DOI: 10.37863/umzh.v74i5.7081

UDC 517.5

D. Leviatan (Raymond and Beverly Sackler School Math. Sci., Tel Aviv Univ., Israel),
O. V. Motorna, I. A. Shevchuk (Taras Shevchenko Nat. Univ. Kyiv, Ukraine)

NO JACKSON-TYPE ESTIMATES FOR PIECEWISE q-MONOTONE $q \geq 3$, TRIGONOMETRIC APPROXIMATION*

НЕМОЖЛИВІ ОЦІНКИ ТИПУ ДЖЕКСОНА ДЛЯ КУСКОВО q-МОНОТОННОЇ, $q \geq 3$, ТРИГОНОМЕТРИЧНОЇ АПРОКСИМАЦІЇ

We say that a function $f \in C[a,b]$ is q-monotone, $q \geq 2$, if $f \in C^{q-2}(a,b)$, the space of functions possessing a $(q-2)$nd continuous derivative in (a,b), and $f^{(q-2)}$ is convex there. Let f be continuous and 2π-periodic, and change its q-monotonicity finitely many times in $[-\pi,\pi]$. We are interested in estimating the degree of approximation of f by trigonometric polynomials which are co-q-monotone with it, namely, trigonometric polynomials that change their q-monotonicity exactly at the points where f does. Such Jackson-type estimates are valid for piecewise monotone ($q=1$) and piecewise convex ($q=2$) approximations. However, we prove, that no such estimates are valid, in general, for co-q-monotone approximation, when $q \geq 3$.

Кажуть, що функція $f \in C[a,b]$ є q-монотонною, $q \geq 2$, якщо вона має $(q-2)$-ту неперервну похідну в (a,b) і $f^{(q-2)}$ там опукла. Нехай f — неперервна 2π-періодична функція, яка змінює свою q-монотонність скінченне число разів на $[-\pi,\pi]$. Нас цікавлять оцінки порядку наближення функції f тригонометричними поліномами, які змінюють свою q-монотонність саме в тих точках, де і f. Такі оцінки типу Джексона справедливі для кусково-монотонного $(q=1)$ та кусково-опуклого $(q=2)$ наближень. Однак ми доводимо, що жодна з таких оцінок не є можливою, взагалі кажучи, у ко-q-монотонній апроксимації, якщо $q \geq 3$.

1. Introduction and the main results. A function $f \in C[a,b]$ is called q-monotone, $q \geq 2$, $q \in \mathbb{N}$, if $f \in C^{q-2}(a,b)$, the space of functions possessing a $(q-2)$nd continuous derivative in (a,b), and $f^{(q-2)}$ is convex there. For the sake of uniformity, for $q=1$, we say that $f \in C[a,b]$ is 1-monotone, if it is nondecreasing in $[a,b]$.

Let $s \in \mathbb{N}$ and $\mathbb{Y}_s := \{Y_s\}$ where $Y_s = \{y_i\}_{i=1}^{2s}$ such that $y_{2s} < \ldots < y_1 < y_{2s} + 2\pi =: y_0$. We say that a 2π-periodic function $f \in C(\mathbb{R})$ is piecewise q-monotone with respect to Y_s, if it changes its q-monotonicity at the points Y_s, that is, if $(-1)^{i-1} f$ is q-monotone on $[y_i, y_{i-1}]$, $1 \leq i \leq 2s$. We denote by $\Delta^{(q)}(Y_s)$ the collection of all such piecewise q-monotone functions. Note that if, in addition, $f \in C^q(\mathbb{R})$, then $f \in \Delta^{(q)}(Y_s)$ if and only if

$$f^{(q)}(t) \prod_{i=1}^{2s} (t - y_i) \geq 0, \quad t \in [y_{2s}, y_0].$$

***Remark* 1.1.** We do not consider the case where Y consists of an odd number of points, since the only trigonometric polynomials in $\Delta^{(q)}(Y)$ are constants.

We also need the notation W^r, $r \in \mathbb{N}$, for the Sobolev class of 2π-periodic functions $f \in AC^{(r-1)}(\mathbb{R})$, such that

$$\left\| f^{(r)} \right\| \leq 2.$$

* Supported by the National Research Foundation of Ukraine (Project #2020.02/0155).

