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CALCULATING HEAT AND WAVE PROPAGATION FROM LATERAL CAUCHY DATA

РОЗРАХУНОК ПОШИРЕННЯ ТЕПЛА І ХВИЛЬ ЗА ДАНИМИ КОШІ НА БІЧНІЙ МЕЖІ

We give an overview of recent methods based on semi-discretization in time for the inverse ill-posed problem of calculating the solution of evolution equations from time-like Cauchy data. Specifically, the function value and normal derivative are given on a portion of the lateral boundary of a space-time cylinder and the corresponding data is to be generated on the remaining lateral part of the cylinder for either the heat or wave equation. The semi-discretization in time constitutes of applying the Laguerre transform or the Rothe method (finite difference approximation), and has the feature that the similar sequence of elliptic problems is obtained for both the heat and wave equation, only the values of certain parameters change. The elliptic equations are solved numerically by either a boundary integral approach involving the Nyström method or a method of fundamental solutions. Theoretical properties are stated together with discretization strategies in space. Systems of linear equations are obtained for finding values of densities or coefficients. Tikhonov regularization is incorporated for the stable solution of the linear equations. Numerical results included show that the proposed strategies give good accuracy with an economical computational cost.

Наведено огляд останніх методів, що ґрунтуються на частковій дискретизації за часом, для обернених некоректних задач обчислення розв’язку еволюційних рівнянь за нестаціонарними даними Коші. Зокрема, значення функції та її нормальної похідної задані на частині бічної межі просторово-часового циліндра і необхідно згенерувати відповідні дані на решті бічної межі для випадків рівняння теплопровідності та хвильового рівняння. Часткова дискретизація за часом полягає у застосуванні перетворення Лагерра або методу Роте (скінченнорізницева апроксимація) і має ту особливість, що для рівняння теплопровідності і хвильового рівняння отримано однакові послідовності еліптичних задач, які відрізняються лише значеннями певних параметрів. Еліптичні рівняння розв’язано чисельно за допомогою граничних інтегральних рівнянь методом Нистрьома або методом фундаментальних розв’язків. Теоретичні властивості викладені разом із стратегіями дискретизації за просторовими змінними. Отримано системи лінійних рівнянь для знаходження значень густин або коефіцієнтів. Для одержання стійкого розв’язку лінійних рівнянь застосовано регуляризацію Тихонова. Наведені числові результати показують, що запропоновані підходи дають хорошу точність при економних обчислювальних затратах.

1. Introduction. We consider the lateral Cauchy problem for the heat equation:

$$
\begin{aligned}
&\frac{1}{c}\frac{\partial u}{\partial t} = \Delta u \quad \text{in} \quad D \times (0,T),\\
&u = f_2 \quad \text{on} \quad \Gamma_2 \times (0,T),\\
&\frac{\partial u}{\partial \nu} = g_2 \quad \text{on} \quad \Gamma_2 \times (0,T),\\
&u(x,0) = 0 \quad \text{for} \quad x \in D,
\end{aligned}
\tag{1.1}
$$

with $c > 0$ a given constant specifying the heat diffusivity, together with the lateral Cauchy problem for the wave equation

$$\begin{aligned}
&\frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} = \Delta u \quad \text{in} \quad D \times (0,T), \\
&u = f_2 \quad \text{on} \quad \Gamma_2 \times (0,T), \\
&\frac{\partial u}{\partial \nu} = g_2 \quad \text{on} \quad \Gamma_2 \times (0,T), \\
&\frac{\partial u}{\partial t}(x,0) = u(x,0) = 0 \quad \text{for} \quad x \in D,
\end{aligned} \tag{1.2}$$

where $a > 0$ is the given constant speed of sound. In both problems, Γ_2 is a portion of the boundary of the bounded domain D, and the final time $T > 0$.

We assume that the given lateral data f_2 and g_2 are sufficiently smooth and compatible such that there exists a solution. For the heat equation, the solution is unique by the Holmgren uniqueness theorem. However, for the wave equation, uniqueness is more subtle due to finite speed of propagation. The solution can be shown to be unique in the region described by $(x,t) \in D \times (0,T)$ with $\operatorname{dist}(x, \Gamma_2) < T - t$ (geodesic distance). For both equations, the respective solution does not in general depend continuously on the data, that is both these problems are ill-posed. For proof of uniqueness and additional properties of lateral Cauchy problems, see [18] (Chapt. 3) and [22] (Chapt. 4) and references therein (for overview and references to other inverse ill-posed problems for parabolic and hyperbolic equations, see, for example, [3, 15, 16, 21]).

