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2-QUASI CROSSED MODULES OF COMMUTATIVE ALGEBRAS

2-КВАЗІ СХРЕЩЕНІ МОДУЛІ КОМУТАТИВНИХ АЛГЕБР

We define 2-quasi crossed modules of commutative algebras obtained by relaxing some 2-crossed module conditions. Moreover, we prove that there exists a functorial relationship between these two structures which enables us to construct the coproduct object in the category of 2-crossed modules of commutative algebras.

Дано визначення 2-квазі схрещених модулів комутативних алгебр на базі послаблення деяких умов для 2-схрещених модулів. Крім того, доведено, що існує функторне співвідношення між цими двома структурами, яке дозволяє збудувати об'єкт ко-добутку у категорії 2-схрещених модулів комутативних алгебр.

1. Introduction. Crossed modules of groups [14] are given by a group homomorphism $\partial\colon E \to G$, together with an action $\triangleright$ of G on E, such that the following Peiffer relations:

$$\text{XM1: } \partial(g \triangleright e) = g\,\partial(e)\,g^{-1}, \qquad \text{XM2: } \partial(e) \triangleright f = e\,f\,e^{-1}$$

are satisfied, for all $e, f \in E,\ g \in G$. Without the second condition, we call it a precrossed module.

2-crossed modules of groups [7] are given by a group complex $L \xrightarrow{\delta} E \xrightarrow{\partial} G$, satisfying certain conditions together with the actions of G on L and E, making it a complex of G-modules, where G acts on itself by conjugation. The first Peiffer relation for the map $\partial\colon E \to G$ automatically holds, thus $\partial\colon E \to G$ is a precrossed module. The second Peiffer relation does not hold in general. However the Peiffer lifting $\{-,-\}\colon E \times E \to L$ measures how far the second Peiffer relation is from being satisfied, namely: $\delta(\{e,f\}) = \big(efe^{-1}\big)\big(\partial(e) \triangleright f^{-1}\big)$, for all $e, f \in E$. The category of 2-crossed modules is equivalent to a reflexive subcategory of the category of simplicial groups with Moore complex of length two [11].

As for the group case, 2-crossed modules of commutative algebras are introduced in [8] to obtain a method for computing the (co)homology groups of a commutative algebra with coefficients which coincides with the Andre – Quillen theory for $n = 0, 1, 2, 3$. Consequently, without simplicial theory, they get the Jacobi – Zariski sequence. The construction of 2-crossed modules of commutative algebras depends on, essentially switching actions by automorphisms to actions by multipliers under certain conditions. A 2-crossed module of commutative algebras $L \xrightarrow{\partial_2} E \xrightarrow{\partial_1} R$ has an underlying complex of commutative algebras and the following data: we have the algebra actions $\triangleright$ of R on E, L; and a Peiffer lifting map $\{-,-\}\colon E \times E \to L$ satisfying the conditions given in Definition 2.4.

As in the group case, simplicial commutative algebras and 2-crossed modules of commutative algebras are closely related. A simplicial commutative algebra [2, 10] $A = (A_n, d_n^i, s_n^i)$, i.e., a simplicial object in the category of commutative algebras, is given by a collection of algebra morphisms $d_n^i\colon A_n \to A_{n-1},\ i = 0, \ldots, n$, and $s_n^i\colon A_n \to A_{n+1},\ i = 0, \ldots, n$, called boundaries and degeneracies respectively, such that satisfying the well known simplicial identities. The Moore complex of the simplicial commutative algebra A is the complex

$$N(A) = \Big(\ldots \xrightarrow{d_{(n+1)}} N(A)_n \xrightarrow{d_n} \ldots \xrightarrow{d_3} N(A)_2 \xrightarrow{d_2} N(A)_1 \xrightarrow{d_1} A_0\Big),$$

where $N(A)_n = \bigcap_{i=0}^{n-1} \ker(d_n^i) \subset A_n$ at level n and the boundary $d_n : N(A)_n \to N(A)_{n-1}$ is the restriction of $d_n^n : A_n \to A_{(n-1)}$. We say that the Moore complex of a simplicial commutative algebra A has length n if $N(A)_i$ is trivial for all $i > n$.

If A has Moore complex of length one, then $N(A)_1 \xrightarrow{d_1} A_0$ defines a crossed module [12]. One level further, a simplicial commutative algebra A with Moore complex of length two, corresponds to a 2-crossed module $N(A)_2 \xrightarrow{d_2} N(A)_1 \xrightarrow{d_1} A_0$; see [3] for details. Conversely, one can get the corresponding simplicial commutative algebra by using a 2-crossed module. This gives an equivalence between the categories of simplicial commutative algebras with Moore complex of length two, and that of 2-crossed modules of commutative algebras [9].

The crossed modules of groups with a fixed codomain G are called crossed G-modules. For any two crossed G-modules of groups $\partial : E \to G$ and $\partial' : E' \to G$, the coproduct is defined via the quotient of the free group $E * E'$ by Brown in [4]. However, we should replace the free group structure by the semi-direct product when we work in the category of commutative algebras [13].