© D. LEVIATAN, O. V. MOTORNA, I. A. SHEVCHUK, 2022

For a 2π-periodic function g, denote

$$\|g\| := \operatorname*{ess\,sup}_{x\in\mathbb{R}} |g(x)|.$$

If, in addition, g is continuous, then, of course,

$$\|g\| = \max_{x\in\mathbb{R}} |g(x)|.$$

Similarly, for a function g, defined on the interval $[a,b]$, we denote $\|g\|_{[a,b]} := \operatorname{ess\,sup}_{x\in[a,b]}|g(x)|$, and if $g \in C[a,b]$, then $\|g\|_{[a,b]} = \max_{x\in[a,b]} |g(x)|$.

Let $\mathcal{T}_n$ be the space of trigonometric polynomials

$$T_n(t) = \alpha_0 + \sum_{k=1}^{n} (\alpha_k \cos kt + \beta_k \sin kt), \quad \alpha_k, \beta_k \in \mathbb{R},$$

of degree $\le n$ (of order $2n+1$) and, for 2π-periodic function $g \in C(\mathbb{R})$, let

$$E_n(g) := \inf_{T_n\in\mathcal{T}_n} \|g - T_n\|$$

denote the error of the best approximation of the function g. If $g \in \Delta^{(q)}(Y_s)$, then we would like to approximate it by trigonometric polynomials that change their q-monotonicity together with g, namely, are in $\Delta^{(q)}(Y_s)$. We call it co-q-monotone approximation. Denote by

$$E_n^{(q)}(g, Y_s) := \inf_{T_n\in\mathcal{T}_n\cap\Delta^{(q)}(Y_s)} \|g - T_n\|$$

the error of the best co-q-monotone approximation of the function g.

It is well-known that for $q = 1$ and $q = 2$, if $f \in \Delta^{(q)}(Y_s) \cap W^r$, $r \ge 1$, then

$$E_n^{(q)}(f, Y_s) = O\left(1/n^r\right), \quad n \to \infty \tag{1.1}$$

(see [2, 4–6, 9] for details and references).

It turns out, and proving this is the main purpose of this article, that for $q \ge 3$, (1.1) is, in general, invalid for any $r, s \in \mathbb{N}$ and every $Y_s \in \mathbb{Y}_s$.

Main result of this paper is the following theorem.

Theorem 1.1. *For each* $q \ge 3$, $r \in \mathbb{N}$, $s \in \mathbb{N}$ *and any* $Y_s \in \mathbb{Y}_s$, *there exists a function* $f \in \Delta^{(q)}(Y_s) \cap W^r$ *such that*

$$\limsup_{n\to\infty} n^r E_n^{(q)}(f, Y_s) = \infty.$$

We will also prove the following less general but more precise statements. Combining all of them, in particular yields Theorem 1.1.

Theorem 1.2. *For each $q \ge 3,\ s \in \mathbb{N}$ and any $Y_s \in \mathbb{Y}_s$, there exists a function*

$$f \in \Delta^{(q)}(Y_s) \cap W^{q-2}$$

such that

$$E_n^{(q)}(f, Y_s) \ge C(q, Y_s), \quad n \in \mathbb{N}, \tag{1.2}$$

where $C(q, Y_s) > 0$ depends only on q and Y_s.

***Corollary* 1.1.** *For each $q \ge 3,\ r \le q-2,\ s \in \mathbb{N}$ and any $Y_s \in \mathbb{Y}_s$, there exists a function $f \in \Delta^{(q)}(Y_s) \cap W^r$ such that*

$$E_n^{(q)}(f, Y_s) \ge C(q, Y_s), \quad n \in \mathbb{N},$$

where $C(q, Y_s) > 0$ depends only on q and Y_s.

Theorem 1.3. *For each $q \ge 3,\ s \in \mathbb{N}$ and any $Y_s \in \mathbb{Y}_s$, there exists a function*

$$f \in \Delta^{(q)}(Y_s) \cap W^{q-1}$$

such that

$$nE_n^{(q)}(f, Y_s) \ge C(q, Y_s), \quad n \in \mathbb{N}, \tag{1.3}$$

where $C(q, Y_s) > 0$ depends only on q and Y_s.

Final result is the following theorem.

