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Rochdi Jebari (Dep. Math., College Sci. and Humanities-Al Quwayiyah, Shaqra Univ., Riyadh, Kingdom of Saudi Arabia; Dep. Math., Faculté Sci. Tunis, Univ. Tunis El-Manar, Tunisie)

ON THE HIGH ENERGY SOLITARY WAVES SOLUTIONS FOR A GENERALIZED KP EQUATION IN BOUNDED DOMAIN

РОЗВ'ЯЗКИ У ВИГЛЯДІ СОЛІТОНОВИХ ХВИЛЬ ДЛЯ УЗАГАЛЬНЕНОГО РІВНЯННЯ КАДОМЦЕВА – ПЕТВІАШВІЛІ В ОБМЕЖЕНІЙ ОБЛАСТІ

We are mainly concerned with the existence of infinitely many high energy solitary waves solutions for a class of generalized Kadomtsev – Petviashvili equation (KP equation) in bounded domain. The aim of this paper is to fill the gap in the relevant literature stated in a previous paper (J. Xu, Z. Wei, Y. Ding, *Stationary solutions for a generalized Kadomtsev – Petviashvili equation in bounded domain*, Electron. J. Qual. Theory Differ. Equ., **2012**, № 68, 1 – 18 (2012)). Under more relaxed assumption on the nonlinearity involved in KP equation, we obtain a new result on the existence of infinitely many high energy solitary waves solutions via a variant fountain theorem.

Розглядається, головним чином, існування нескінченної кількості розв'язків у вигляді солітонових хвиль для узагальненого рівняння Кадомцева – Петвіашвілі в обмеженій області. Мета цієї роботи — заповнити пробіли в результатах, які вказані у попередній роботі (J. Xu, Z. Wei, Y. Ding, *Stationary solutions for a generalized Kadomtsev – Petviashvili equation in bounded domain*, Electron. J. Qual. Theory Differ. Equ., **2012**, № 68, 1 – 18 (2012)). При більш слабких обмеженнях на нелінійність у рівнянні Кадомцева – Петвіашвілі за допомогою варіанта теореми про фонтан отримано новий результат щодо існування нескінченного числа розв'язків у вигляді солітонових хвиль.

1. Introduction. The Kadomtsev – Petviashvili equation (KP equation) with variable coefficients has been proposed some time ago [1 – 4]. The motivation was to describe water waves that propagate in straits, or rivers, rather than on unbounded surfaces, like oceans. This equation appear in many physic fields, see for example [5, 6] and the references therein. There are two distinct versions of the KP equation, which can be written in normalized form as follows:

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0 \tag{1.1}$$

or, in the "integrated" form

$$u_t + 6uu_x + u_{xxx} + 3\sigma^2 \partial_x^{-1} u_{yy} = 0, \tag{1.2}$$

where $u = u(t, x, y)$ is a scalar function, x and y are respectively the longitudinal and transverse spatial coordinates, subscripts x, y, t denote partial derivatives,

$$\partial_x^{-1} f(x) = \frac{1}{2}\left(\int_{-\infty}^{x} f(t)\,dt - \int_{x}^{\infty} f(t)\,dt \right)$$

and $\sigma^2 =^{+}_{-} 1$. The case $\sigma = 1$ is known as the KPII equation, and models, for instance, water waves with small surface tension. The case $\sigma = i$ is known as the KPI equation, and may be used to model waves in thin films with high surface tension. The presence of the nonlocal operator $\partial_x^{-1}\partial_y^2$ imposes a constraint on the solution u of the KP equation, which, in some sense, has to be an x-derivative (see [7, 8]). This last equation (among other completely integrable systems) was studied extensively by means of algebro-geometric techniques [9], Hirota bilinear method [10] and reduction method [11].

A solitary wave or solitary wave solution of (1.2) is a solution of the form $u(t,x,y)=v(x-ct,y)$, where $c>0$ is fixed, were studied by Ablowitz et al. [12]. Consequently, solitary wave solution are important, because their properties can provide a useful platform for explaining many unusual dynamical behaviors of various physical equations (see [13 – 16]).

The generalized KP equation is written in the following form:

$$u_t+u_{xxx}+(f(u))_x=D_x^{-1}\Delta_y u, \tag{1.3}$$

where $(t,x,y):=(t,x,y_1,\ldots,y_{n-1})\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^{n-1}$, $n\geq 2$, $D_x^{-1}h(x,y)=\int_{-\infty}^{x}h(s,y)ds$ and $\Delta_y=\sum_{k=1}^{n-1}\frac{\partial^2}{\partial y_k^2}$. The equation (1.3) were studied by many authors (see [17 – 23]).