However, the construction of the coproduct of 2-crossed module is definitely more complicated than crossed modules. Because 2-crossed modules have much more data than crossed modules. To overcome this difficulty, it was necessary to define something weaker than a 2-crossed module, yet with some functorial relations (adjunction) again with 2-crossed modules. For this aim, in this paper, we first define 2-quasi crossed modules in the category of commutative algebras inspired by [6]. Afterwards, we give an adjunction between the category of 2-crossed modules and the category of 2-quasi crossed modules. This adjunction allow us to define the coproduct object with the category theoretical point of view.

2. Preliminaries. We fix a commutative ring κ, not necessarily with 1. All algebras considered will be associative and commutative over κ, but not necessarily with a multiplicative identity.

If E and R are two algebras, a bilinear map $(r, e) \in R \times E \longmapsto r \triangleright e \in E$ is called an algebra action of R on E if, for all $e, e' \in e$ and $r, r' \in R$, we have

$$\mathrm{A1}: \; r \triangleright (ee') = (r \triangleright e)\, e' = e\, (r \triangleright e'), \qquad \mathrm{A2}: \; (rr') \triangleright e = r \triangleright (r' \triangleright e).$$

Then we get the semidirect product $E \rtimes R$ with

$$(e, r)\,(e', r') = (r \triangleright e' + r' \triangleright e + ee', rr'),$$

for all $e, e' \in E$ and $r, r' \in R$.

***Convention*:** Let $L \to E \to R$ be a chain complex of R-algebras. The actions of R on E and L will be both denoted by "$\triangleright$"in the rest of the paper. We say that the subalgebra E' of E is R-invariant if $r \triangleright e' \in E'$ for all $e' \in E'$ and $r \in R$. A function $f : L \to E$ is said to be R-equivariant if $f(r \triangleright l) = r \triangleright f(l)$, for all $l \in L$ and $r \in R$. Remark that R has a natural R-algebra structure where the action is defined via its multiplication.

2.1. Crossed modules of algebras.

Definition 2.1. *A precrossed module of algebras (E, R, ∂), is given by an algebra homomorphism $\partial : E \to R$, together with an action $\triangleright$ of R on E, such that the following relation, called the "first Peiffer relation", holds*

(XM1) $\partial(r \triangleright e) = r\, \partial(e)$, *for all $e \in E$ and $r \in R$.*

A crossed module of algebras (E, R, ∂) is a precrossed module satisfying, furthermore, the "second Peiffer relation"

(XM2) $\partial(e) \triangleright e' = e\,e'$, *for all* $e, e' \in E$.

***Example* 2.1.** Let R be an algebra and $E \trianglelefteq R$ be any ideal of R. Then (E, R, i), where $i\colon E \to R$ is the inclusion map, is a crossed module. We use the multiplication in R to define the action of R on E.

Definition 2.2. *A crossed module morphism* $(f_1, f_0)\colon (E, R) \to (E', R')$ *consists of algebra homomorphisms* $f_0 : R \to R'$ *and* $f_1 : E \to E'$ *such that the following diagram commutes*:

$$\begin{tikzcd} E \arrow[r, "\partial"] \arrow[d, "f_1"'] & R \arrow[d, "f_0"] \\ E' \arrow[r, "\partial'"'] & R' \end{tikzcd}$$

and preserve the action, *namely* $f_1(r \triangleright e) = f_0(e) \triangleright f_1(e)$, *for all* $r \in R$ *and* $e \in E$.

Thus we get the category of crossed modules of algebras denoted by XMod.

Definition 2.3. *The category of crossed modules with fixed codomain* R *is the full subcategory of* XMod *that is denoted by* $\mathrm{XMod/R}$. *These crossed modules will be called crossed* R*-modules.*

2.2. 2-crossed modules of algebras.

Definition 2.4. *A* 2*-crossed module of algebras* $L \xrightarrow{\partial_2} E \xrightarrow{\partial_1} R$ *is given by a chain complex of* R*-algebra homomorphisms* $(\partial_1 \circ \partial_2 = 0)$, *equipped with an* R*-equivariant bilinear map namely* $r \triangleright \{e, e'\} = \{r \triangleright e, e'\} = \{e, r \triangleright e'\}$, *called Peiffer lifting*

$$\{-,-\}\colon E \otimes_R E \longrightarrow L,$$

such that satisfying:

(2XM1) $\partial_2\{e, e'\} = ee' - \partial_1(e') \triangleright e$,

(2XM2) $\{\partial_2(l), \partial_2(l')\} = ll'$,

(2XM3) $\{e, e'e''\} = \{ee', e''\} + \partial_1(e'') \triangleright \{e, e'\}$,

(2XM4) $\{e, \partial_2(l)\} - \{\partial_2(l), e\} = \partial_1(e) \triangleright l$,

for all $l, l' \in L$, $e, e', e'' \in E$, *and* $r \in R$.

***Remark* 2.1.** Note that $\partial_2 : L \to E$ is a crossed module, where E acts on L with

$$e \triangleright' l = \{e, \partial_2(l)\}.$$

However, $\partial_1 : E \to R$ is a precrossed module in general. The Peiffer lifting in E measures exactly the failure of $\partial_1 : E \to R$ to be a crossed module.