In applications, lateral Cauchy problems occur for example when a part of the boundary is inaccessible for measurements (this part can be too hot to place sensors on or too risky to approach like in measurements of heart activity). To model a typical situation, for the remaining part of this work, let D be the annular region between two bounded simply connected domains D_1 and D_2, with $\bar{D}_1 \subset D_2$, in $\mathbb{R}^d$, $d = 2, 3$. The boundary of D_1 is denoted by Γ_1 and the boundary of D_2 by Γ_2. It is assumed that each boundary part is a simple closed sufficiently smooth surface (curve when $d = 2$). The aim is then to calculate the solution to (1.1) or (1.2) and, in particular, to find the corresponding data on the inner inaccessible lateral boundary $\Gamma_1 \times (0,T)$.

In [4, 5, 7, 10], numerical methods are derived, based on various time-transformations, for the stable reconstructions of the solution to the respective lateral Cauchy problem. We shall survey these methods here and present the main findings together with some additional numerical results.

We begin in Section 2 by presenting the two semidiscretizations in time; the Laguerre transform respectively the Rothe method (finite difference approximation). Both these transformations applied to either the heat or wave equation render the similar sequence of elliptic equations. Included in Section 2 is the definition and explicit representation of what is known as a fundamental sequence of the obtained elliptic equations. In Section 3, it is outlined how to generate a numerical approximation to the sequence of elliptic equations based on integral equations and the Nyström method. As an alternative, in Section 4, a method of fundamental solutions (MFS) is given for the numerical solution of the elliptic equations. The discretization strategies in space in combination with the semidiscretization in time render explicit expressions for the sought Cauchy data on the inner lateral boundary part. Numerical results are presented in Section 5.

2. Semidiscretization in time of (1.1) and (1.2). ***2.1. The Laguerre transform.*** The Laguerre transformation with respect to the time-variable of an element $u(x,t)$ has the following representation:

$$u(x,t) = \kappa \sum_{p=0}^{\infty} \widetilde{u}_p(x) L_p(\kappa t), \tag{2.1}$$

where $L_p(t) = \sum_{k=0}^{p} \binom{p}{k} \frac{(-t)^k}{k!}$ is the Laguerre polynomial of order p [1] (Chapt. 22), $\kappa > 0$ is a given constant and the Fourier–Laguerre coefficients $\widetilde{u}_p$ are defined as

$$\widetilde{u}_p(x) = \int_0^{\infty} e^{-\kappa t} L_p(\kappa t) u(x,t)\, dt, \quad p = 0, 1, 2, \ldots. \tag{2.2}$$

Assuming that the solution to (1.1) and (1.2) has been extended into the time interval $(0, \infty)$, then applying the transform (2.2) with respect to the time-variable, we obtain (details are given in [7, 13]) the following theorem.

Theorem 2.1. *The function u defined in* (2.1) *is a solution of the lateral Cauchy problem for the heat equation* (1.1) *respectively the wave equation* (1.2) *when $T = \infty$ provided that the Fourier–Laguerre coefficients $\widetilde{u}_p$, $p = 0, 1, 2, \ldots$, are the solution of the following sequence of elliptic Cauchy problems:*

$$\begin{gathered} \Delta \widetilde{u}_p - \gamma_0 \widetilde{u}_p = \sum_{m=0}^{p-1} \gamma_{p-m} \widetilde{u}_m \quad \textit{in} \quad D, \\ \widetilde{u}_p = \widetilde{f}_{2,p} \quad \textit{on} \quad \Gamma_2, \\ \frac{\partial \widetilde{u}_p}{\partial \nu} = \widetilde{g}_{2,p} \quad \textit{on} \quad \Gamma_2, \end{gathered} \tag{2.3}$$

where

$$\widetilde{f}_{2,p}(x) = \int_0^{\infty} e^{-\kappa t} L_p(\kappa t) f_2(x,t)\, dt, \quad p = 0, 1, 2, \ldots,$$

$$\widetilde{g}_{2,p}(x) = \int_0^{\infty} e^{-\kappa t} L_p(\kappa t) g_2(x,t)\, dt, \quad p = 0, 1, 2, \ldots,$$

with the coefficients γ_p being in the case of the heat equation: $\gamma_p = \dfrac{\kappa}{c}$, $p = 0, 1, 2, \ldots$, and in the case of the wave equation: $\gamma_p = \dfrac{\kappa^2}{a^2}(p+1)$, $p = 0, 1, 2, \ldots$.