In [17], B. Xuan studied the existence of multiple stationary solutions of Generalized KP equation in a bounded domain with smooth boundary and for superlinear conditions of nonlinearity $f(u)=$ $=\lambda|u|^{p-2}u+|u|^{q-2}u$ where $1\leq p,q<2^*=\frac{2(2n-1)}{2n-3}$. The techniques used in [17] are based on variational methods. In [18, 19], by means of constrained minimization method, Bouard et al. studied the existence and nonexistence of solitary waves when $f(u)=u^{\frac{k}{l}}$, where $k,\ l$ are relatively prime and l is odd. In the Chapter 7 of [20], Willem extended the results of [18] to the case where $n=2$ and with an autonomous continuous nonlinearity $f(u)$. In [21], Xuan extended the result in [20] to higher spatial dimension with $f\in C(\mathbb{R},\mathbb{R})$. Their results were obtained by applying the mountain pass theorem of Ambrosetti – Rabinowitz [28] and Lusternik – Schnirelman theory.

In [23], J. Xua et al. studied the existence of multiple solitary waves for the generalized KP equation (1.3) in one-dimensional spaces when $f(u)=\mu|u|^{\mu-1}$ and $1<\mu<2$. Their methods were based on variant fountain Theorem [24].

To our knowledge, all known results are concerned with the case that f is autonomous. Except in paper [22], Z. Liang et al. studied the existence of nontrivial solution for the limiting case $f(x,y,u)=$ $=Q(x,y)u^{p-2}u$. Here, some compactness property for the energy functional like the Palais – Smale condition [24] were used.

Inspired by the above facts, in the present paper we consider a more general problem (1.4)

$$\begin{aligned} &u_t+u_{xxx}+(f(x,y,u))_x=D_x^{-1}\Delta_y u \quad \text{in} \quad \Omega,\\ &D_x^{-1}u_{|\partial\Omega}=0, \quad u_{|\partial\Omega}=0. \end{aligned} \tag{1.4}$$

Note here that the nonlinearity f is non autonomous. Such equation are of scientific and practical interest because of the variety of applications involving solitary wave propagation in inhomogeneous media [25 – 27]. We recall that in the above papers, the high energy solitary waves solutions have not been studied. Under more general assumptions on the nonlinearity f which are much strong assumptions than used in paper [23], we obtain a new result on the existence of infinitely many high energy solitary waves solutions for the problem (1.4), (see Theorem 2 in Section 3). Such result are obtained by using some special proof techniques.

This paper is organized as follows. In Section 2, we recall some basic preliminaries. In Section 3 we give some lemmas and finally, we prove our result.

2. Preliminaries and functional setting. In this section we introduce some preliminaries which used in our paper. Let $c>0$, substituting $u(x-ct,y)$ in (1.4), we obtain

$$-cu_x + u_{xxx} + (f(x,y,u))_x = D_x^{-1}\Delta_y u \tag{2.1}$$

or

$$\left(-u_{xx} + D_x^{-2}\Delta_y u + cu - f(x,y,u)\right)_x = 0. \tag{2.2}$$

Note that we can rewrite (1.4) in the following form (see [17, p. 12]):

$$\begin{aligned} &-u_{xx} + D_x^{-2}\Delta_y u + cu = f(x,y,u) \quad \text{in} \quad \Omega, \\ &D_x^{-1}u_{|\partial\Omega} = 0, \quad u_{|\partial\Omega} = 0. \end{aligned} \tag{2.3}$$

Definition 2.1. *For $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ on $Y := \{g_x : g \in C_0^\infty\}$, we define the inner product*

$$(u,v) = \int\limits_\Omega \left[u_x v_x + D_x^{-1}\nabla_y u . D_x^{-1}\nabla_y v + cuv\right] dV \tag{2.4}$$

where $\nabla_y = \left(\dfrac{\partial}{\partial y_1}, \ldots, \dfrac{\partial}{\partial y_{n-1}}\right)$, $dV = dx\,dy$ and the corresponding norm

$$\|u\| := \left(\int\limits_\Omega \left[u_x^2 + |D_x^{-1}\nabla_y u|^2 + cu^2\right] dV\right)^{\frac{1}{2}}. \tag{2.5}$$

A function $u\colon \Omega \to \mathbb{R}$ belongs to E, if there exists $\{u_m\}_{m=1}^{+\infty} \subset Y$ such that:

(a) $u_m \to u$ *a.e. on* Ω,

(b) $\|u_j - u_k\| \to 0$ *as* $j,k \to \infty$.

Note that the space E with inner product (2.4) and norm (2.5) is a Hilbert space, see [22] (Definition) and [17, p. 12, 13].

For each $v \in E$, multiply the both sides of the above equation in (2.3) by $v(x,y)$ and integrate over Ω to obtain

$$\int\limits_\Omega \left(-\frac{\partial^2}{\partial x^2}u\right) v\,dV + \int\limits_\Omega (D_x^{-2}\Delta_y u)v\,dV + c\int\limits_\Omega uv\,dV = \int\limits_\Omega f(x,y,u)\,v\,dV \tag{2.6}$$

and then we obtain by Green formula and integration by parts,

$$\int\limits_\Omega \frac{\partial}{\partial x}u . \frac{\partial}{\partial x}v\,dV + \int\limits_\Omega D_x^{-1}\nabla_y u . D_x^{-1}\nabla_y v\,dV + c\int\limits_\Omega uv\,dV = \int\limits_\Omega f(x,y,u)v\,dV. \tag{2.7}$$

Therefore, on E, define a functional ϕ as

$$\begin{aligned} \phi(u) := \frac{1}{2}\int\limits_\Omega \left[u_x^2 + |D_x^{-1}\nabla_y u|^2 + cu^2\right] dV - \int\limits_\Omega F(x,y,u)\,dV = \\ = \frac{1}{2}\|u\|^2 - \psi(u) \end{aligned} \tag{2.8}$$

where $F(x,y,u) := \int_0^u f(x,y,s)\,ds$ and $\psi(u) := \int_\Omega F(x,y,u)\,dV$.