***Example* 2.2.** Let (E, R, ∂) be a precrossed module. $\ker(\partial) \xrightarrow{i} E \xrightarrow{\partial} R$, where $i\colon \ker(\partial) \to$ $\to E$ is the inclusion map, is a 2-crossed module, where

$$\{-,-\}\colon (e, e') \in E \otimes_R E \longmapsto \{e, e'\} = ee' - \partial(e) \triangleright e' \in \ker(\partial).$$

Notation. Any 2-crossed module of algebras will be denoted by $(L, E, R, \partial_1, \partial_2)$.

Definition 2.5. *Given* 2*-crossed modules* $(L, E, R, \partial_1, \partial_2)$ *and* $(L', E', R', \partial_1', \partial_2')$, *a* 2*-crossed module morphism consists of algebra homomorphisms* $f_0 : R \to R'$, $f_1 : E \to E'$ *and* $f_2 : L \to L'$, *making the diagram*

$$\begin{tikzcd} L \arrow[r, "\partial_2"] \arrow[d, "f_2"] & E \arrow[r, "\partial_1"] \arrow[d, "f_1"] & R \arrow[d, "f_0"] \\ L' \arrow[r, "\partial_2'"'] & E' \arrow[r, "\partial_1'"'] & R' \end{tikzcd}$$

commutative and preserving the actions of R, R', *and the Peiffer lifting, namely*

$$f_1(r \triangleright e) = f_0(r) \triangleright f_1(e), \quad \textit{for all} \quad e \in E \quad \textit{and} \quad r \in R,$$

$$f_2(r \triangleright l) = f_0(r) \triangleright f_2(l), \quad \textit{for all} \quad l \in L \quad \textit{and} \quad r \in R,$$

$$f_2\{e, e'\} = \{f_1(e), f_1(e')\}, \quad \textit{for all} \quad e, e' \in E.$$

Thus we get the category of 2*-crossed modules of algebras that is denoted by* $\mathrm{X_2Mod}$.

Definition 2.6. *The category of* 2*-crossed modules with fixed tail* $(E \to R)$ *is the full subcategory of* $\mathrm{X_2Mod}$ *that is denoted by* $\mathrm{X_2Mod/(E \to R)}$. *This type of* 2*-crossed modules will be called* 2*-crossed* $(R \to E)$*-modules.*

3. 2-quasi crossed modules of algebras.

Definition 3.1. *A* 2*-quasi crossed module of algebras is a chain complex of* R*-algebra homomorphisms* $L \xrightarrow{\partial_2} E \xrightarrow{\partial_1} R$, *together with an* R*-equivariant bilinear map*

$$\{-,-\} : E \otimes_R E \longrightarrow L,$$

satisfying the following axioms:

(2QX1) $\partial_2\{e, e'\} = ee' - \partial_1(e') \triangleright e$,

(2QX2) $\{e, e'e''\} = \{ee', e''\} + \partial_1(e'') \triangleright \{e, e'\}$,

(2QX3) $\{e', e\}\, \partial_1(e') \triangleright \{e, \partial_2(l)\} = \{ee'e, \partial_2(\partial_1(e') \triangleright l)\} - (\partial_1(e') \triangleright e)\{e, \partial_2(\partial_1(e') \triangleright l)\}$

for all $l \in L$ *and* $e, e', e'' \in E$.

Definition 3.2. 2*-quasi crossed module morphisms can be defined in a similar way. Therefore, we get the category of* 2*-quasi crossed modules of algebras denoted by* $\mathrm{QX_2Mod}$.

The category of 2*-quasi* $(E \to R)$*-modules, namely* $\mathrm{QX_2Mod/(E \to R)}$ *can also be defined according to Definition* 2.6.

3.1. 2-crossed modules vs 2-quasi crossed modules.

Lemma 3.1. *Any* 2*-crossed module is a* 2*-quasi crossed module. This leads an inclusion functor*

$$\mathrm{X_2Mod} \longrightarrow \mathrm{QX_2Mod}. \tag{1}$$

Proof. Let $L \xrightarrow{\partial_2} E \xrightarrow{\partial_1} R$ be a 2-crossed module. We only have to prove that axiom $2QX3$ is verified. So we obtain

$$\{e', e\}\partial_1(e') \triangleright \{e, \partial_2(l)\} = (e'e - \partial_1(e) \triangleright e')\{e, \partial_2(\partial_1(e') \triangleright l)\} =$$

$$= e'e\,\{e, \partial_2(\partial_1(e') \triangleright l)\} - (\partial_1(e) \triangleright e')\{e, \partial_2(\partial_1(e') \triangleright l)\} =$$

$$= e'e\,(e \cdot (\partial_1(e') \triangleright l)) - (\partial_1(e') \triangleright e) \cdot (e \cdot (\partial_1(e') \triangleright l)) =$$

$$= e\left(e'e\,(\partial_1(e')\triangleright l) - e((\partial_1(e')\triangleright e)\cdot(\partial_1(e')\triangleright l)\right) =$$

$$= e\left(\{e'e, \partial_2(\partial_1(e')\triangleright l)\} - \{\partial_1(e')\triangleright e, \partial_2(\partial_1(e')\triangleright l)\}\right) =$$

$$= \{ee'e, \partial_2(\partial_1(e')\triangleright l)\} - \{e(\partial_1(e')\triangleright e), \partial_2(\partial_1(e')\triangleright l)\} =$$

$$= \{ee'e, \partial_2(\partial_1(e')\triangleright l)\} - (\partial_1(e')\triangleright e)\{e, \partial_2(\partial_1(e')\triangleright l)\},$$

for all $l \in L$ and $e, e' \in E$, that completes the proof.