We remark that when $T = \infty$ uniqueness of a solution to the lateral Cauchy problem (1.2) can be shown by time-transformation in combination with uniqueness of a solution to elliptic equations with Cauchy data, for assumptions and details see [19].

2.2. The Rothe method. We present an alternative to the time-transformation of the previous section, which operates directly on the time interval $(0, T)$ that is without any time extension of the solution.

The time derivatives in the heat equation (1.1) and the wave equation (1.2) are discretized by a finite difference approximation [12]. Thus, on the equidistant mesh

$$\{t_p = (p+1)h_t,\ p = -1,\ldots,N_t-1,\ h_t = T/N_t,\ N_t \in \mathbb{N}\}$$

we approximate the solution u by the sequence $\hat{u}_p \approx u(\cdot,t_p),\ p = 0,\ldots,N_t-1$; the elements of this sequence satisfy the equations

$$\Delta\hat{u}_p - \alpha_0\hat{u}_p = \alpha_2\hat{u}_{p-1} + \alpha_1\hat{u}_{p-2} \quad \text{in} \quad D$$

with the coefficients α_p being: in the case of the heat equation: $\alpha_0 = \dfrac{1}{ch_t}$, $\alpha_2 = -\dfrac{1}{ch_t}$, $\alpha_1 = 0$ and in the case of the wave equation: $\alpha_0 = \dfrac{1}{a^2h_t^2}$, $\alpha_2 = -\dfrac{2}{a^2h_t^2}$ and $\alpha_1 = \dfrac{1}{a^2h_t^2}$.

Note that other higher order finite difference approximations of the time derivatives can also be applied (such as [17] applied, for example, in [5] and [6]).

2.3. A fundamental sequence. Interestingly, the described semidiscretization approaches (the Laguerre transform respectively the Rothe method) for the lateral Cauchy problems for the heat and wave equation, all lead to stationary elliptic problems that can be written into the following form:

$$\Delta u_p - \gamma^2 u_p = \sum_{m=0}^{p-1} \beta_{p-m} u_m \quad \text{in} \quad D, \tag{2.4}$$

$$u_p = f_{2,p} \quad \text{on} \quad \Gamma_2, \quad \frac{\partial u_p}{\partial \nu} = g_{2,p} \quad \text{on} \quad \Gamma_2, \tag{2.5}$$

with given functions $f_{2,p}$ and $g_{2,p}$, $p = 0,\ldots,N$, $N \in \mathbb{N}$ and with the constants γ^2 and β_i being explicitly known with their values depending on the type of semidiscretization used together with the type of the underlying governing partial differential equation (heat or wave equation).

In order to generate solutions to (2.4), (2.5), we shall need what is known as a fundamental sequence.

Definition 2.1. *The sequence of functions* $\{\Phi_p\}_{p=0}^N$ *is a fundamental sequence for* (2.3) *provided that*

$$\Delta_x\Phi_p(x,y) - \gamma^2\Phi_p(x,y) - \sum_{m=0}^{p-1} \beta_{p-m}\Phi_m(x,y) = \delta(x-y),$$

where δ *is the Dirac delta function.*

It is possible to derive explicit expressions for the elements in this fundamental sequence in $\mathbb{R}^d$, details can be found in [7, 8, 10] and we recall the result.

Theorem 2.2. *The functions* Φ_p *specified by:*

a) *in the two-dimensional case* $(d = 2)$

$$\Phi_p(x,y) = K_0(\gamma|x-y|)v_p(|x-y|) + K_1(\gamma|x-y|)w_p(|x-y|), \quad x \neq y; \tag{2.6}$$

b) *in the three-dimensional case* $(d = 3)$

$$\Phi_p(x,y) = \frac{e^{-\gamma|x-y|}}{|x-y|}\,\widetilde{v}_p(|x-y|), \quad x \neq y \tag{2.7}$$

for $p = 0,1,2,\ldots,N$, *constitute a fundamental sequence of the elliptic equations* (2.3) *in the sense of Definition* 2.1.

