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EIGENFUNCTION AND GREEN'S FUNCTION ASYMPTOTICS FOR HILL'S EQUATION WITH SYMMETRIC SINGLE WELL POTENTIAL

АСИМПТОТИКА ВЛАСНОЇ ФУНКЦІЇ ТА ФУНКЦІЇ ГРІНА ДЛЯ РІВНЯННЯ ХІЛЛА ІЗ СИМЕТРИЧНИМ ПОТЕНЦІАЛОМ ОДНІЄЇ СВЕРДЛОВИНИ

This paper is devoted to determine the asymptotic formulae for eigenfunctions of the Hill's equation with symmetric single well potential under periodic and semi-periodic boundary conditions. The obtained results for eigenvalues by H. Coşkun and the others (2019) are used. With these estimates on the eigenfunctions, Green's functions related to the Hill's equation are obtained. The method is based on the work of C. T. Fulton (1977) to derive Green's functions in an asymptotical manner. We need the derivatives of the solutions in this method. Therefore, the asymptotic approximations for the derivatives of the eigenfunctions are also calculated with different types of restrictions on the potential.

Статтю присвячено встановленню асимптотичних формул для власних функцій рівняння Хілла із симетричним потенціалом однієї свердловини при періодичних та напівперіодичних граничних умовах. При цьому використано результати для власних значень, отримані в роботі H. Coşkun та ін. (2019). За допомогою відповідних оцінок для власних функцій отримано функції Гріна, пов'язані з рівнянням Хілла. Метод базується на роботі Ч. Т. Фултона (1977) щодо асимптотичного отримання функцій Гріна. У цьому методі нам потрібні похідні розв'язків, тому обчислено також асимптотичні наближення похідних власних функцій із різними типами обмежень на потенціал.

1. Introduction. The Hill's equation is the second-order linear differential equation

$$y'' + q(x)y = 0, \tag{1.1}$$

where $q(x)$ is a real-valued and periodic function. This equation has numerous applications in engineering and physics. Some of them contain the problems in mechanics, astronomy, circuits, electric conductivity of metals, cyclotrons, quadrupole mass spectrometers, quantum optics of two-level systems and accelerator physics.

Furthermore, the theory related to the Hill's equation can be extended to every differential equation written in the general form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \tag{1.2}$$

such that the coefficients $a_0,\ a_1$ and a_2 have enough regularity. This is due to the fact that, with a suitable transformation, (1.2) can be reduced into one of the type of (1.1) (details can be seen in [10, 11]).

As an example let us consider a mathematical (or inverted) pendulum studied in [9]. If we assume that the oscillations of the pendulum are small and that the suspension point of the string vibrates vertically with an acceleration $a(t)$, then, as it is proved in [9], the movement would be modelled by the equation (which follows the form (1.1))

$$\theta''(t) - \frac{1}{l}(g + a(t))\theta(t) = 0,$$

where g denotes the gravity, l is the length of the string and θ represents the angle between the string and the perpendicular line to the base [4].

Another equation that fit on the framework of Hill's equation is Mathieu's equation

$$y''(x) + (a + b\cos x)y(x) = 0$$

(see [14]).

At the moment of studying oscillation phenomena of the solutions of (1.1), it is observed that these are determined by the potential $q(x)$. In particular, solutions of (1.1) do not oscillate when $q(x) < 0$ but they do it infinite times for $q(x) > 0$ large enough. Moreover, the larger the potential $q(x)$ is, the faster the solutions of (1.1) oscillate [4].

The theory on Hill's equation takes on a new significance when the equation (1.1) involves a real parameter λ in the form

$$y'' + [\lambda - q(x)]y = 0, \quad x \in [0, a]. \tag{1.3}$$

If we consider (1.3) coupled with suitable boundary value conditions, we have a spectral problem. We introduce here two eigenvalue problems associated with (1.3) and the interval $[0, a]$, where λ is regarded as the eigenvalue parameter. The periodic eigenvalue problem is defined with (1.3) and boundary conditions $y(0) = y(a),\ y'(0) = y'(a)$ and the semiperiodic eigenvalue problem is given with (1.3) and boundary conditions $y(0) = -y(a),\ y'(0) = -y'(a)$. The periodic and semiperiodic eigenvalue problems are self-adjoint and they have a countable infinity of eigenvalues denoted by λ_n and $\mu_n,\ n = 0, 1, 2, \ldots,$ respectively. It is known [11] that the eigenvalues of periodic problem satisfy

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots, \quad \lambda_n \to \infty \quad \text{as} \quad n \to \infty$$

and the eigenvalues of semiperiodic problem satisfy

$$\mu_0 \le \mu_1 \le \mu_2 \le \ldots, \quad \mu_n \to \infty \quad \text{as} \quad n \to \infty.$$

The periodic and semiperiodic problems with various types of restrictions on the potential have been widely studied in the literature [1, 2, 6, 7, 11, 15, 17]. Important results about the eigenvalues and instability intervals were obtained in [1, 6, 13, 16]. The properties of the Green's functions and some criteria for the maximum and antimaximum principles were investigated in [3 – 5]. In addition, Coşkun in [8] has studied on the inverse problem.