Lemma 2.1 (see ([17], Lemma 1). *The embedding from the space* $(E, \|.\|)$ *into the space* $(L^p(\Omega), \|.\|_p)$ *is compact for* $1 \le p < \overline{p}$ *with* $\overline{p} = \dfrac{2(2n-1)}{2n-3} > 2$. *In add, there exists* $\tau_p > 0$ *such that*

$$\|u\|_p \le \tau_p \|u\|, \quad p \in [1, \overline{p}), \quad \textit{for all } u \in E \tag{2.9}$$

where $\|u\|_p = \left(\int_\Omega |u|^p \, dV\right)^{\frac{1}{p}}$.

We assume that the nonlinearity f, satisfying the following hypotheses:

$(f1)$ $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $f(x, y, u)u \ge 0$ for all $u \in \mathbb{R}$, $(x, y) \in \Omega$ and there exists a constants $C > 0$ and $p \in (2, \overline{p})$ such that
$|f(x, y, u)| \le C(1 + |u|^{p-1})$, for all $u \in \mathbb{R}$ and $(x, y) \in \Omega$.

$(f2)$ $f(x, y, u) = o(|u|)$ as $|u| \to 0$ uniformly for $(x, y) \in \Omega$.

$(f3)$ $\lim_{|u|\to\infty} \dfrac{F(x, y, u)}{|u|^2} = +\infty$ uniformly for $(x, y) \in \Omega$.

$(f4)$ There exists $\theta \ge 1$ such that $\theta\varphi(u) \ge \varphi(\tau u)$ for all $\tau \in [0, 1]$ and $(x, y, u) \in \Omega \times \mathbb{R}$ where $\varphi(u) = u\, f(x, y, u) - 2F(x, y, u)$.

$(f5)$ $f(x, y, -u) = -f(x, y, u)$ for all $u \in \mathbb{R}$ and $(x, y) \in \Omega$.

Example of a function f satisfying the above assumptions is

$$f(x, y, t) = a(x, y)|t|^{\nu-2} t$$

for all $(x, y) \in \Omega$ and $t \in \mathbb{R}$ where $\nu \in (2, \overline{p})$ and a is a continuous bounded function with positive lower bound.

Lemma 2.2 (see [23]). *Let* $(f1)$ *holds. Then* $\phi \in C^1(E, \mathbb{R})$. *Moreover, we have*

$$\langle \psi'(u), v \rangle = \int\limits_\Omega f(x, y, u) v \, dV \tag{2.10}$$

and

$$\langle \phi'(u), v \rangle = (u, v) - (\psi'(u), v) = (u, v) - \int\limits_\Omega f(x, y, u) v \, dV \tag{2.11}$$

for all $u, v \in E$. *We note that a critical points of* ϕ *on* E *are weak solutions of* (2.3).

For the convenience of the readers, we recall some notation which will be used later.

Let X be a Banach space with the norm $\|.\|$ and let $\{X_j\}$ be a sequence of subspaces of X with $\dim X_j < \infty$ for each $j \in \mathbb{N}$.

Further, $X = \overline{\oplus_{j\in\mathbb{N}} X_j}$ the closure of the direct sum of all $\{X_j\}$.

Set $W_k := \oplus_{j=0}^{k} X_j$ and $Z_k := \overline{\oplus_{j=k+1}^{\infty} X_j}$, for $\rho_k > r_k > 0$

$$B_k = \{u \in W_k : \|u\| \le \rho_k\} \qquad \text{and} \qquad S_k = \{u \in Z_k : \|u\| = r_k\}.$$

Consider a family of C^1-functionals $\phi_\lambda : X \to \mathbb{R}$ defined by

$$\phi_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1, 2]. \tag{2.12}$$

Theorem 2.1 (see [24]). *Assume that the functional* ϕ_λ *defined above satisfies*

(A1) ϕ_λ *maps bounded sets into bounded sets uniformly for* $\lambda \in [0, 1]$, *and* $\phi_\lambda(-u) = \phi_\lambda(u)$ *for all* $(\lambda, u) \in [1, 2] \times X$;

(A2) $B(u) \geq 0$ *for all* $u \in X$, *and* $B(u) \to \infty$ *as* $\|u\| \to \infty$ *on any finite dimensional subspace of* X, *or*

(A3) $B(u) \leq 0$ *for all* $u \in X$, *and* $B(u) \to -\infty$ *as* $\|u\| \to \infty$.