Lemma 3.2. *Let $L \xrightarrow{\partial_2} E \xrightarrow{\partial_1} R$ be a 2-quasi crossed module and let $[L, L]$ be the ideal of L generated by the elements of the form*

$$e \star l = \partial_1(e)\triangleright l - \{e, \partial_2(l)\} + \{\partial_2(l), e\},$$

$$l_0 \sharp l_1 = l_0\, l_1 - \{\partial_2(l_0), \partial_2(l_1)\},$$

for all $l, l_0, l_1 \in L$ and $e \in E$. Then $[L, L]$ is an R-invariant ideal of L.

Proof. For all $r \in R$, $e \in E$ and $l_0, l_1, l_2 \in L$, we get

$$r\triangleright(e\star l) = r\triangleright(\partial_1(e)\triangleright l - \{e, \partial_2(l)\} + \{\partial_2(l), e\}) =$$

$$= r\triangleright(\partial_1(e)\triangleright l) - r\triangleright(\{e, \partial_2(l)\} + \{\partial_2(l), e\}) =$$

$$= r\partial_1(e)\triangleright l - r\triangleright\{e, \partial_2(l)\} + r\triangleright\{\partial_2(l), e\} =$$

$$= \partial_1(r\triangleright e)\triangleright l - \{r\triangleright e, \partial_2(l)\} + \{\partial_2(l), r\triangleright e\},$$

Fix $r\triangleright e = e' \in E$; it follows

$$\partial_1(e')\triangleright l - \{e', \partial_2(l)\} + \{\partial_2(l), e'\} = e' * l \in [L, L]\,.$$

If we handle the second type of elements, we get

$$r\triangleright(l_0\sharp l_1) = r\triangleright(l_0\, l_1 - \{\partial_2(l_0), \partial_2(l_1)\}) =$$

$$= r\triangleright(l_0\, l_1) - r\triangleright(\{\partial_2(l_0), \partial_2(l_1)\}) =$$

$$= (r\triangleright l_0)l_1 - \{r\triangleright\partial_2(l_0), \partial_2(l_1)\} =$$

$$= (r\triangleright l_0)l_1 - \{\partial_2(r\triangleright l_0), \partial_2(l_1)\},$$

Fix $r\triangleright l_0 = l_2 \in L$; then it follows

$$l_2\, l_1 - \{\partial_2(l_2), \partial_2(l_1)\} = l_2\sharp l_1 \in [L, L]$$

and proves that $[L, L]$ is an R-invariant ideal.

Proposition 3.1. *Hence we get the quotient R-algebra*

$$L^{cr} = L/[L, L].$$

Lemma 3.3. *The quotient map $\phi\colon L \to L/[L, L]$ provides an induced functor*

$$(\ )^*_{cr} : \mathrm{QX_2Mod} \to \mathrm{X_2Mod}$$

which maps any 2*-quasi crossed module* $L \xrightarrow{\partial_2} E \xrightarrow{\partial_1} R$ *to a* 2*-crossed module*

$$L^{cr} \xrightarrow{\partial_2^{cr}} E \xrightarrow{\partial_1} R \tag{2}$$

with the new Peiffer lifting $\{-,-\}^{cr} : E \otimes_R E \longrightarrow L^{cr}$ *given by the composition*

$$E \otimes_R E \xrightarrow{\{-,-\}} L \xrightarrow{\phi} L^{cr}.$$

Proof. Axioms 2XM2 and 2XM4 are satisfied as the elements $e \star l$ and $l_0 \sharp l_1$ are already quotiented out in the definition of L^{cr}.

Since $\partial_2(e * l) = \partial_2(l_0 \sharp l_1) = 0$, we also have

$$\partial_2([L, L]) = 0,$$

for all $e \in E$ and $l, l_0, l_1 \in L$.