Throughout this paper, the equation (1.3) under the periodic and semiperiodic boundary conditions is considered when the potential $q(x)$ is a real-valued, absolutely continuous and periodic function with period a. Here, the first purpose is to derive asymptotic formulae for the eigenfunctions of the periodic and semiperiodic problems with $q(x)$ being of a symmetric single well potential with mean value zero. By a symmetric single well potential on $[0, a]$, we mean a continuous function $q(x)$ on $[0, a]$ which is symmetric about $x = \dfrac{a}{2}$ and nonincreasing on $\left[0, \dfrac{a}{2}\right]$.

It was shown in [6] that the periodic and semiperiodic eigenvalues of (1.3) having symmetric single well potential are, as $n \to \infty$,

$$\begin{matrix}\lambda_{2n+1}^{1/2}\\ \lambda_{2n+2}^{1/2}\end{matrix} = \frac{2(n+1)\pi}{a} \mp \frac{a}{8(n+1)^2\pi^2}\left|\int_0^{a/2} q'(t)\sin\left(\frac{4(n+1)\pi}{a}t\right)dt\right| - \frac{a^2}{64(n+1)^3\pi^3}\times$$

$$\times\left[aq^2(a) + 2a\int_0^{a/2} q(t)q'(t)dt - 4\int_0^{a/2} tq(t)q'(t)dt\right] + o(n^{-3}) \tag{1.4}$$

and

$$\begin{matrix}\mu_{2n}^{1/2}\\ \mu_{2n+1}^{1/2}\end{matrix} = \frac{(2n+1)\pi}{a} \mp \frac{a}{2(2n+1)^2\pi^2}\left|\int_0^{a/2} q'(t)\sin\left(\frac{2(2n+1)\pi}{a}t\right)dt\right| -$$

$$-\frac{a^2}{8(2n+1)^3\pi^3}\left[aq^2(a) + 2a\int_0^{a/2} q(t)q'(t)dt - 4\int_0^{a/2} tq(t)q'(t)dt\right] + o(n^{-3}). \tag{1.5}$$

In this work, the eigenfunctions corresponding to λ_n and μ_n given by (1.4) and (1.5) are investigated.

The following results obtained in [11] will be used to determine the eigenfunctions.

We define the linearly independent solutions $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ of (1.3) with the initial conditions

$$\phi_1(0,\lambda) = 1, \quad \phi_1'(0,\lambda) = 0 \tag{1.6}$$

and

$$\phi_2(0,\lambda) = 0, \quad \phi_2'(0,\lambda) = 1. \tag{1.7}$$

Theorem 1.1 ([11], § 4.3). *Let* $\phi_1(x,\lambda)$ *and* $\phi_2(x,\lambda)$ *be the solutions of* (1.3) *satisfying* (1.6) *and* (1.7), *respectively. Assume that* $q(x)$ *is an absolutely continuous function. Then, as* $\lambda\to\infty$,

$$\phi_1(x,\lambda) = \cos\left(x\sqrt{\lambda}\right) + \frac{1}{2}\lambda^{-\frac{1}{2}}Q(x)\sin\left(x\sqrt{\lambda}\right) +$$

$$+\frac{1}{4}\lambda^{-1}\left\{q(x) - q(0) - \frac{1}{2}Q^2(x)\right\}\cos\left(x\sqrt{\lambda}\right) + o(\lambda^{-1}), \tag{1.8}$$

$$\phi_2(x,\lambda) = \lambda^{-\frac{1}{2}}\sin\left(x\sqrt{\lambda}\right) - \frac{1}{2}\lambda^{-1}Q(x)\cos\left(x\sqrt{\lambda}\right) +$$

$$+\frac{1}{4}\lambda^{-\frac{3}{2}}\left\{q(x) + q(0) - \frac{1}{2}Q^2(x)\right\}\sin\left(x\sqrt{\lambda}\right) + o(\lambda^{-\frac{3}{2}}), \tag{1.9}$$

where

$$Q(x) = \int_0^x q(t)dt. \tag{1.10}$$

Theorem 1.2 ([11], § 4.3). *Let $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ be the solutions of* (1.3) *satisfying* (1.6) *and* (1.7), *respectively. Assume that $q(x)$ is a piecewise continuous function. Then*, *as $\lambda\to\infty$*,

$$\phi_1(x,\lambda)=\cos\left(x\sqrt{\lambda}\right)+\lambda^{-\frac{1}{2}}\int\limits_0^x\sin\left\{(x-t)\sqrt{\lambda}\right\}q(t)\cos(t\sqrt{\lambda})dt+$$

$$+\lambda^{-1}\int\limits_0^x\sin\left\{(x-t)\sqrt{\lambda}\right\}q(t)dt\int\limits_0^t\sin\{(t-u)\sqrt{\lambda}\}q(u)\cos(u\sqrt{\lambda})du+O(\lambda^{-\frac{3}{2}}),\qquad(1.11)$$

$$\phi_2(x,\lambda)=\lambda^{-\frac{1}{2}}\sin\left(x\sqrt{\lambda}\right)+\lambda^{-1}\int\limits_0^x\sin\left\{(x-t)\sqrt{\lambda}\right\}q(t)\sin(t\sqrt{\lambda})dt+$$

$$+\lambda^{-\frac{3}{2}}\int\limits_0^x\sin\left\{(x-t)\sqrt{\lambda}\right\}q(t)dt\int\limits_0^t\sin\{(t-u)\sqrt{\lambda}\}q(u)\sin(u\sqrt{\lambda})du+O(\lambda^{-2}).\qquad(1.12)$$