(A4) *There exists* $\rho_k > r_k > 0$ *such that*

$$b_k(\lambda) := \inf_{u \in Z_k, \|u\| = r_k} \phi_\lambda(u) > a_k(\lambda) := \max_{u \in W_k, \|u\| = \rho_k} \phi_\lambda(u) \quad \textit{for all} \quad \lambda \in [1, 2].$$

Then $b_k(\lambda) \leq c_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \phi_\lambda(\gamma(u))$ *for all* $\lambda \in [1, 2]$, *where*

$$\Gamma_k = \Big\{ \gamma \in C(B_k, X) \colon \gamma \textit{ odd}, \ \gamma_{|\partial B_k} = id \Big\}, \quad k \geq 2.$$

Moreover, for almost every $\lambda \in [1, 2]$ *there exists a sequence* $u_n^k(\lambda)$ *such that*

$$\sup_n \|u_n^k(\lambda)\| < \infty, \qquad \phi'_\lambda(u_n^k(\lambda)) \to 0 \qquad \textit{and} \qquad \phi_\lambda(u_n^k(\lambda)) \to c_k(\lambda) \quad \textit{as} \quad n \to \infty.$$

3. Existence of infinitely many high solitary waves energy solutions. In order to apply the above theorem to prove our main results, we define the functional ϕ_λ on our working space E by

$$\phi_\lambda(u) := \frac{1}{2} \int\limits_\Omega \left[u_x^2 + |D_x^{-1} \nabla_y u|^2 + c u^2 \right] dV - \lambda \int\limits_\Omega F(x, y, u) \, dV = \frac{1}{2} \|u\|^2 - \lambda \psi(u) \tag{3.1}$$

for all $u \in E$ and $\lambda \in [0, 1]$. We use the some lemma to show the existence

Lemma 3.1. *For the finite dimensional subspace* $F \subset E$ *of* E, *there exists a constant* $\varepsilon_0 > 0$ *such that*

$$\text{meas}\big\{ (x, y) \in \Omega \colon |u(x, y)| \geq \varepsilon_0 \|u\| \big\} \geq \varepsilon_0 \quad \forall u \in F \backslash \{0\}. \tag{3.2}$$

Proof. If not, for any $n \in \mathbb{N}^*$, there exists $u_n \in F \backslash \{0\}$ such that

$$\text{meas}\left\{ (x, y) \in \Omega \colon |u_n(x, y)| \geq \frac{1}{n} \|u_n\| \right\} < \frac{1}{n} \quad \forall n \in \mathbb{N}^*. \tag{3.3}$$

Let $v_n = \dfrac{u_n}{\|u_n\|}$ for all $n \in \mathbb{N}^*$, then $\|v_n\| = 1$ for all $n \in \mathbb{N}^*$, and

$$\text{meas}\left\{ (x, y) \in \Omega \colon |v_n(x, y)| \geq \frac{1}{n} \right\} < \frac{1}{n} \quad \forall n \in \mathbb{N}^*. \tag{3.4}$$

By the boundedness of $\{v_n\}$, passing to a subsequence if necessary, we may assume that $v_n \to v$ with $\|v\| = 1$ in E for some $v \in E$ since E is a finite dimension. By Lemma 2.1, we have

$$\int\limits_\Omega |v_n(x, y) - v(x, y)|^2 \, dV \to 0 \quad \text{as} \quad n \to \infty. \tag{3.5}$$

Since $v \neq 0$, there exists a constant $\delta_0 > 0$ such that

$$\text{meas}\big\{ (x, y) \in \Omega \colon |v(x, y)| \geq \delta_0 \big\} \geq \delta_0. \tag{3.6}$$

For any $n \in \mathbb{N}^*$, we set

$$D_n = \Big\{(x,y) \in \Omega \colon |v_n(x,y)| < \frac{1}{n}\Big\}, \qquad D_n^c = \Big\{(x,y) \in \Omega \colon |v_n(x,y)| \geq \frac{1}{n}\Big\}$$

and $D_0 = \big\{(x,y) \in \Omega \colon |v(x,y)| \geq \delta_0\big\}$. Thus for n large enough, by (3.4) and (3.6), we get

$$\mathrm{meas}(D_n \cap D_0) \geq \mathrm{meas}(D_0) - \mathrm{meas}(D_n^c) \geq \frac{2\delta_0}{3}. \tag{3.7}$$

Consequently, for n large enough, we have

$$\int\limits_{\Omega} |v_n(x,y) - v(x,y)|^2\, dV \geq \int\limits_{D_n \cap D_0} |v_n(x,y) - v(x,y)|^2\, dV \geq$$
$$\geq \int\limits_{D_n \cap D_0} \big[|v(x,y)|^2 - 2v_n(x,y)v(x,y)\big]\, dV \geq$$
$$\geq \int\limits_{D_n \cap D_0} \big[|v(x,y)|^2 - 2|v_n(x,y)||v(x,y)|\big]\, dV \geq$$
$$\geq \delta_0\Big(\delta_0 - \frac{2}{n}\Big)\mathrm{meas}(D_n \cap D_0) \geq \frac{2}{9}\delta_0^3 > 0. \tag{3.8}$$

This is in contradiction with (3.4). Therefore (3.2) holds.