Moreover we get

$$\text{(2XM1)} \qquad \begin{aligned} \partial_2^{cr}\{e, e'\}^{cr} &= \partial_2^{cr}\big(\phi\{e, e'\}\big) = \\ &= \partial_2\big(\{e, e'\} + [L, L]\big) = \\ &= \partial_2\{e, e'\} + \partial_2([L, L]) = \\ &= ee' - \partial_1(e') \triangleright e \qquad (\because \text{2QX1}) \end{aligned}$$

and

$$\text{(2XM3)} \qquad \begin{aligned} \{e, e'e''\}^{cr} &= \phi\{e, e'e''\} = \{e, e'e''\} + [L, L] = \\ &= \Big(\{ee', e''\} + \partial_1(e'') \triangleright \{e, e'\}\Big) + [L, L] \qquad (\because \text{2QX2}) = \\ &= \Big(\{ee', e''\} + [L, L]\Big) + \Big(\partial_1(e'') \triangleright \{e, e'\} + [L, L]\Big) = \\ &= \Big(\{ee', e''\} + [L, L]\Big) + \partial_1(e'') \triangleright \Big(\{e, e'\} + [L, L]\Big) = \\ &= \phi\{ee', e''\} + \partial_1(e'') \triangleright \phi\{e, e'\} = \\ &= \{ee', e''\}^{cr} + \partial_1(e'') \triangleright \{e, e'\}^{cr} \end{aligned}$$

for all $e, e', e'' \in E$, that completes the proof.

Remark that, we used the fact that $[L, L]$ is R-invariant, in the above calculations.

***Corollary* 3.1.** *We get the following adjunction*:

$$\begin{tikzcd} \mathrm{QX_2Mod} \arrow[r, bend left, "(\ )^*_{cr}"] \arrow[r, phantom, "\perp"] & \mathrm{X_2Mod}. \arrow[l, bend left, hook'] \end{tikzcd} \tag{3}$$

3.2. Simplicial algebras vs 2-quasi crossed modules of algebras. We know that the category of 2-crossed modules of algebras is equivalent to the category of simplicial algebras with Moore complex of length two. This equivalence is proven by higher dimensional Peiffer elements in [3] with the method introduced in [5]. Briefly, for a given simplicial algebra $A = (A_n, d_n^i, s_n^i)$ with Moore complex of length two, the subalgebras generated by

$$(\ker d_0)(\ker d_1 \cap \ker d_2), \qquad (\ker d_1)(\ker d_0 \cap \ker d_2), \qquad (\ker d_2)(\ker d_0 \cap \ker d_1)$$

of A_2 are all trivial. However, the notion of 2-quasi crossed modules arises by weakening some of these conditions as follows:

– define A_1' be the subalgebra of $\ker(d_1 \colon A_2 \to A_1)$ generated by the elements in the form $s_1(x) - s_0(x)$, for all $x \in A_1$,

– define A_2' be the subalgebra of $\ker(d_2 \colon A_2 \to A_1)$ generated by the elements in the form $s_0(x) - s_1 s_0 d_1(x)$, for all $x \in A_1$.

Then we get the definition of 2-quasi crossed modules corresponding to the 2-truncated simplicial algebras with the following trivial subalgebras of A_2:

$$(A_1')\,(\ker d_0 \cap \ker d_2), \qquad (A_2')\,(\ker d_0 \cap \ker d_1).$$

4. Coproduct of 2-quasi crossed modules. Let us recall the coproduct of crossed modules of algebras from [13].

4.1. Coproduct of crossed modules. Let (A, R, ∂_1) and (B, R, ∂_2) be two crossed R-modules. There exists an action of B on A with

$$b \triangleright a = \partial_2(b) \triangleright a\,. \tag{4}$$

Then, we have the semidirect product $B \rtimes A$. Define $\partial \colon B \rtimes A \to R$ by

$$\partial(b, a) = \partial_2(b) + \partial_1(a).$$

for all $(b, a) \in B \rtimes A$. Here ∂ becomes a precrossed module where R acts on $B \rtimes A$ in a natural way, since

$$\partial(r \triangleright (b, a)) = \partial(r \triangleright b, r \triangleright a) = \partial_2(r \triangleright b) + \partial_1(r \triangleright a) =$$
$$= r\,\partial_2(b) + r\,\partial_1(a) = r\,\big(\partial_2(b) + \partial_1(a)\big) = r\,\partial(b, a),$$

for all $r \in R$ and $(b, a) \in B \rtimes A$.

Let P be the ideal of $B \rtimes A$ generated by the elements of the form

$$(b, a)(b', a') - \partial(b, a) \triangleright (b', a')$$

for all $(b, a), (b', a') \in B \rtimes A$. On the other hand, we have (by using (4))

$$(b, a)(b', a') - \partial(b, a) \triangleright (b', a') = (b, a) \cdot (b', a') - (\partial_2(b) + \partial_1(a)) \triangleright (b', a') =$$
$$= (bb', \partial_2(b) \triangleright a' + \partial_2(b') \triangleright a + aa') - \big((\partial_2(b) + \partial_1(a)) \triangleright b', (\partial_2(b) + \partial_1(a)) \triangleright a'\big) =$$
$$= (bb', \partial_2(b) \triangleright a' + \partial_2(b') \triangleright a + aa') - \big(\partial_2(b) \triangleright b' + \partial_1(a) \triangleright b', \partial_2(b) \triangleright a' + \partial_1(a) \triangleright a'\big) =$$
$$= (bb', \partial_2(b) \triangleright a' + \partial_2(b') \triangleright a + aa') - \big(bb' + \partial_1(a) \triangleright b', \partial_2(b) \triangleright a' + aa'\big) =$$
$$= (-\partial_1(a) \triangleright b', \partial_2(b') \triangleright a).$$