The second goal of this paper is to determine the Green's function asymptotics related to the Hill's equation with the estimates on the eigenfunctions. The method developed by Fulton [12] is followed. In this method since the derivatives of the solutions are needed, the asymptotic approximations for the derivatives of $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ are also calculated with different types of restrictions on the potential $q(x)$.

2. Approximations for the eigenfunctions. In this section, we obtain approximations for the solutions $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ of (1.3) satisfying the initial conditions (1.6) and (1.7), respectively.

Before, we give the following lemma for $q(x)$ being of a symmetric single well potential.

Lemma 2.1. *If $q(x)$ is a symmetric single well potential on $[0,a]$, then*

$$\int\limits_0^x q(t)dt=xq(x)+\int\limits_{a/2}^a(a-t)q'(t)dt-\int\limits_{a/2}^x tq'(t)dt.\qquad(2.1)$$

Proof. Using integration by parts gives

$$\int\limits_0^x q(t)dt=tq(t)\Big|_{t=0}^x-\int\limits_0^x tq'(t)dt=$$

$$=xq(x)-\left[\int\limits_0^{a/2}tq'(t)dt+\int\limits_{a/2}^x tq'(t)dt\right]=xq(x)-\left[-\int\limits_0^{a/2}tq'(a-t)dt+\int\limits_{a/2}^x tq'(t)dt\right]=$$

$$=xq(x)-\int\limits_a^{a/2}(a-t)q'(t)dt-\int\limits_{a/2}^x tq'(t)dt=xq(x)+\int\limits_{a/2}^a(a-t)q'(t)dt-\int\limits_{a/2}^x tq'(t)dt.$$

Theorem 2.1. *Let $q(x)$ be a symmetric single well potential on $[0, a]$. Then, as $\lambda \to \infty$, for the solutions of* (1.3) *with the initial conditions* (1.6) *and* (1.7), *respectively, we have*

$$\phi_1(x,\lambda) = \cos\left(x\sqrt{\lambda}\right) + \frac{1}{2}\lambda^{-\frac{1}{2}}\left[xq(x) + \int\limits_{a/2}^{a}(a-t)q'(t)dt - \int\limits_{a/2}^{x} tq'(t)dt\right]\sin\left(x\sqrt{\lambda}\right) +$$

$$+\frac{1}{4}\lambda^{-1}\left\{q(x) - q(0) - \frac{1}{2}\left[xq(x) + \int\limits_{a/2}^{a}(a-t)q'(t)dt - \int\limits_{a/2}^{x} tq'(t)dt\right]^2\right\}\cos\left(x\sqrt{\lambda}\right) + o(\lambda^{-1}), \tag{2.2}$$

$$\phi_2(x,\lambda) = \lambda^{-\frac{1}{2}}\sin\left(x\sqrt{\lambda}\right) - \frac{1}{2}\lambda^{-1}\left[xq(x) + \int\limits_{a/2}^{a}(a-t)q'(t)dt - \int\limits_{a/2}^{x} tq'(t)dt\right]\cos\left(x\sqrt{\lambda}\right) +$$

$$+\frac{1}{4}\lambda^{-\frac{3}{2}}\left\{q(x) + q(0) - \frac{1}{2}\left[xq(x) + \int\limits_{a/2}^{a}(a-t)q'(t)dt - \int\limits_{a/2}^{x} tq'(t)dt\right]^2\right\}\sin\left(x\sqrt{\lambda}\right) + o(\lambda^{-\frac{3}{2}}). \tag{2.3}$$

Proof. If we use Theorem 1.1 and substitute (2.1) in (1.10), the proof is done.

Theorem 2.2. *The eigenfunctions of the periodic problem having symmetric single well potential satisfy, as $n \to \infty$,*

$$\phi_1(x,n) = \cos\frac{2(n+1)\pi x}{a} +$$

$$+\frac{a}{4(n+1)\pi}\left[xq(x) + \int\limits_{a/2}^{a}(a-t)q'(t)dt - \int\limits_{a/2}^{x} tq'(t)dt\right]\sin\frac{2(n+1)\pi x}{a} +$$

$$+\frac{a^2}{16(n+1)^2\pi^2}\left\{q(x) - q(0) - \frac{1}{2}\left[xq(x) + \int\limits_{a/2}^{a}(a-t)q'(t)dt - \int\limits_{a/2}^{x} tq'(t)dt\right]^2\right\}\times$$

