$. If with every starting point z_0 in the above squares numerical iterative methods generate a sequence that converges to a zero z^* of the function with a tolerance $-|F(z_n)| < 5 \times 10^{-2}$ and a maximum of 21 iterations, then we say that z_0 will lie in the basin of attraction of this zero and we assign a fixed color to this point z_0, i.e., z_0 would be the part of the basin if both the above criterion are satisfied. We have described the basins

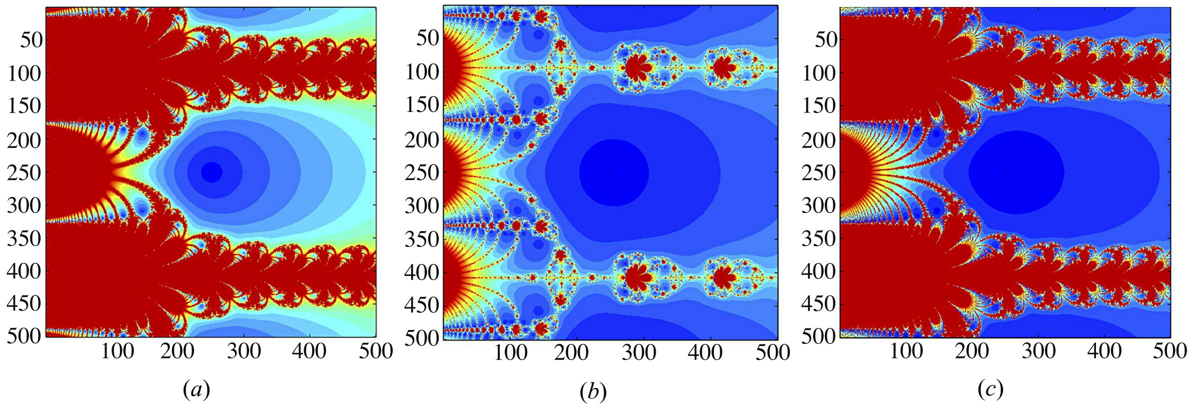

Fig. 1. Basin of attraction for $e^x - 1$ by Newton method (*a*), Jarratt method (*b*) and proposed method (*c*).

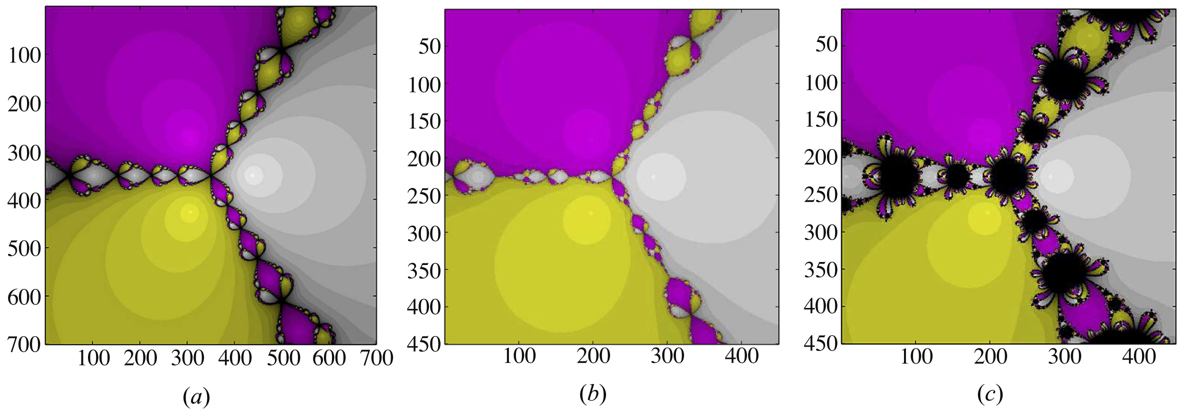

Fig. 2. Basin of attraction for $f_2 = z^3 - 1$ by Newton 2nd order method (*a*), Jarratt 4th order method (*b*), proposed 4th order method (*c*).

of attraction for 2nd order Newton method (2), 4th order Jarratt method (14) and 4th order proposed frozen-type Newton-like method (3) for finding complex roots of above mentioned function in Fig. 1.

1. The basins of attraction for all the iterative methods contains fractal Julia set and basin of Newton method looks almost similar to that of proposed method.

2. The Fatou set with basins of attraction of the Jarratt method is larger in comparison to the other methods shown in blue color.

3. Again the Fatou set with bigger orbits of proposed method in comparison to the other is showing the faster convergence of the proposed method to the roots.

4. Julia set with blue color shows the chaotic behavior and instability in the case of Newton method and proposed frozen-type Newton-like method.

8.2. For Example 2. We have also considered Example 2 for the illustrations of the dynamics of the iterative methods under the same previous conditions. We have plotted the fractal patterns graph of the Example 2 $(F(z) = z^3 - 1)$ for the different iterative methods with a fixed different color to each root of the basins of attraction.

Following points may be concluded by the study of the basins of attraction for 2nd order Newton method (2), 4th order Jarratt method (14) and 4th order proposed frozen-type Newton-like method (3) in Fig. 2.