The elements K_0 and K_1 are what is known as modified Bessel functions [1] (Chapt. 9.6 – 9.11), which for $\ell = 0, 1, 2, \ldots,$ have the following representation:

$$K_\ell(z) = \frac{1}{2}\left(\frac{z}{2}\right)^{-\ell} \sum_{k=0}^{\ell-1} \frac{(\ell-k-1)!}{k!}\left(-\frac{z^2}{4}\right)^k + (-1)^{\ell+1}\ln\left(\frac{z}{2}\right) I_\ell(z) +$$

$$+\frac{(-1)^\ell}{2}\left(\frac{z}{2}\right)^\ell \sum_{k=0}^{\infty}[\psi(k+1)+\psi(\ell+k+1)]\,\frac{\left(\frac{z^2}{4}\right)^k}{k!(\ell+k)!},$$

$$I_\ell(z) = \left(\frac{z}{2}\right)^\ell \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!\Gamma(\ell+k+1)},$$

$$\psi(1) = -\zeta, \quad \psi(\ell) = -\zeta + \sum_{k=1}^{\ell-1}\frac{1}{k}, \quad \ell = 2, 3, \ldots,$$

with $\Gamma(\ell)$ the gamma function and $\zeta \approx 0.57721$ the Euler constant.

The polynomials v_p and w_p for $p = 0, 1, \ldots, N$ are given by

$$v_p(r) = \sum_{m=0}^{\left[\frac{p}{2}\right]} a_{p,2m} r^{2m} \quad \text{and} \quad w_p(r) = \sum_{m=0}^{\left[\frac{p-1}{2}\right]} a_{p,2m+1} r^{2m+1}, \quad w_0 = 0,$$

with $[q]$ being the largest integer not greater than q. The coefficients a_p for $p = 0, 1, \ldots, N$ are obtained from the recurrence relations

$$a_{p,0} = 1,$$

$$a_{p,p} = -\frac{1}{2\gamma p}\,\beta_1 a_{p-1,p-1},$$

$$a_{p,k} = \frac{1}{2\gamma k}\left\{4\left[\frac{k+1}{2}\right]^2 a_{p,k+1} - \sum_{m=k-1}^{p-1}\beta_{p-m} a_{m,k-1}\right\}, \quad k = p-1, \ldots, 1.$$

The polynomials $\widetilde{v}_p$ for $p = 0, 1, \ldots$ are given by

$$\widetilde{v}_p(r) = \sum_{m=0}^{p} \widetilde{a}_{p,m} r^m,$$

where the coefficients $\widetilde{a}_p$ for $p = 0, 1, \ldots$ are obtained from the recurrence relations

$$\widetilde{a}_{p,0} = 1,$$

$$\widetilde{a}_{p,p} = -\frac{1}{2\gamma p}\,\beta_1\,\widetilde{a}_{p-1,p-1},$$

$$\widetilde{a}_{p,k} = \frac{1}{2\gamma k}\left\{k(k+1)\widetilde{a}_{p,k+1} - \sum_{m=k-1}^{p-1}\beta_{p-m}\,\widetilde{a}_{m,k-1}\right\}, \quad k = p-1, \ldots, 1.$$

We then turn to the numerical solution of (2.4), (2.5).

3. Numerical solution of the stationary problems via a boundary integral equation method (BIEM). Following the integral approach [9, 11] for the Cauchy problem for the Laplace equation, we search for the solution of the Cauchy problem (2.4), (2.5) in the following potential-layer form:

$$u_p(x) = \frac{1}{\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{p}\int_{\Gamma_\ell} q_m^\ell(y)\Phi_{p-m}(x,y)\,ds(y), \quad x \in D, \tag{3.1}$$

with the unknown densities q_m^1 and q_m^2, $m = 0,\ldots,N$, defined on the two boundary parts Γ_1 and Γ_2, respectively, and Φ_p is given by (2.6) or (2.7).

The boundary integral operators in (3.1) have the similar jump properties as the classical single-layer operator for the Laplace equation. This can be verified by noticing from the above expansion of the elements K_0 and K_1 that the functions in the fundamental sequence each have at most a logarithmic singularity in the 2-dimensional case, and in the 3-dimensional case from (2.7) we see the presence of a weak singularity. Therefore, matching (3.1) against the data (2.5) and employing the corresponding jump properties, we obtain the following system of boundary integral equations:

$$\begin{aligned} &\frac{1}{\pi}\sum_{\ell=1}^{2}\int_{\Gamma_\ell} q_p^\ell(y)\Phi_0(x,y)\,ds(y) = F_p(x), \quad x \in \Gamma_2, \\ &q_p^2(x) + \frac{1}{\pi}\sum_{\ell=1}^{2}\int_{\Gamma_\ell} q_p^\ell(y)\frac{\partial \Phi_0(x,y)}{\partial \nu(x)}\,ds(y) = G_p(x), \quad x \in \Gamma_2, \end{aligned} \tag{3.2}$$