$$\times\cos\frac{2(n+1)\pi x}{a} + o(n^{-2}),$$

$$\phi_2(x,n) = \frac{a}{2(n+1)\pi}\sin\frac{2(n+1)\pi x}{a} -$$

$$-\frac{a^2}{8(n+1)^2\pi^2}\left[xq(x) + \int\limits_{a/2}^{a}(a-t)q'(t)dt - \int\limits_{a/2}^{x} tq'(t)dt\right]\cos\frac{2(n+1)\pi x}{a} +$$

$$+\frac{a^3}{32(n+1)^3\pi^3}\left\{q(x)+q(0)-\frac{1}{2}\left[xq(x)+\int\limits_{a/2}^{a}(a-t)q'(t)dt-\int\limits_{a/2}^{x}tq'(t)dt\right]^2\right\}\times$$

$$\times\sin\frac{2(n+1)\pi x}{a}+o(n^{-3}).$$

Theorem 2.3. *The eigenfunctions of the semiperiodic problem having symmetric single well potential satisfy, as* $n\to\infty$,

$$\phi_1(x,n)=\cos\frac{(2n+1)\pi x}{a}+$$

$$+\frac{a}{2(2n+1)\pi}\left[xq(x)+\int\limits_{a/2}^{a}(a-t)q'(t)dt-\int\limits_{a/2}^{x}tq'(t)dt\right]\sin\frac{(2n+1)\pi x}{a}+$$

$$+\frac{a^2}{4(2n+1)^2\pi^2}\left\{q(x)-q(0)-\frac{1}{2}\left[xq(x)+\int\limits_{a/2}^{a}(a-t)q'(t)dt-\int\limits_{a/2}^{x}tq'(t)dt\right]^2\right\}\times$$

$$\times\cos\frac{(2n+1)\pi x}{a}+o(n^{-2}),$$

$$\phi_2(x,n)=\frac{a}{(2n+1)\pi}\sin\frac{(2n+1)\pi x}{a}-$$

$$-\frac{a^2}{2(2n+1)^2\pi^2}\left[xq(x)+\int\limits_{a/2}^{a}(a-t)q'(t)dt-\int\limits_{a/2}^{x}tq'(t)dt\right]\cos\frac{(2n+1)\pi x}{a}+$$

$$+\frac{a^3}{4(2n+1)^3\pi^3}\left\{q(x)+q(0)-\frac{1}{2}\left[xq(x)+\int\limits_{a/2}^{a}(a-t)q'(t)dt-\int\limits_{a/2}^{x}tq'(t)dt\right]^2\right\}\times$$

$$\times\sin\frac{(2n+1)\pi x}{a}+o(n^{-3}).$$

To prove Theorems 2.2 and 2.3, the related eigenvalues given by (1.4) and (1.5) are substituted in Theorem 2.1.

We have also some approximations for the derivatives of $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$. We will use them in calculation of the Green's functions.

Lemma 2.2. *If* $q(x)$ *is a piecewise continuous function, then the derivative of* (1.11) *is, as* $\lambda\to\infty$,

$$\phi_1'(x,\lambda) = -\lambda^{\frac{1}{2}} \sin\left(x\sqrt{\lambda}\right) + \int\limits_0^x \cos\left\{(x-t)\sqrt{\lambda}\right\} q(t)\cos(t\sqrt{\lambda})dt +$$

$$+\lambda^{-\frac{1}{2}} \int\limits_0^x \cos\left\{(x-t)\sqrt{\lambda}\right\} q(t)dt \int\limits_0^t \sin\left\{(t-u)\sqrt{\lambda}\right\} q(u)\cos(u\sqrt{\lambda})du + O(\lambda^{-1}). \tag{2.4}$$

Proof. The usual variation of constants formula [10] (§ 2.5) gives

$$\phi_1(x,\lambda) = \cos\left(x\sqrt{\lambda}\right) + \lambda^{-\frac{1}{2}} \int\limits_0^x \sin\left\{(x-t)\sqrt{\lambda}\right\} q(t)\phi_1(t,\lambda)dt.$$

If we arrange this formula, one can write

$$\phi_1(x,\lambda) = \cos\left(x\sqrt{\lambda}\right) +$$

$$+\lambda^{-\frac{1}{2}} \left\{ \sin\left(x\sqrt{\lambda}\right) \int\limits_0^x \cos(t\sqrt{\lambda})q(t)\phi_1(t,\lambda)dt - \cos\left(x\sqrt{\lambda}\right) \int\limits_0^x \sin(t\sqrt{\lambda})q(t)\phi_1(t,\lambda)dt \right\}. \tag{2.5}$$