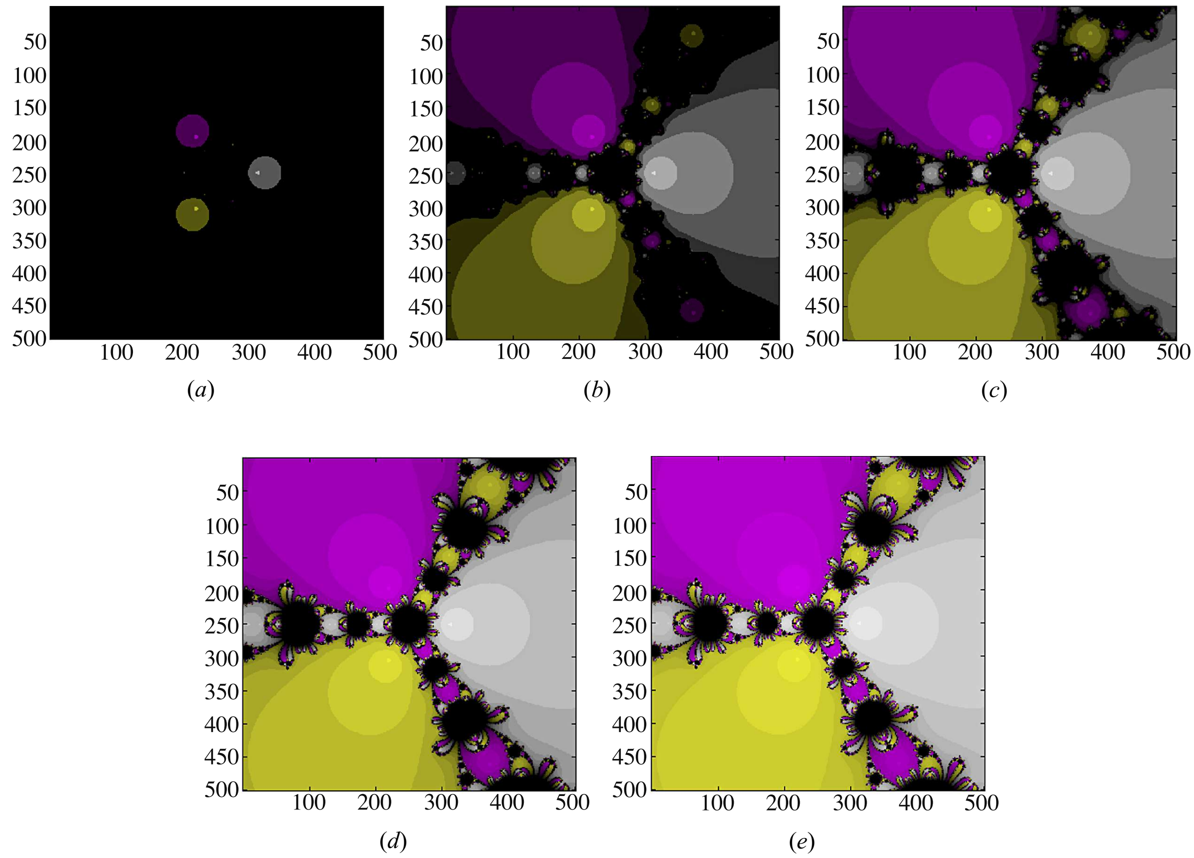

Fig. 3. Dynamics of the proposed method with 03 (a), 06 (b), 09 (c), 15 (d), and 21 (e) iterations for $f_2 = z^3 - 1$.

1. We can see the extraneous fixed points for the Newton-like methods in the basins of attraction plot of Example 2 ($z^3 - 1$) (see Fig. 2). These fixed points are repelling (the derivative at these points has its magnitude > 1). Clearly there is no extraneous fixed point for Newton method. Again there are 6 extraneous fixed points for Jarratt 4th order method and 27 extraneous fixed points for proposed 4th order frozen-type Newton-like method.

2. The basins in the 4th order proposed frozen method contains a lot of black area with black points which shows some defects of the method.

8.3. Effect of stop criterion on the dynamics of the proposed method. To study the effect of the stopping criterion such that the tolerance $|F(z_n)| < 5 \times 10^{-2}$ and maximum number of iterations, we have potted the fractal patterns graph for the proposed method with a fixed value of tolerance, e.g., $|F(z_n)| < 5 \times 10^{-2}$ and variable value of iterations in the Fig. 3. We can observe from this figure the following.

1. In the proposed method with only three iterations, very few number of starting points are fulfilling the tolerance and hence the Fatou set with the basin having the colored region is very small and the Juia set with non converging area having the black region is very large (Fig. 3(a)).

2. As the number of iterations increases Fatou set with the basin get increases and the Juia set with non converging area decreases.

3. It happen because of the fact that with increasing number of iterations large number of starting points pass the stopping criterion and become the part of the basin.

4. Thus we conclude that the tolerance $-|F(z_n)| < 5 \times 10^{-2}$ and number of iterations both are affecting the dynamics of the proposed method.

9. Compliance with ethical standards. Author M. K. Singh declares that he has no conflict of interest. This article does not contain any studies with animals performed by any of the authors. This article does not contain any studies with human participants or animals performed by any of the authors.

10. Conclusion. We have discussed a three-steps 4th order frozen-type Newton-like method for solving nonlinear equation in Banach space. We have performed the local and semilocal convergence analysis for the method. Local convergence analysis demand the fourth-order differentiability while the semilocal convergence analysis need only the second-order derivative using the recurrence relations technique under the same conditions of Kantorovich theorem for Newton method. We have studied about the extraneous fixed points and they are repulsive. Theoretical results are checked by the numerical examples and numerical results are examined with the basins for some selected examples. Theoretical and numerical results show about the faster convergence and ease of not calculating the inverse of the Jacobian of the proposed method at each step, but dynamical analysis divulge some internal hidden defects of frozen-type Newton-like method.

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