Lemma 3.2. *Assume that $(f1)$ and $(f3)$ hold. Then $\psi(u) \geq 0$ for all $u \in E$, and $\psi(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E.*

Proof. Evidently, from $(f1)$, we have $\psi(u) \geq 0$ for all $u \in E$. Let $H \subset E$ be any finite dimensional subspace of E, next we will show that $\psi(u) \to \infty$ as $\|u\| \to \infty$ on H.

By $(f3)$, there exists $R > 0$ such that

$$F(x,y,u) \geq |u|^2 \quad \text{for all} \quad (x,y) \in \Omega \quad \text{and} \quad |u| \geq R. \tag{3.9}$$

Let $D_u = \big\{(x,y) \in \Omega \colon |u(x,y)| \geq \varepsilon_0\|u\|\big\}$ for $u \in E\backslash\{0\}$. By Lemma 3.1, we see that for any $u \in E$ with $\|u\| \geq \dfrac{R}{\varepsilon_0}$ we have $|u(x,y)| \geq R$, for all $(x,y) \in D_u$. Hence, for any $u \in E$ with $\|u\| \geq \frac{R}{\varepsilon_0}$, from $(f1)$ and (3.9), we get

$$\psi(u) \geq \int\limits_{D_u} F(x,y,u)\, dV \geq \int\limits_{D_u} |u|^2\, dV \geq$$
$$\geq \varepsilon_0^2\|u\|^2 \mathrm{meas}(D_u) \geq \varepsilon_0^3\|u\|^2. \tag{3.10}$$

This implies that $\psi(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E.

The proof is completed.

Let $\{e_j\}$ be a total orthonormal basis of E and $X_j = \mathbb{R}e_j$, $W_k := \oplus_{j=0}^{k} X_j$ and $Z_k := := \overline{\oplus_{j=k+1}^{\infty} X_j}$.

Lemma 3.3. *If $p \in [1, \overline{p})$, then one has $\alpha_k(p) := \sup_{u \in Z_k, \|u\|=1} \|u\|_p \to 0$ as $k \to \infty$.*

Proof. Firstly, $\alpha_k(p)$ is convergent science $\alpha_k(p) \geq 0$ and $\alpha_k(p)$ is decreasing in k. Furthermore, for any $k \in \mathbb{N}$, by the definition of $\alpha_k(p)$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_p \geq \dfrac{\alpha_k(p)}{2}$.

For any $v \in E$, $v = \sum_{n=1}^{\infty} a_n e_n$, it has

$$|\langle u_k, v\rangle| = \left|\left\langle u_k, \sum_{n=1}^{\infty} a_n e_n \right\rangle\right| \leq$$

$$\leq \|u_k\| \left\| \sum_{k=n+1}^{\infty} a_n e_n \right\| \leq \left\| \sum_{k=n+1}^{\infty} a_n e_n \right\| \to 0, \quad \text{as} \quad k \to \infty.$$

which implies that $u_k \rightharpoonup 0$ weakly in E. By virtue of Lemma 2.1, we can conclude $u_k \to 0$ strongly in $L^p(\Omega)$. The combination with implies that $\alpha_k(p) \to 0$.

Lemma 3.4. *Assume that* $(f1)$ *and* $(f2)$ *hold. Then there exists a sequences* $r_k > 0$, $k \in \mathbb{N}$ *such that*

$$b_k(\lambda) := \inf_{u \in Z_k, \|u\| = r_k} \phi_\lambda(u) > 0 \tag{3.11}$$

uniformly for $\lambda \in [1, 2]$.

Proof. By $(f1)$ and $(f2)$, for any $\epsilon > 0$, there exists a $C_\epsilon > 0$ such that

$$|f(x, y, u)| \leq \epsilon|u| + C_\epsilon |u|^{p-1} \quad \text{for all} \quad u \in \mathbb{R}. \tag{3.12}$$

Let $\alpha_k(p) := \sup_{u \in Z_k, \|u\|=1} \|u\|_p$, from Lemma 3.3, we see that $\alpha_k(p) \to 0$. Therefore, for $u_k \in Z_k$ and ϵ small enough, by (3.12), we have

$$\phi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \frac{\lambda\epsilon}{2}\|u\|_2^2 - \frac{\lambda\epsilon}{p}\|u\|_p^p \geq$$

$$\geq \frac{1}{4}\|u\|^2 - c_4\|u\|_p^p \geq \frac{1}{4}\|u\|^2 - c_4\alpha_k^p(p)\|u\|^p. \tag{3.13}$$

If we choose $r_k = \left(8c_4\alpha_k^p(p)\right)^{\frac{1}{2-p}}$ then for any $u \in Z_k$ with $\|u\| = r_k$, we get that

$$\phi_\lambda(u) \geq \frac{1}{8}\left(8c_4\alpha_k^p(p)\right)^{\frac{1}{2-p}} > 0. \tag{3.14}$$