It is obtained by differentiating (2.5) with respect to x and substituting $\phi_1(t,\lambda)$ from (1.11) in the integral that

$$\phi_1'(x,\lambda) = -\lambda^{\frac{1}{2}} \sin\left(x\sqrt{\lambda}\right) + \lambda^{-\frac{1}{2}} \left\{ \lambda^{\frac{1}{2}} \cos\left(x\sqrt{\lambda}\right) \int\limits_0^x \cos(t\sqrt{\lambda})q(t)\phi_1(t,\lambda)dt + \right.$$

$$\left. +\lambda^{\frac{1}{2}} \sin\left(x\sqrt{\lambda}\right) \int\limits_0^x \sin(t\sqrt{\lambda})q(t)\phi_1(t,\lambda)dt \right\} =$$

$$= -\lambda^{\frac{1}{2}} \sin\left(x\sqrt{\lambda}\right) + \int\limits_0^x \cos\left\{(x-t)\sqrt{\lambda}\right\} q(t)\phi_1(t,\lambda)dt =$$

$$= -\lambda^{\frac{1}{2}} \sin\left(x\sqrt{\lambda}\right) + \int\limits_0^x \cos\left\{(x-t)\sqrt{\lambda}\right\} q(t)\cos(t\sqrt{\lambda})dt +$$

$$+\lambda^{-\frac{1}{2}} \int\limits_0^x \cos\left\{(x-t)\sqrt{\lambda}\right\} q(t)dt \int\limits_0^t \sin\left\{(t-u)\sqrt{\lambda}\right\} q(u)\cos(u\sqrt{\lambda})du + O(\lambda^{-1}).$$

In [11] (§ 4.3), it is determined similarly

$$\phi_2'(x,\lambda) = \cos\left(x\sqrt{\lambda}\right) + \lambda^{-\frac{1}{2}} \int\limits_0^x \cos\left\{(x-t)\sqrt{\lambda}\right\} q(t)\sin(t\sqrt{\lambda})dt +$$

$$+\lambda^{-1}\int\limits_0^x \cos\{(x-t)\sqrt{\lambda}\}q(t)dt\int\limits_0^t \sin\{(t-u)\sqrt{\lambda}\}q(u)\sin(u\sqrt{\lambda})du+O(\lambda^{-\frac{3}{2}}).$$

Lemma 2.3. *Let* $q(x)$ *be absolutely continuous. Then the derivatives of* (1.8) *and* (1.9) *are*, *as* $\lambda\to\infty$,

$$\phi_1'(x,\lambda)=-\lambda^{\frac{1}{2}}\sin\left(x\sqrt{\lambda}\right)+\frac{1}{2}Q(x)\cos\left(x\sqrt{\lambda}\right)+$$

$$+\frac{1}{4}\lambda^{-\frac{1}{2}}\left\{q(x)+q(0)+\frac{1}{2}Q^2(x)\right\}\sin\left(x\sqrt{\lambda}\right)+o(\lambda^{-\frac{1}{2}}), \tag{2.6}$$

$$\phi_2'(x,\lambda)=\cos\left(x\sqrt{\lambda}\right)+\frac{1}{2}\lambda^{-\frac{1}{2}}Q(x)\sin\left(x\sqrt{\lambda}\right)-$$

$$-\frac{1}{4}\lambda^{-1}\left\{q(x)-q(0)+\frac{1}{2}Q^2(x)\right\}\cos\left(x\sqrt{\lambda}\right)+o(\lambda^{-1}). \tag{2.7}$$

Proof. If $q(x)$ is absolutely continuous, this implies that $q'(x)$ exists p.p. and is integrable. Under these conditions, let consider the second term on the right-hand side of (2.4). We have

$$\int\limits_0^x \cos\{(x-t)\sqrt{\lambda}\}q(t)\cos(t\sqrt{\lambda})dt=$$

$$=\frac{1}{2}\int\limits_0^x\left[\cos\left(x\sqrt{\lambda}\right)+\cos\{(x-2t)\sqrt{\lambda}\}\right]q(t)dt=$$

$$=\frac{1}{2}Q(x)\cos\left(x\sqrt{\lambda}\right)+\frac{1}{2}\int\limits_0^x\cos\{(x-2t)\sqrt{\lambda}\}q(t)dt=$$

$$=\frac{1}{2}Q(x)\cos\left(x\sqrt{\lambda}\right)+\frac{1}{2}\left[-\frac{1}{2}\lambda^{-\frac{1}{2}}q(t)\sin\{(x-2t)\sqrt{\lambda}\}\Big|_{t=0}^{x}+\right.$$

$$\left.+\frac{1}{2}\lambda^{-\frac{1}{2}}\int\limits_0^x q'(t)\sin\{(x-2t)\sqrt{\lambda}\}dt\right]=$$

$$=\frac{1}{2}Q(x)\cos\left(x\sqrt{\lambda}\right)+\frac{1}{4}\lambda^{-\frac{1}{2}}\left[q(x)+q(0)\right]\sin\left(x\sqrt{\lambda}\right)+$$

$$+\frac{1}{4}\lambda^{-\frac{1}{2}}\int\limits_0^x q'(t)\sin\{(x-2t)\sqrt{\lambda}\}dt.$$