That means P is generated by the elements

$$(-\partial_1(a) \triangleright b', \partial_2(b') \triangleright a).$$

Remark that $\partial(P) = 0$. Hence we have the induced morphism

$$\bar{\partial} \colon (B \rtimes A)/P \to R,$$

defined by

$$\bar{\partial}\big((b,a) + P\big) = \partial_2(b) + \partial_1(a),$$

gives us a crossed module since, for all $(b,a),(b',a') \in B \rtimes A$ we have

$$\begin{aligned}\bar{\partial}((b,a)+P) \triangleright ((b',a')+P) &= (\partial_2(b)+\partial_1(a)) \triangleright ((b',a')+P) = \\ &= \big(\partial_2(b) \triangleright b' + \partial_1(a) \triangleright b', \partial_2(b) \triangleright a' + \partial_1(a) \triangleright a'\big) = \\ &= (bb' + \partial_1(a) \triangleright b', \partial_2(b) \triangleright a' + aa').\end{aligned}$$

We know that $(-\partial_1(a) \triangleright b', \partial_2(b') \triangleright a) \in P$, it follows

$$\begin{aligned}(bb' + \partial_1(a) \triangleright b', \partial_2(b) \triangleright a' + aa') &= (bb', \partial_2(b') \triangleright a + \partial_2(b) \triangleright a' + aa') = \\ &= ((b,a)(b',a')) + P = ((b,a)+P)((b',a')+P).\end{aligned}$$

Therefore we get the crossed module $((B \rtimes A)/P, R, \partial)$ which is the coproduct in $\mathrm{XMod/R}$.

4.2. Coproduct of 2-quasi crossed modules. Let us fix two 2-quasi crossed $(E \to R)$-modules

$$\mathcal{A} = L_1 \xrightarrow{\partial_2} E \xrightarrow{\partial_1} R, \quad \{-,-\}_1 \colon E \times E \to L_1,$$

$$\mathcal{A}' = L_2 \xrightarrow{\partial_2'} E \xrightarrow{\partial_1} R, \qquad \{-,-\}_2 \colon E \times E \to L_2$$

throughout the entire section.

Remark that, we have

$$\partial_2\{e,e'\}_1 = \partial_2'\{e,e'\}_2 = ee' - \partial_1(e') \triangleright e,$$

when we consider $\mathcal{A}$ and $\mathcal{A}'$.

Construct $L_1 \rtimes L_2$ in the sense of subsection 4.1. Then let P be the ideal of $L_1 \rtimes L_2$ generated by the elements

$$(\epsilon_1\{e,e'\}_1, \epsilon_2\{e,e'\}_2),$$

where $\epsilon_i = \pm 1$ and $\epsilon_1 \neq \epsilon_2$.

Define the Peiffer lifting

$$\{-,-\} \colon E \times E \to (L_1 \rtimes L_2)/P,$$

with

$$\{e,e'\} = (\{e,e'\}_1, 0) + P = (0, \{e,e'\}_2) + P,$$

by considering $(\{e,e'\}_1, -\{e,e'\}_2) \in P$.

E acts on $(L_1 \rtimes L_2)/P$ in a natural way, namely

$$e \triangleright ((l, l') + P) = (e \triangleright l, e \triangleright l') + P,$$

for all $e \in E$ and $(l, l') \in L_1 \rtimes L_2$.

There exists an induced morphism

$$\bar{\partial} \colon (L_1 \rtimes L_2)/P \to E,$$

where

$$\bar{\partial}\big((l_1, l_2) + P\big) = \partial_2(l_1) + \partial_2'(l_2),$$

by using the fact that $\partial(P) = 0$.

Thus we get a 2-quasi crossed module

$$(L_1 \rtimes L_2)/P \xrightarrow{\bar{\partial}} E \xrightarrow{\partial_1} R$$

since

$$\text{(2QX1)} \qquad \bar{\partial}\{e, e'\} = \bar{\partial}(\{e, e'\}_1, 0) + P =$$
$$= \partial_2\{e, e'\}_1 = ee' - \partial_1(e') \triangleright e,$$
$$\text{(2QX2)} \qquad \{e, e'e''\} = (\{e, e'e''\}_1, 0) + P =$$
$$= \big(\{ee', e''\}_1 + \partial_1(e'') \triangleright \{e, e'\}_1, 0\big) + P =$$
$$= (\{ee', e''\}_1, 0) + P + (\partial_1(e'') \triangleright \{e, e'\}_1, 0) + P =$$
$$= \{ee', e''\} + \partial_1(e'') \triangleright \{e, e'\}$$