This inequality implies that

$$b_k(\lambda) := \inf_{u \in Z_k, \|u\| = r_k} \phi_\lambda(u) \geq \frac{1}{8}\left(8c_4\alpha_k^p(p)\right)^{\frac{1}{2-p}} > 0 \quad \text{for all} \quad \lambda \in [1, 2]. \tag{3.15}$$

Lemma 3.5. *Assume that* $(f1)$, $(f2)$, *and* $(f3)$ *hold. Then for the positive integer* k_1 *and the sequence* r_k *obtained in Lemma 3.4, there exists* $\rho_k > r_k > 0$ *for any* $k \geq k_1$ *such that*

$$a_k(\lambda) := \max_{u \in W_k, \|u\| = \rho_k} \phi_\lambda(u) < 0 \tag{3.16}$$

uniformly for $\lambda \in [1, 2]$.

Proof. By Lemma 3.1, for any $k \in \mathbb{N}$, there exists $\varepsilon_k > 0$ constant such that

$$\operatorname{meas}(S_u) \geq \varepsilon_k \quad \forall u \in W_k \backslash \{0\}, \tag{3.17}$$

where $S_u = \big\{(x,y) \in \Omega \colon |u(x,y)| \geq \varepsilon_k \|u\|\big\}$. By $(f3)$, for any $k \in \mathbb{N}$, there exists a constant $R_k > 0$ such that

$$F(x,y,u) \geq \frac{1}{\varepsilon_k^3}|u|^2 \quad \forall u \geq R_k. \tag{3.18}$$

Hence, by (3.17), we see that for any $u \in W_k$ with $\|u\| \geq \dfrac{R_k}{\varepsilon_k}$, we have $|u(x,y)| \geq R_k$ for all $(x,y) \in S_u$. Therefore, for any $u \in W_k$ with $\|u\| \geq \dfrac{R_k}{\varepsilon_k}$ and $\lambda \in [1,2]$, by (3.17) and (3.18), we have

$$\phi_\lambda(u) \leq \frac{1}{2}\|u\|^2 - \int\limits_{\Omega} F(x,y,u)\, dV \leq \frac{1}{2}\|u\|^2 - \int\limits_{S_u} F(x,y,u)\, dV \leq$$

$$\leq \frac{1}{2}\|u\|^2 - \int\limits_{S_u} \frac{1}{\varepsilon_k^3}|u|^2\, dV \leq \frac{1}{2}\|u\|^2 - \varepsilon_k^2\|u\|^2 \frac{\operatorname{meas}(S_u)}{\varepsilon_k^3} \leq$$

$$\leq \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\|u\|^2. \tag{3.19}$$

If we choose $\rho_k > \max\left\{r_k, \dfrac{R_k}{\varepsilon_k}\right\}$, we get that

$$a_k(\lambda) := \max_{u \in W_k, \|u\| = \rho_k} \phi_\lambda(u) \leq -\frac{r_k^2}{2} < 0 \quad \forall k \in \mathbb{N} \quad \text{and for all} \quad \lambda \in [1,2].$$

The proof is completed.

By using (3.12) and Lemma 2.1 we can see that ϕ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Moreover, by $(f5)$, ϕ_λ is even. Then condition $(A1)$ in Theorem 2.1 is satisfied. Condition $(A2)$ is clearly true, while $(A4)$ follows by Lemma 3.4 and Lemma 3.5. Then, by Theorem 2.1, for any $k \geq k_1$ and $\lambda \in [1,2]$ there exists a sequence $\{u_n^k(\lambda)\}_n$ such that

$$\sup_n \|u_n^k(\lambda)\| < \infty, \qquad \phi'_\lambda(u_n^k(\lambda)) \to 0 \qquad \text{and} \qquad \phi_\lambda(u_n^k(\lambda)) \to c_k(\lambda) \quad \text{as} \quad n \to \infty,$$

where $c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \phi_\lambda(\gamma(u))$, $\forall \lambda \in [1,2]$ and B_k, Γ_k are given by

$$B_k = \big\{u \in W_k \colon \|u\| \leq \rho_k\big\} \qquad \text{and} \qquad \Gamma_k = \Big\{\gamma \in C(B_k, X) \colon \gamma \text{ odd}, \ \gamma_{|\partial B_k} = id\Big\}, \quad k \geq 2.$$

In particular, from the proof of Lemma 3.3, we deduce that for any $k \geq k_1$ and $\lambda \in [1,2]$

$$\frac{1}{8}\big(8c_4\alpha_k^p(p)\big)^{\frac{2}{2-p}} = : \overline{b_k} \leq b_k \leq c_k.$$

Also since

$$c_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \phi_\lambda(\gamma(u)) \leq \max_{u \in B_k} \phi_\lambda(\gamma(u)) = \overline{c_k}.$$

Hence,

$$\overline{b_k} \leq c_k(\lambda) \leq \overline{c_k}. \tag{3.20}$$

As a consequence, for any $k \geq k_1$, we can choose $\lambda_m \to 1$ (depending on k) and get the corresponding sequences satisfying

$$\sup_n \left\| u_n^k(\lambda_m) \right\| < \infty, \qquad \phi'_{\lambda_m}\big(u_n^k(\lambda_m)\big) \to 0 \qquad \text{and} \qquad \phi_{\lambda_m}\big(u_n^k(\lambda_m)\big) \to c_k(\lambda_m) \tag{3.21}$$

as $n \to \infty$.