The last integral on the right-hand side is $o(1)$ as $\lambda\to\infty$ by the Riemann – Lebesgue lemma. So,

$$\int_0^x \cos\left\{(x-t)\sqrt{\lambda}\right\} q(t)\cos(t\sqrt{\lambda})dt =$$

$$= \frac{1}{2}Q(x)\cos\left(x\sqrt{\lambda}\right) + \frac{1}{4}\lambda^{-\frac{1}{2}}\left[q(x)+q(0)\right]\sin\left(x\sqrt{\lambda}\right) + o(\lambda^{-\frac{1}{2}}). \tag{2.8}$$

Also, from [11] (§ 4.3)

$$\int_0^x \sin\left\{(x-t)\sqrt{\lambda}\right\} q(t)\cos(t\sqrt{\lambda})dt =$$

$$= \frac{1}{2}Q(x)\sin\left(x\sqrt{\lambda}\right) + \frac{1}{4}\lambda^{-\frac{1}{2}}\left[q(x)-q(0)\right]\cos\left(x\sqrt{\lambda}\right) + o(\lambda^{-\frac{1}{2}}). \tag{2.9}$$

For the third term on the right-hand side of (2.4), together with (2.9) we find

$$\lambda^{-\frac{1}{2}}\int_0^x \cos\left\{(x-t)\sqrt{\lambda}\right\} q(t)dt \int_0^t \sin\{(t-u)\sqrt{\lambda}\} q(u)\cos(u\sqrt{\lambda})du =$$

$$= \frac{1}{2}\lambda^{-\frac{1}{2}}\int_0^x \cos\left\{(x-t)\sqrt{\lambda}\right\} q(t)Q(t)\sin(t\sqrt{\lambda})dt + O(\lambda^{-1}) =$$

$$= \frac{1}{4}\lambda^{-\frac{1}{2}}\int_0^x \left[\sin\left(x\sqrt{\lambda}\right) - \sin\left\{(x-2t)\sqrt{\lambda}\right\}\right] q(t)Q(t)dt + O(\lambda^{-1}) =$$

$$= \frac{1}{4}\lambda^{-\frac{1}{2}}\sin\left(x\sqrt{\lambda}\right)\left[\frac{Q^2(t)}{2}\right]\Bigg|_{t=0}^{x} + o(\lambda^{-\frac{1}{2}}) =$$

$$= \frac{1}{8}\lambda^{-\frac{1}{2}}Q^2(x)\sin\left(x\sqrt{\lambda}\right) + o(\lambda^{-\frac{1}{2}}), \tag{2.10}$$

again by using the Riemann – Lebesgue lemma. (2.8) and (2.10) prove (2.6). The proof of (2.7) is similar.

Lemma 2.4. *Consider the equation* (1.3) *having symmetric single well potential. As* $\lambda \to \infty$, *for the derivatives of its solutions* $\phi_1(x,\lambda)$ *and* $\phi_2(x,\lambda)$ *which satisfy* (1.6) *and* (1.7), *respectively, we have*

$$\phi_1'(x,\lambda) = -\lambda^{\frac{1}{2}}\sin\left(x\sqrt{\lambda}\right) + \frac{1}{2}\left[xq(x) + \int_{a/2}^{a}(a-t)q'(t)dt - \int_{a/2}^{x} tq'(t)dt\right]\cos\left(x\sqrt{\lambda}\right) +$$

$$+\frac{1}{4}\lambda^{-\frac{1}{2}}\left\{q(x)+q(0)+\frac{1}{2}\left[xq(x) + \int_{a/2}^{a}(a-t)q'(t)dt - \int_{a/2}^{x} tq'(t)dt\right]^2\right\}\sin\left(x\sqrt{\lambda}\right) + o(\lambda^{-\frac{1}{2}}), \tag{2.11}$$

$$\phi_2'(x,\lambda)=\cos\left(x\sqrt{\lambda}\right)+\frac{1}{2}\lambda^{-\frac{1}{2}}\left[xq(x)+\int\limits_{a/2}^{a}(a-t)q'(t)dt-\int\limits_{a/2}^{x}tq'(t)dt\right]\sin\left(x\sqrt{\lambda}\right)-$$

$$-\frac{1}{4}\lambda^{-1}\left\{q(x)-q(0)+\frac{1}{2}\left[xq(x)+\int\limits_{a/2}^{a}(a-t)q'(t)dt-\int\limits_{a/2}^{x}tq'(t)dt\right]^2\right\}\cos\left(x\sqrt{\lambda}\right)+o(\lambda^{-1}). \tag{2.12}$$

Proof. By using (2.1) in (1.10) and substituting this in (2.6) and (2.7), we prove the lemma.

3. Approximations for Green's functions. In this section, we aim to improve asymptotic formulae for Green's functions of the periodic and semiperiodic problems with symmetric single well potential. The Green's function $G(x,\xi,\lambda)$ is given by

$$G(x,\xi,\lambda)=\begin{cases}\dfrac{\phi_1(\xi,\lambda)\phi_2(x,\lambda)}{w(\lambda)}, & 0\le\xi\le x\le a,\\[2mm] \dfrac{\phi_1(x,\lambda)\phi_2(\xi,\lambda)}{w(\lambda)}, & 0\le x\le\xi\le a\end{cases} \tag{3.1}$$

(see [12]). Here, $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ are linearly independent solutions of (1.3) satisfying (1.6) and (1.7), respectively. And, we define $w(\lambda)$ as follows:

$$w(\lambda):=\phi_1(x,\lambda)\phi_2'(x,\lambda)-\phi_1'(x,\lambda)\phi_2(x,\lambda). \tag{3.2}$$

It is known as the Wronskian function of $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$.