and also

$$\text{2QX3)} \quad \{e, e'\}\partial_1(e) \triangleright \{e', \bar{\partial}((l, l') + P)\} = \{e, e'\}\partial_1(e) \triangleright \{e', \partial_2(l) + \partial_2'(l')\} =$$
$$= \{e, e'\}\partial_1(e) \triangleright (\{e', \partial_2(l)\} + \{e', \partial_2'(l')\}) =$$
$$= \{e, e'\}\big(\partial_1(e) \triangleright \{e', \partial_2(l)\} + \partial_1(e) \triangleright \{e', \partial_2'(l')\}\big) =$$
$$= \{e, e'\}\partial_1(e) \triangleright \{e', \partial_2(l)\} + \{e, e'\}\partial_1(e) \triangleright \{e', \partial_2'(l')\} =$$
$$= ((\{e, e'\}_1, 0) + P)\partial_1(e) \triangleright ((\{e', \partial_2(l)\}_1, 0) + P) +$$
$$+ ((0, \{e, e'\}_2) + P)\partial_1(e) \triangleright ((0, \{e', \partial_2'(l')\}) + P) =$$
$$= ((\{e, e'\}_1, 0)\partial_1(e) \triangleright (\{e', \partial_2(l)\}_1, 0) + P) +$$
$$+ ((0, \{e, e'\}_2)\partial_1(e) \triangleright (0, \{e', \partial_2'(l')\}) + P) =$$
$$= ((\{e, e'\}_1\partial_1(e) \triangleright \{e', \partial_2(l)\}_1, 0) + P) +$$
$$+ ((0, \{e, e'\}_2\partial_1(e) \triangleright \{e', \partial_2'(l')\}_2) + P) =$$
$$= ((\{e'ee', \partial_2(\partial_1(e) \triangleright l)\}_1 - (\partial_1(e) \triangleright e')\{e', \partial_2(\partial_1(e) \triangleright l)\}_1, 0) + P) +$$
$$+ \big((0, \{e'ee', \partial_2'(\partial_1(e) \triangleright l')\}_2 - (\partial_1(e) \triangleright e')\{e', \partial_2'(\partial_1(e) \triangleright l')\}_2\big) + P\big) =$$

$$
\begin{gathered}
= ((\{e'ee', \partial_2(\partial_1(e) \triangleright l)\}_1, 0) + P) - (\partial_1(e) \triangleright e')((\{e', \partial_2(\partial_1(e) \triangleright l)\}_1, 0) + P) + \\
+((0, \{e'ee', \partial'_2(\partial_1(e) \triangleright l')\}_2) + P) - (\partial_1(e) \triangleright e')((0, \{e', \partial'_2(\partial_1(e) \triangleright l')\}_2)) + P) = \\
= \{e'ee', \partial_2(\partial_1(e) \triangleright l)\} - (\partial_1(e) \triangleright e')\{e', \partial_2(\partial_1(e) \triangleright l)\} + \\
+\{e'ee', \partial'_2(\partial_1(e) \triangleright l')\} - (\partial_1(e) \triangleright e')\{e', \partial'_2(\partial_1(e) \triangleright l')\} = \\
= \{e'ee', \partial_2(\partial_1(e) \triangleright l) + \partial'_2(\partial_1(e) \triangleright l')\} - \\
-(\partial_1(e) \triangleright e')\{e', \partial_2(\partial_1(e) \triangleright l) + \partial'_2(\partial_1(e) \triangleright l')\} = \\
= \{e'ee', \bar{\partial}((\partial_1(e) \triangleright l, \partial_1(e) \triangleright l') + P)\} - \\
-(\partial_1(e) \triangleright e')\{e', \bar{\partial}(\partial_1(e) \triangleright l, \partial_1(e) \triangleright l') + P)\} = \\
= \{e'ee', \bar{\partial}(\partial_1(e) \triangleright ((l, l') + P))\} - (\partial_1(e) \triangleright e')\{e', \bar{\partial}(\partial_1(e) \triangleright ((l, l') + P))\},
\end{gathered}
$$

for all $e, e', e'' \in E$ and $((l, l') + P) \in (L_1 \rtimes L_2)/P$.

Theorem 4.1. *Given two 2-quasi crossed $(E \to R)$-modules $\mathcal{A}$ and $\mathcal{A}'$, we have the coproduct*

$$
\mathcal{A} \coprod_{\mathrm{QX_2Mod}} \mathcal{A}' = (L_1 \rtimes L_2)/P \xrightarrow{\bar{\partial}} E \xrightarrow{\partial_1} R
$$

in the category $\mathrm{QX_2Mod}/(\mathrm{E} \to \mathrm{R})$.

Proof. Let

$$
\begin{tikzcd}
L_1 \arrow[r, "\partial_2"] \arrow[d, "\alpha"] & E \arrow[r, "\partial_1"] \arrow[d, "id"] & R \arrow[d, "id"] \\
D \arrow[r, "\partial''_2"'] & E \arrow[r, "\partial_1"'] & R
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
L_2 \arrow[r, "\partial'_2"] \arrow[d, "\beta"] & E \arrow[r, "\partial_1"] \arrow[d, "id"] & R \arrow[d, "id"] \\
D \arrow[r, "\partial''_2"'] & E \arrow[r, "\partial_1"'] & R
\end{tikzcd}
$$

be two 2-quasi crossed module morphisms. Then there exists a unique 2-quasi crossed module morphism

$$
\begin{tikzcd}
(L_1 \rtimes L_2)/P \arrow[r, "\bar{\partial}"] \arrow[d, "\phi"] & E \arrow[r, "\partial_1"] \arrow[d, "id"] & R \arrow[d, "id"] \\
D \arrow[r, "\partial''_2"'] & E \arrow[r, "\partial_1"'] & R
\end{tikzcd}
$$

where the morphism

$$
\phi \colon (L_1 \rtimes L_2)/P \longrightarrow D,
$$

is given by

$$
\phi((l_1, l_2) + P) = \alpha(l_1) + \beta(l_2),
$$

which satisfies the universal property of the coproduct object with the following diagram, and completes the proof.