Lemma 3.6. *For each λ_m given in $[1, 2]$ such that $\lambda_m \to 1$, the sequence $\{u_n^k(\lambda_m)\}_{n=1}^{\infty}$ has a strong convergent subsequence $\{u^k(\lambda_m)\}_m$ such that $\phi'_{\lambda_m}(u^k(\lambda_m)) = 0$ and $\phi_{\lambda_m}(u^k(\lambda_m)) \in$ $\in [\overline{b_k}, \overline{c_k}]$ for all $m \in \mathbb{N}$, $k \geq k_1$.*

Proof. By (3.21) we may assume, without loss of generality, that as $n \to \infty$,

$$u_n^k(\lambda_m) \rightharpoonup u^k(\lambda_m) \quad \text{in} \quad E. \tag{3.22}$$

By Lemma 2.1 we have

$$u_n^k(\lambda_m) \to u^k(\lambda_m) \quad \text{in} \quad L^p(\Omega). \tag{3.23}$$

By $(f1)$ and $(f2)$, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|f(x, y, u)| \leq \epsilon|u| + C_\epsilon|u|^{p-1} \quad \text{for all} \quad u \in \mathbb{R} \tag{3.24}$$

and Hölder inequality it follows that

$$\left| \int_\Omega f(x, y, u_n^k(\lambda_m))(u_n^k(\lambda_m) - u^k(\lambda_m))\, dV \right| \leq$$

$$\leq \epsilon \|u_n^k(\lambda_m)\|_2 \|u_n^k(\lambda_m) - u^k(\lambda_m)\|_2 + C_\epsilon \|u_n^k(\lambda_m)\|_p^{p-1} \|u_n^k(\lambda_m) - u^k(\lambda_m)\|_p$$

so, by using (3.23), we obtain

$$\lim_{n \to \infty} \int_\Omega f(x, y, u_n^k(\lambda_m))(u_n^k(\lambda_m) - u^k(\lambda_m))\, dV = 0$$

and

$$\lim_{n \to \infty} \int_\Omega \big[f(x, y, u_n^k(\lambda_m)) - f(x, y, u^k(\lambda_m))\big](u_n^k(\lambda_m) - u^k(\lambda_m))\, dV = 0.$$

Observe that

$$\|u_n^k(\lambda_m) - u^k(\lambda_m)\|^2 = \Big\langle \phi'_{\lambda_m}(u_n^k(\lambda_m) - \phi'_{\lambda_m}(u^k(\lambda_m) \Big\rangle +$$

$$+ \int_\Omega \big[\lambda_m f(x, y, u_n^k(\lambda_m)) - f(x, y, u^k(\lambda_m))\big](u_n^k(\lambda_m) - u^k(\lambda_m))\, dV \tag{3.25}$$

it is clear that

$$\Big\langle \phi'_{\lambda_m}(u_n^k(\lambda_m) - \phi'_{\lambda_m}(u^k(\lambda_m), u_n^k(\lambda_m) - u^k(\lambda_m) \Big\rangle \to 0 \tag{3.26}$$

as $n \to \infty$.

By (3.25), we have $\|u_n^k(\lambda_m) - u^k(\lambda_m)\| \to 0$ as $n \to \infty$.

As a consequence, we obtain

$$\phi'_{\lambda_m}(u^k(\lambda_m)) = 0 \quad \text{and} \quad \phi_{\lambda_m}(u^k(\lambda_m)) \in [\overline{b_k}, \overline{c_k}] \tag{3.27}$$

for all $m \in \mathbb{N},\ k \geq k_1$.

Lemma 3.7. *For any $k \geq k_1$, the sequence $\{u^k(\lambda_m)\}_{m=1}^{\infty}$ is bounded in E.*

Proof. For simplicity we set $u_m = u^k(\lambda_m)$. We suppose by contradiction that, up to a subsequence,

$$\|u_m\| \to \infty \quad \text{as} \quad m \to \infty. \tag{3.28}$$

Let $w_m = \dfrac{u_m}{\|u_m\|}$ for any $m \in \mathbb{N}$. Then, up to subsequence, we may assume that

$$\begin{aligned} w_m &\rightharpoonup w \quad \text{in} \quad E,\\ w_m &\to w \quad \text{in} \quad L^p(\Omega),\\ w_m &\to w \quad \text{a.e. in} \quad \Omega. \end{aligned} \tag{3.29}$$

Now we distinguish two cases.