for $p = 0,\ldots,N$, with the right-hand sides

$$F_p(x) = f_{2,p}(x) - \frac{1}{\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{p-1}\int_{\Gamma_\ell} q_m^\ell(y)\Phi_{p-m}(x,y)\,ds(y)$$

and

$$G_p(x) = g_{2,p}(x) - \sum_{m=0}^{p-1} q_m^2(x) - \frac{1}{\pi}\sum_{\ell=1}^{2}\sum_{m=0}^{p-1}\int_{\Gamma_\ell} q_m^\ell(y)\frac{\partial \Phi_{p-m}(x,y)}{\partial \nu(x)}\,ds(y).$$

The following is shown in [7].

Theorem 3.1. *The system* (3.2) *has a unique solution for a dense set of data F_p and G_p with the solution and data in corresponding L_2-spaces on the boundary.*

The full discretization of the sequence of systems (3.2) of ill-posed integral equations can be realized by a Nyström method based on trigonometrical quadratures when the dimension $d = 2$ and by a discrete projection method with spherical harmonics as basis functions when $d = 3$. In both cases, parametric representations of the given boundary parts are needed. Additionally, when $d = 3$, it is assumed that the boundary surfaces Γ_1 and Γ_2 can each be smoothly mapped bijectively onto the unit sphere (for details we refer to [8], Sect. 4). In [10], for 3-dimensional domains, discretization using the boundary element method is instead used.

The discretization renders a set of linear equations to solve for the values of the densities at a finite number of points on the respective boundary part. Due to ill-posedness of the lateral Cauchy problems, Tikhonov regularization is applied to obtain a stable solution of the linear equations.

Using (3.1) together with the obtained values of the densities, explicit formulas can be given for the numerical approximation of the sought Cauchy data on the interior lateral boundary $\Gamma_1 \times (0,T)$, see [7] (Sect. 4.2).

4. Numerical solution of the stationary problems via a MFS. The method of fundamental solutions has become a popular choice for approximating solutions to elliptic equations having an explicitly known fundamental solution, see [14, 20]. As an alternative to the BIEM of the previous section, an MFS can be derived for (2.4), (2.5).

Following [4], the function u_p solving (2.3) is approximated by the element $u_{p,n}$, where

$$u_p(x) \approx u_{p,n}(x) = \sum_{m=0}^{p} \sum_{k=1}^{n} \alpha_{mk} \Phi_{p-m}(x, y_k), \quad x \in D, \tag{4.1}$$

for $n > 0$ with Φ_p given by (2.6) for 2-dimensional domains and by (2.7) for 3-dimensional domains, and with the coefficients $\alpha_{mk} \in \mathbb{R},\ k = 1, 2, \ldots, n,\ m = 0, 1, \ldots, p$, to be determined. The so-called source points $y_k,\ k = 1, 2, \ldots, n$, are located outside of the domain D (on what is known as artificial boundaries).

The coefficients α_{mk} in (4.1) is determined by collocating on the boundary of the solution domain D using a set of so-called collocation points. To select source and collocation points in an efficient way, we assume that the boundaries of the domain D have the following parametrisation:

$$\Gamma_\ell = \{x_\ell(s) = (x_{1\ell}(s), x_{2\ell}(s)),\ s \in [0, 2\pi]\}, \quad \ell = 1, 2,$$

in the 2-dimensional case and

$$\Gamma_\ell = \{x_\ell(\theta, \phi) = \rho_\ell(\theta, \phi)(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),\ \theta \in [0, \pi],\ \phi \in [0, 2\pi]\}, \quad \ell = 1, 2,$$

in the 3-dimensional case.

Note that since D is an annular domain, source points have to be placed both in the unbounded exterior region of D and in the bounded region enclosed by Γ_1. We construct an artificial boundary curve (surface when the dimension $d = 3$) in each of these two regions, and place evenly distributed source points y_k on these boundaries. For 2-dimensional domains, source points are distributed according to the rule

$$y_k = \begin{cases} 2x_2\left(\widetilde{s}_k\right) & \text{for even} \quad k, \\ 0.5x_1\left(\widetilde{s}_k\right) & \text{for odd} \quad k, \end{cases} \tag{4.2}$$

where

$$\widetilde{s}_k = \frac{2\pi}{n} k \quad \text{for} \quad k = 1, \ldots, n. \tag{4.3}$$