$$\begin{tikzcd}
 & D \xrightarrow{\partial_2''} E \xrightarrow{\partial_1} R & \\
L_1 \xrightarrow{\partial_2} E \xrightarrow{\partial_1} R \arrow[ur, "{(\alpha,id,id)}"] \arrow[r, "{(i_1,id,id)}"'] & (L_1 \rtimes L_2)/P \xrightarrow{\bar{\partial}} E \xrightarrow{\partial_1} R \arrow[u, dotted, "{(\phi,id,id)}"] & L_2 \xrightarrow{\partial_2'} E \xrightarrow{\partial_1} R\,. \arrow[ul, "{(\beta,id,id)}"'] \arrow[l, "{(i_2,id,id)}"]
\end{tikzcd}$$

5. Coproduct of 2-crossed modules. In this section, we construct the coproduct object in the category $\mathrm{X_2Mod/(E \to R)}$ through 2-quasi crossed modules and their functorial relationship with 2-crossed modules.

First of all, let us denote two fixed 2-crossed $(E \to R)$-modules

$$\mathcal{A} = \ L_1 \xrightarrow{\partial_2} E \xrightarrow{\partial_1} R, \qquad \{-,-\}_1 \colon E \times E \to L_1,$$

$$\mathcal{A}' = \ L_2 \xrightarrow{\partial_2'} E \xrightarrow{\partial_1} R, \qquad \{-,-\}_2 \colon E \times E \to L_2.$$

Theorem 5.1. *Given two 2-crossed $(E \to R)$-modules $\mathcal{A}$ and $\mathcal{A}'$, we have the coproduct*

$$\mathcal{A} \coprod_{\mathrm{X_2Mod}} \mathcal{A}' = \Big(\mathcal{A} \coprod_{\mathrm{QX_2Mod}} \mathcal{A}'\Big)^*_{cr}$$

in the category $\mathrm{X_2Mod/(E \to R)}$.

Proof. Suppose that we have 2-crossed $(E \to R)$-modules $\mathcal{A}$, $\mathcal{A}'$. Considering the inclusion functor $\mathrm{X_2Mod} \to \mathrm{QX_2Mod}$ given in (1), we naturally have 2-quasi crossed $(E \to R)$-modules $\mathcal{A}$ and $\mathcal{A}'$. Thus, from Theorem 4.1, we obtain

$$\mathcal{A} \coprod_{\mathrm{QX_2Mod}} \mathcal{A}' = (L_1 \rtimes L_2)/P \xrightarrow{\bar{\partial}} E \xrightarrow{\partial_1} R \tag{5}$$

which is the coproduct object in the category $\mathrm{QX_2Mod/(E \to R)}$.

Recall from (3) that the functor

$$(\ )^*_{cr} \colon \mathrm{QX_2Mod} \longrightarrow \mathrm{X_2Mod} \tag{6}$$

is left adjoint to the inclusion functor (1).

From the categorical point of view, it is a well-known property that left adjoints preserve colimits; moreover, the coproduct object is defined as a colimit over the diagram that consists of just two objects [1]. Consequently, the functor $(\ )^*_{cr}$ maps coproducts to coproducts.

Then, when we apply the functor (6) to (5) we get the 2-crossed module

$$\Big(\mathcal{A} \coprod_{\mathrm{QX_2Mod}} \mathcal{A}'\Big)^*_{cr} = \ \Big((L_1 \rtimes L_2)/P\Big)^{cr} \xrightarrow{\bar{\partial}^{cr}} E \xrightarrow{\partial_1} R \tag{7}$$

that follows from (2); which gives the coproduct $\mathcal{A} \coprod\limits_{\mathrm{X_2Mod}} \mathcal{A}'$ in the category $\mathrm{X_2Mod/(E \to R)}$.

***Remark* 5.1.** In fact, (7) defines the coproduct of $(\mathcal{A})^*_{cr}$ and $(\mathcal{A}')^*_{cr}$ in general. However, the inclusion map in our adjunction (3) provides that: when we have a 2-crossed module and apply the inclusion functor and $(\ )^*_{cr}$ respectively, we obtain the same 2-crossed module up to isomorphism. Summarily, $\mathcal{A} \in \mathrm{X_2Mod} \longhookrightarrow \mathcal{A} \in \mathrm{QX_2Mod} \xrightarrow{(\ )^*_{cr}} (\mathcal{A})^*_{cr} \cong A \in \mathrm{X_2Mod}$.

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