Theorem 3.1. *Suppose that the equation* (1.3) *has the symmetric single well potential and its independent solutions* $\phi_1(x,\lambda)$ *and* $\phi_2(x,\lambda)$ *satisfy the initial conditions* (1.6) *and* (1.7), *respectively. Then the Green's function of the problem is*, *as* $\lambda\to\infty$,

$$G(x,\xi,\lambda)=\lambda^{-\frac{1}{2}}\cos\left(\xi\sqrt{\lambda}\right)\sin\left(x\sqrt{\lambda}\right)-$$

$$-\frac{1}{2}\lambda^{-1}\left[A(x)\cos\left(\xi\sqrt{\lambda}\right)\cos\left(x\sqrt{\lambda}\right)-A(\xi)\sin\left(\xi\sqrt{\lambda}\right)\sin\left(x\sqrt{\lambda}\right)\right]+$$

$$+\frac{1}{4}\lambda^{-\frac{3}{2}}\left\{\left[q(\xi)+q(x)-\frac{1}{2}\left(A^2(\xi)+A^2(x)\right)\right]\cos\left(\xi\sqrt{\lambda}\right)\sin\left(x\sqrt{\lambda}\right)-\right.$$

$$\left.-A(\xi)A(x)\sin\left(\xi\sqrt{\lambda}\right)\cos\left(x\sqrt{\lambda}\right)\right\}+o(\lambda^{-\frac{3}{2}}),\quad 0\le\xi\le x\le a,$$

where

$$A(x):=xq(x)+\int\limits_{a/2}^{a}(a-t)q'(t)dt-\int\limits_{a/2}^{x}tq'(t)dt.$$

Similar result holds for $0\le x\le\xi\le a$ *changing the role of* ξ *and* x.

Proof. We begin to the proof by evaluating the Wronskian function $w(\lambda)$. For this, we substitute (2.2), (2.3), (2.11) and (2.12) into (3.2). Hence,

$$w(\lambda) = 1 - \frac{1}{4}\lambda^{-1}\left[q(x) - q(0) + \frac{1}{2}A^2(x)\right]\cos^2\left(x\sqrt{\lambda}\right) +$$

$$+\frac{1}{4}\lambda^{-1}\left[q(x) + q(0) - \frac{1}{2}A^2(x)\right]\sin^2\left(x\sqrt{\lambda}\right) +$$

$$+\frac{1}{4}\lambda^{-1}A^2(x) + \frac{1}{4}\lambda^{-1}\left[q(x) - q(0) - \frac{1}{2}A^2(x)\right]\cos^2\left(x\sqrt{\lambda}\right) -$$

$$-\frac{1}{4}\lambda^{-1}\left[q(x) + q(0) + \frac{1}{2}A^2(x)\right]\sin^2\left(x\sqrt{\lambda}\right) + o(\lambda^{-1}) =$$

$$= 1 - \frac{1}{4}\lambda^{-1}A^2(x) + \frac{1}{4}\lambda^{-1}A^2(x) + o(\lambda^{-1}) = 1 + o(\lambda^{-1}).$$

From that, we can write

$$\frac{1}{w(\lambda)} = \frac{1}{1 + o(\lambda^{-1})} = 1 + o(\lambda^{-1}). \tag{3.3}$$

Finally, by using (2.2), (2.3), (3.3) in (3.1), we find

$$\frac{\phi_1(\xi,\lambda)\phi_2(x,\lambda)}{w(\lambda)} = \Bigg\{\cos\left(\xi\sqrt{\lambda}\right) + \frac{1}{2}\lambda^{-\frac{1}{2}}A(\xi)\sin\left(\xi\sqrt{\lambda}\right) +$$

$$+\frac{1}{4}\lambda^{-1}\left[q(\xi) - q(0) - \frac{1}{2}A^2(\xi)\right]\cos\left(\xi\sqrt{\lambda}\right) + o(\lambda^{-1})\Bigg\}\times$$

$$\times\Bigg\{\lambda^{-\frac{1}{2}}\sin\left(x\sqrt{\lambda}\right) - \frac{1}{2}\lambda^{-1}A(x)\cos\left(x\sqrt{\lambda}\right) +$$

$$+\frac{1}{4}\lambda^{-\frac{3}{2}}\left[q(x) + q(0) - \frac{1}{2}A^2(x)\right]\sin\left(x\sqrt{\lambda}\right) + o(\lambda^{-\frac{3}{2}})\Bigg\}\left\{1 + o(\lambda^{-1})\right\} =$$

$$= \Bigg\{\lambda^{-\frac{1}{2}}\cos\left(\xi\sqrt{\lambda}\right)\sin\left(x\sqrt{\lambda}\right) - \frac{1}{2}\lambda^{-1}A(x)\cos\left(\xi\sqrt{\lambda}\right)\cos\left(x\sqrt{\lambda}\right) +$$