For 3-dimensional domains, source points are distributed as

$$y_k = \begin{cases} 2x_2\left(\widetilde{\theta}_k, \widetilde{\phi}_k\right) & \text{for even} \quad k, \\ 0.5x_1\left(\widetilde{\theta}_k, \widetilde{\phi}_k\right) & \text{for odd} \quad k, \end{cases} \tag{4.4}$$

where

$$\widetilde{\theta}_k = \pi \, \frac{\left\{ \dfrac{k-1}{\widetilde{n}} \right\} + 1}{\widetilde{n}+1}, \quad \widetilde{\phi}_k = \pi \, \frac{\left[\dfrac{k-1}{\widetilde{n}} \right] + 1}{\widetilde{n}+1},$$

$$\widetilde{n} = \sqrt{\frac{n}{2}}, \quad \text{with} \quad \{q\} = q - [q] \quad \text{for} \quad k = 1, \ldots, n.$$

In two dimensions, we assume that n is even, $n = 2\xi$, and correspondingly in three dimensions, $n = 2\xi^2$, where $\xi \in \mathbb{N}$.

The collocation points are generated in a similar way but with the constants 2 and 0.5 in (4.2) and (4.4) both replaced by unity. The approximation (4.1) is assumed to satisfy the boundary conditions in (2.3) at the collocation points on the outer boundary part Γ_2. This in combination with the observation that it is only the coefficients in front of Φ_0 in $u_{p,n}$ in (4.1) which are not present in $u_{p-1,n}$ $p > 0$, render the following recursive system to determine the coefficients α_{mk}:

$$\begin{gathered}
\sum_{k=1}^{n} \alpha_{pk} \Phi_0 \left(\widetilde{x}_j, y_k \right) = \widetilde{f}_{2,p} \left(\widetilde{x}_j \right) - \sum_{m=0}^{p-1} \sum_{k=1}^{n} \alpha_{mk} \Phi_{p-m} \left(\widetilde{x}_j, y_k \right), \\
\sum_{k=1}^{n} \alpha_{pk} \frac{\partial \Phi_0}{\partial \nu(x)} \left(\widetilde{x}_j, y_k \right) = \widetilde{g}_{2,p} \left(\widetilde{x}_j \right) - \sum_{m=0}^{p-1} \sum_{k=1}^{n} \alpha_{mk} \frac{\partial \Phi_{p-m}}{\partial \nu(x)} \left(\widetilde{x}_j, y_k \right),
\end{gathered} \tag{4.5}$$

where the collocation points $\widetilde{x}_j$ are given by when $d = 2$:

$$\widetilde{x}_j = x_2(s_j), \quad s_j = \frac{4\pi}{n+1} j \quad \text{for} \quad j = 1, \ldots, n/2,$$

and when $d = 3$:

$$\widetilde{x}_j = x_2(\theta_j, \phi_j), \quad \widetilde{\theta}_j = \pi \, \frac{\left\{ \dfrac{2j-1}{\widetilde{n}} \right\} + 1}{\widetilde{n}+1}, \quad \widetilde{\phi}_j = \pi \, \frac{\left[\dfrac{2j-1}{\widetilde{n}} \right] + 1}{\widetilde{n}+1} \quad \text{for} \quad j = 1, \ldots, n/2.$$

Note that when the parameter $p = 0$, the sums in the right-hand side of (4.5) are set to zero.

The system (4.5) is ill-conditioned due to the ill-posedness of the Cauchy problem (2.3) and, therefore, in order to obtain a stable solution, we apply Tikhonov regularization.

The following is shown in [4] building in particular on results in [2] and gives a theoretical justification of the derived MFS.

Theorem 4.1. *Let y_k be a dense set of source points distributed evenly over the artificial boundary parts. Then the corresponding basis elements used in the described MFS is a linearly independent and dense set on Γ_1 respectively on Γ_2 in the L_2-sense. The same holds for the normal derivatives of the basis elements on those two boundary parts.*

Combining the MFS approximation with either the Laguerre expansion or the Rothe method, explicit approximation formulations are obtained for the sought Cauchy data on the inner lateral boundary part $\Gamma_1 \times (0, T)$, see [4] (Sect. 3.2) and [5] (Sect. 4), respectively.