Case $w = 0$. As in [29], we can say that for any $m \in \mathbb{N}$ there exists $t_m \in [0,1]$ such that

$$\phi_{\lambda_m}(t_m u_m) = \max_{t \in [0,1]} \phi_{\lambda_m}(t u_m). \tag{3.30}$$

Since (3.28) holds, for any $j \in \mathbb{N}$, we can choose $r_j = 2\sqrt{j} w_m$ such that

$$r_j \|u_m\|^{-1} \in (0,1) \tag{3.31}$$

provided m is large enough. By (3.29), $F(.,0) = 0$ and the continuity of F, we can see that

$$F(x,y,r_j w_m) \to F(x,y,r_j w) = 0 \quad \text{a.e.} \quad (x,y) \in \Omega \tag{3.32}$$

as $m \to \infty$ for any $j \in \mathbb{N}$. Then, taking into account (3.24), (3.29), (3.32), $(A4)$ and by using the Dominated Convergence Theorem we deduce that

$$F(x,y,r_j w_m) \to 0 \quad \text{in} \quad L^1(\Omega) \tag{3.33}$$

as $m \to \infty$ for any $j \in \mathbb{N}$. Then (3.30), (3.31) and (3.33) yield

$$\phi_{\lambda_m}(t_m u_m) \geq \phi_{\lambda_m}(r_j w_m) \geq 2j - \lambda_m \int\limits_{\Omega} F(x,y,r_j w_m)\, dV \geq j$$

for m is large enough and for any $j \in \mathbb{N}$. As a consequence

$$\phi_{\lambda_m}(t_m u_m) \to \infty \quad \text{as} \quad m \to \infty. \tag{3.34}$$

Since $\phi_{\lambda_m}(0) = 0$ and $\phi_{\lambda_m}(u_m) \in \left[\overline{b_k}, \overline{c_k}\right]$, we deduce that $t_m \in (0,1)$ for m large enough. Thus, by (3.30) we have

$$\left\langle \phi'_{\lambda_m}(t_m u_m), t_n u_m \right\rangle = t_m \frac{d}{dt}_{|t=t_m} \phi_{\lambda_m}(t u_m) = 0. \tag{3.35}$$

Taking into account $(f4)$, (3.35) and (2.11) we obtain

$$\frac{1}{\theta}\phi_{\lambda_m}(t_m u_m) = \frac{1}{\theta}\left(\phi_{\lambda_m}(t_m u_m) - \frac{1}{2}\left\langle \phi'_{\lambda_m}(t_m u_m), t_m u_m \right\rangle\right) =$$

$$= \frac{\lambda_m}{2\theta}\int_\Omega \varphi(t_m u_m)\,dV \le \frac{\lambda_m}{2}\int_\Omega \varphi(u_m)\,dV =$$

$$= \phi_{\lambda_m}(u_m) - \frac{1}{2}\left\langle \phi'_{\lambda_n}(u_m), u_m \right\rangle = \phi_{\lambda_m}(u_m)$$

which contradicts (3.27) and (3.34).

Case $w \neq 0$. Thus the set $\Omega' := \{(x,y) \in \Omega \colon w(x,y) \neq 0\}$ has positive Lebesgue measure. By using (3.28) and that $w \neq 0$, we have

$$|u_m(x,y)| \to \infty \quad \text{a.e.} \quad (x,y) \in \Omega' \quad \text{as} \quad m \to \infty. \tag{3.36}$$

Putting together (3.27), (3.36) and $(f3)$, and by applying Fatou's Lemma, we can easily deduce that

$$\frac{1}{2} - \frac{\phi_{\lambda_m}(u_m)}{\|u_m\|^2} = \lambda_m \int_\Omega \frac{F(x,y,u_m)}{\|u_m\|^2}\,dV \ge$$

$$\ge \lambda_m \int_{\Omega'} |w_m|^2 \frac{F(x,y,u_m)}{|u_m|^2}\,dV \to \infty \quad \text{as} \quad m \to \infty$$

which gives a contradiction because of (3.27). Then, we have proved that the sequence $\{u_m\}$ is bounded in E.

Theorem 3.1. *Assume that* $(f2)$, $(f3)$ – $(f5)$ *hold. Then problem* (2.3) *possesses infinitely many high energy solutions* $u_k \in E$ *for every* $k \in \mathbb{N}$, *in the sense that*

$$\frac{1}{2}\int_\Omega \left[(u_k)_x^2 + |D_x^{-1}\nabla_y u_k|^2 + c u_k^2\right] dV - \int_\Omega F(x,y,u_k)\,dV \to +\infty \tag{3.37}$$

as $k \to \infty$.

Proof. Taking into account Lemma 3.7 and (3.27), for each $k \ge k_1$, we can use similar arguments to those in the proof of Lemma 3.6, to show that the sequence $\{u^k(\lambda_m)\}_{m=1}^{\infty}$ admits a strong convergent subsequence with the limit u^k being just a critical point of $\phi_1 = \phi$. Clearly, $\phi(u^k) \in \left[\overline{b_k}, \overline{c_k}\right]$ for all $k \ge k_1$. Since $\overline{b_k} \to \infty$ as $k \to \infty$ in (3.20), we deduce the existence of infinitely many nontrivial critical points of ϕ. As a consequence, we have that (2.3) possesses infinitely many nontrivial weak solutions.

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