$$+\frac{1}{4}\lambda^{-\frac{3}{2}}\left[q(x) + q(0) - \frac{1}{2}A^2(x)\right]\cos\left(\xi\sqrt{\lambda}\right)\sin\left(x\sqrt{\lambda}\right) +$$

$$+\frac{1}{2}\lambda^{-1}A(\xi)\sin\left(\xi\sqrt{\lambda}\right)\sin\left(x\sqrt{\lambda}\right) - \frac{1}{4}\lambda^{-\frac{3}{2}}A(\xi)A(x)\sin\left(\xi\sqrt{\lambda}\right)\cos\left(x\sqrt{\lambda}\right) +$$

$$+\frac{1}{4}\lambda^{-\frac{3}{2}}\left[q(\xi) - q(0) - \frac{1}{2}A^2(\xi)\right]\cos\left(\xi\sqrt{\lambda}\right)\sin\left(x\sqrt{\lambda}\right) + o(\lambda^{-\frac{3}{2}})\Bigg\}\left\{1 + o(\lambda^{-1})\right\} =$$

$$= \lambda^{-\frac{1}{2}} \cos\left(\xi\sqrt{\lambda}\right) \sin\left(x\sqrt{\lambda}\right) - \frac{1}{2}\lambda^{-1}\left[A(x)\cos\left(\xi\sqrt{\lambda}\right)\cos\left(x\sqrt{\lambda}\right) - A(\xi)\sin\left(\xi\sqrt{\lambda}\right)\sin\left(x\sqrt{\lambda}\right)\right] +$$

$$+\frac{1}{4}\lambda^{-\frac{3}{2}}\left\{\left[q(\xi)+q(x)-\frac{1}{2}\left(A^2(\xi)+A^2(x)\right)\right]\cos\left(\xi\sqrt{\lambda}\right)\sin\left(x\sqrt{\lambda}\right)-\right.$$

$$\left.-A(\xi)A(x)\sin\left(\xi\sqrt{\lambda}\right)\cos\left(x\sqrt{\lambda}\right)\right\} + o(\lambda^{-\frac{3}{2}}).$$

Theorem 3.2. *Green's functions of the periodic problem with symmetric single well potential satisfy, as $n \to \infty$,*

$$G(x,\xi,n) = \frac{a}{2(n+1)\pi}\cos\frac{2(n+1)\pi\xi}{a}\sin\frac{2(n+1)\pi x}{a} - \frac{a^2}{8(n+1)^2\pi^2}\times$$

$$\times\left[A(x)\cos\frac{2(n+1)\pi\xi}{a}\cos\frac{2(n+1)\pi x}{a} - A(\xi)\sin\frac{2(n+1)\pi\xi}{a}\sin\frac{2(n+1)\pi x}{a}\right]+$$

$$+\frac{a^3}{32(n+1)^3\pi^3}\left\{\left[q(\xi)+q(x)-\frac{1}{2}\left(A^2(\xi)+A^2(x)\right)\right]\times\right.$$

$$\left.\times\cos\frac{2(n+1)\pi\xi}{a}\sin\frac{2(n+1)\pi x}{a} - A(\xi)A(x)\sin\frac{2(n+1)\pi\xi}{a}\cos\frac{2(n+1)\pi x}{a}\right\} + o(n^{-3})$$

for $0 \le \xi \le x \le a$. Similar result holds for $0 \le x \le \xi \le a$ changing the role of ξ and x.

Theorem 3.3. *Green's functions of the semiperiodic problem with symmetric single well potential satisfy, as $n \to \infty$,*

$$G(x,\xi,n) = \frac{a}{(2n+1)\pi}\cos\frac{(2n+1)\pi\xi}{a}\sin\frac{(2n+1)\pi x}{a}-$$

$$-\frac{a^2}{2(2n+1)^2\pi^2}\left[A(x)\cos\frac{(2n+1)\pi\xi}{a}\cos\frac{(2n+1)\pi x}{a}-\right.$$

$$\left.-A(\xi)\sin\frac{(2n+1)\pi\xi}{a}\sin\frac{(2n+1)\pi x}{a}\right]+$$

$$+\frac{a^3}{4(2n+1)^3\pi^3}\left\{\left[q(\xi)+q(x)-\frac{1}{2}\left(A^2(\xi)+A^2(x)\right)\right]\cos\frac{(2n+1)\pi\xi}{a}\sin\frac{(2n+1)\pi x}{a}-\right.$$

$$\left.-A(\xi)A(x)\sin\frac{(2n+1)\pi\xi}{a}\cos\frac{(2n+1)\pi x}{a}\right\} + o(n^{-3})$$

for $0 \le \xi \le x \le a$. Similar result holds for $0 \le x \le \xi \le a$ changing the role of ξ and x.

To prove Theorems 3.2 and 3.3, the related eigenvalues given by (1.4) and (1.5) are used together with Theorem 3.1.

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