5. Numerical experiments. In this section, we illustrate the considered approaches for the lateral Cauchy problems in a 2-dimensional doubly connected domain for both the parabolic and hyperbolic cases. Let the domain $D \subset \mathbb{R}^2$ be bounded by the inner boundary curve

$$\Gamma_1 = \{x_1(s) = (0.6\cos s, 0.5\sin s),\ s \in [0, 2\pi]\}$$

and outer boundary curve

$$\Gamma_2 = \left\{x_2(s) = \left(\cos s, \sin s - 0.5\cos^2 s\right),\ s \in [0, 2\pi]\right\}.$$

As a semidiscretization approach in time, we use the Laguerre transform. Keeping with the above notation, clearly, the approximation of the Cauchy data have then the form

$$u_{N,n}(x_1(s), t) = \kappa \sum_{p=0}^{N} \tilde{u}_{p,n}(x_1(s)) L_p(\kappa t)$$

and

$$\frac{\partial u_{N,n}}{\partial \nu}(x_1(s), t) = \kappa \sum_{p=0}^{N} \frac{\partial \tilde{u}_{p,n}}{\partial \nu}(x_1(s)) L_p(\kappa t),$$

where the values $\tilde{u}_{p,n}$ and $\dfrac{\partial \tilde{u}_{p,n}}{\partial \nu}$ on Γ_1 can be calculated by the MFS or by the suggested integral equation method. The space discretization parameter n, which correspond to the number of source points for the MFS and to the number of quadrature points in the boundary integral equation approach is taken as $n = 32$. We consider the case when the given Cauchy data on the outer boundary Γ_2 has no noise as well as when some noise are added. In the case of noisy data, noise is added to the known function g_2 to render g_2^δ whilst the function f_2 is defined exactly. The noise is such that

$$\left\| g_2 - g_2^\delta \right\|_{L_2(\Gamma_2 \times (0,\infty))} \le \delta,$$

where δ is the noise level.

***Example* 1.** We consider the parabolic Cauchy problem (1.1) with $c = 1$. As the exact solution, we use the restriction of the fundamental solution for the heat equation

$$u_{ex}(x, t) = \frac{100}{4\pi t} e^{-\frac{|x - x^*|^2}{4t}}, \quad (x, t) \in D \times (0, \infty), \quad x^* = (0, 4).$$

Then the Cauchy data on the boundary Γ_2 is

$$f_2(x, t) = u_{ex}(x, t), \quad g_2(x, t) = \frac{\partial u_{ex}}{\partial \nu(x)}(x, t) \quad \text{with} \quad (x, t) \in D \times (0, \infty).$$

Note that in this case the exact solutions of the sequence of Cauchy problems (2.4), (2.5) have the form

$$u_p^{ex}(x) = \frac{100}{2\pi} \Phi_p(x, x^*), \quad x \in D. \tag{5.1}$$

The relative L_2-errors of the reconstruction of the Cauchy data on the inner boundary Γ_1 for exact and 5% noisy data in (2.4), (2.5) are given in Table 1 for various p (compared against the corresponding exact data obtained from (5.1)). The columns e_p contains the error for function values and the columns q_p are for normal derivatives. The corresponding relative L_2-errors e and q of the reconstruction of the Cauchy data on $\Gamma_1 \times (0, T]$ with $T = 5$ to the parabolic equation (1.1) are also presented. All integrals in these errors are calculated using the trapezoidal quadrature rule. The regularization parameters were chosen by trial and error: we calculated the numerical solutions for $\alpha = 10^{-\ell}$ with $\ell = 1, \ldots, 15$ and used the value giving the most accurate result.

Table 1

	Exact data				5% noise			
p	MFS		BIEM		MFS		BIEM	
	e_p	q_p	e_p	q_p	e_p	q_p	e_p	q
0	1.93E–1	2.16E–1	1.18E–4	9.54E–4	8.87E–2	4.88E–1	7.10E–2	1.86E–1
5	3.34E–2	2.33E–1	1.36E–4	6.38E–4	1.10E+0	3.93E+0	1.29E–1	3.16E–1
10	3.32E–2	3.98E–1	1.56E–4	1.63E–3	1.29E+0	1.02E+1	2.77E–1	1.88E+0
15	3.04E–1	1.02E+0	5.39E–4	1.29E–3	1.60E+1	2.90E+1	1.48E+0	3.01E+0
20	1.21E–1	2.35E+0	1.32E–3	1.84E–2	1.30E+1	3.53E+1	1.32E+0	1.71E+1
N	e	q	e	q	e	q	e	q
20	1.31E–2	8.32E–2	1.27E–2	2.68E–2	4.41E–1	5.87E+0	1.69E–2	3.78E–1
α	1E-10		1E-7		1E-4		1E-2	

***Example* 2.** We now solve the hyperbolic lateral Cauchy problem (1.2) with $a = 1$. The Cauchy data is generated by first solving the Dirichlet initial boundary-value problem for the wave equation with boundary functions

$$f_\ell(x,t) = t^2 e^{-t+2}(x_1 + x_2), \quad x \in \Gamma_\ell, \quad t > 0, \quad \ell = 1, 2,$$

and then taking restrictions of the solution and its normal derivative on the outer boundary $\Gamma_2 \times (0,T)$.

The L_2-errors of the reconstruction of the Cauchy data on the inner boundary Γ_1 from (2.4), (2.5) for exact and 2% noisy data are given in Table 2 for various p together with the corresponding reconstruction of the Cauchy data in the wave equation (1.2) for $T = 5$.

Table 2

	Exact data				2% noise			
p	MFS		BIEM		MFS		BIEM	
	e_p	q_p	e_p	q_p	e_p	q_p	e_p	q
0	2.13E–2	3.38E–1	4.71E–4	7.41E–3	1.25E–1	1.09E+0	7.10E–2	1.86E–1
5	4.37E–2	8.27E–1	1.83E–4	2.57E–3	9.36E–2	1.15E+0	1.29E–1	3.16E–1
10	3.24E–1	6.10E+0	4.91E–4	7.90E–3	5.35E–1	3.65E+0	2.77E–1	1.88E+0
15	1.26E+0	2.45E+1	1.22E–3	1.80E–2	4.59E+0	3.59E+1	1.48E+0	3.01E+0
20	3.52E+0	7.06E+1	2.93E–3	4.95E–2	2.48E+1	2.19E+2	1.32E+0	1.71E+1
N	e	q	e	q	e	q	e	q
20	5.24E+0	4.41E+0	1.37E–2	9.56E–2	3.47E+1	1.28E+2	1.69E–2	3.78E–1
α	1E–10		1E–7		1E–4		1E–2	

As can be seen from the two tables the results are more accurate for the BIEM than the MFS. This can to some extent be attributed to the fact that the BIEM involves a rather large amount of analytical work related to the existing singularities in the kernels, special quadratures, etc. Furthermore, the MFS results are for a fixed set of source points (4.2) and (4.4), adjusting these the results can most likely be further improved.

For both methods, it is also seen from the tables that the accuracy of the approximations decreases with increasing values of p. This is natural since errors are propagating forward in (2.4), (2.5) due to the recursive structure of that system. However, it is pleasing to see that the decrease in accuracy is rather mild, and in total accurate solutions to the time-dependent Cauchy problems are obtained. In general, it is known that numerical solution to time-dependent lateral Cauchy problems is less accurate near the final time T. This is not really seen here due to the existence and smoothness of the solution in time.

We point out that numerical solution of parabolic and hyperbolic Cauchy problems in 3-dimensional doubly connected domains via the presented approaches are considered in [5, 10]. Also, semidiscretization in time using finite differences is investigated in [5, 7].

6. Conclusion. We summarized in this paper results by the authors related to the numerical solution of time-dependent lateral Cauchy problems. The general approach consists of the following steps. Firstly, a semidiscretization in time is carried out (by either the Laguerre transform or the Rothe method). This leads to a sequence of Cauchy problems for elliptic equations with a recursive right-hand side. A key property is that the obtained stationary problems are similar for both parabolic and hyperbolic lateral Cauchy problems only values of some parameters changes. The next step is the explicit construction of a special sequence of fundamental solutions. This gives the possibility to apply a standard version of the MFS to the sequence of stationary elliptic Cauchy problems. It renders a sequence of ill-conditioned linear systems having a recurrent right-hand side for finding the coefficients in the MFS expansion. Tikhonov regularization is incorporated for the stable solution. Once these coefficients have been found, the final step is the calculation of the Cauchy data on the inner boundary. As an alternative to the solution steps for the elliptic equations, a BIEM can be applied. This method is based on the single-layer approach incorporation the constructed sequence of fundamental solutions. As a result, the stationary elliptic problems are reduced to a sequence of boundary integral equations. The full discretization by some suitable projection method leads to linear systems for identifying values of densities in the integral equations. Due to the ill-posedness of the Cauchy problem the obtained linear systems are ill-conditioned and also here the Tikhonov regularization is incorporated for the stable solution. Numerical results confirm the applicability of the stated steps for the numerical approximation of solutions to lateral Cauchy problems for evolution equations.

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