Theorem 1.4. *Let $q \ge 3,\ p \ge q,\ s \in \mathbb{N}$ and $Y_s \in \mathbb{Y}_s$. For each sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive numbers, tending to infinity, there is a function $f \in \Delta^{(q)}(Y_s) \cap W^p$ such that*

$$\limsup_{n\to\infty} \varepsilon_n n^{p-q+2} E_n^{(q)}(f, Y_s) = \infty.$$

We prove Theorem 1.2 in Section 2, Theorem 1.3 in Section 4 and Theorem 1.4 in Section 6. In the proofs we apply ideas from [3], and we have to overcome the constraints and challenges of periodicity.

In the sequel, positive constants c and c_i either are absolute or may depend only on $r,\ q,\ p$ and m.

2. Eulerian type ideal splines and proof of Theorem 1.2.

Definition 2.1. *For each $b \in (0, \pi]$ and $r \in \mathbb{N}$ denote by $\varepsilon_{r,b}$ the 2π-periodic function such that*

1) $\varepsilon_{r,b} \in C^{r-1}$,

2) $\displaystyle\int_{-\pi}^{\pi} \varepsilon_{r,b}(x)dx = 0$,

and

3) $\varepsilon_{r,b}^{(r)} = \operatorname{sgn} x - \gamma_b,\ x \in (-b, 2\pi - b) \setminus \{0\}$, *where*

$$\gamma_b = 1 - b/\pi, \tag{2.1}$$

so that

$$\int_{-\pi}^{\pi} \varepsilon_{r,b}^{(r)}(x)dx = 0.$$

***Remark* 2.1.** By its definition, $\varepsilon_{r,b}$ is a spline of minimal defect of degree r, in particular, $\varepsilon_{r,\pi}$ is called an Eulerian ideal spline.

Put

$$F_r(x) := \frac{1}{r!}\,|x|x^{r-1}.$$

The following properties of $\varepsilon_{r,b}$ readily follow from its definition

$$\varepsilon_{r,b}(x) = F_r(x) + p_{r,b}(x), \quad x \in [-b, 2\pi - b], \tag{2.2}$$

where $p_{r,b}$ is an algebraic polynomial of degree $\le r$;

$$1 \le \left\|\varepsilon_{r,b}^{(r)}\right\| < 2, \quad \text{whence} \quad \varepsilon_{r,b} \in W^r, \tag{2.3}$$

and, for each collection Y_s such that $\{-b, 0\} \in Y_s$ and every $q > r$, we have

$$\varepsilon_{r,b} \in \Delta^{(q)}(Y_s). \tag{2.4}$$

We need the following lemma (see [3, Lemma 2.4]).

Lemma 2.1. *For each $q \ge 3$ and any function $g \in C^{q-2}[-1, 1]$ such that $g^{(q-2)}$ is convex on $[0, 1]$ and concave on $[-1, 0]$, we have*

$$\|F_{q-2} - g\|_{[-1,1]} \ge c.$$

***Proof of Theorem* 1.2.** Given $Y_s \in \mathbb{Y}_s$, let

$$b := \min_{1\le j\le 2s}\{y_{j-1} - y_j\},$$

and by shifting the periodic function f, we may assume, without loss of generality, that $y_{2s} = -b$ and $y_{2s-1} = 0$. Obviously, it follows that $y_{2s-2} \ge b$.

We will show that $f := \varepsilon_{q-2,b}$ is the desired function. Indeed, by (2.3) and (2.4), $\varepsilon_{q-2,b} \in \Delta^{(q)}(Y_s) \cap W^{q-2}$. So we have to prove (1.2).

To this end we take an arbitrary polynomial $T_n \in \mathcal{T}_n \cap \Delta^{(q)}(Y_s)$. Then the function $g_n := T_n - p_{q-2,b}$ satisfies $xg_n^{(q)}(x) \ge 0$ for $x \in [-b, b]$, whence $xg_n^{(q)}(x/b) \ge 0$ for $x \in [-1, 1]$. Let $\tilde{F}_{q-2}(x) := F_{q-2}(x/b)$ and $\tilde{g}_n(x) := g_n(x/b)$. By Lemma 2.1, we obtain

$$\|f - T_n\|_{[-\pi,\pi]} = \|F_{q-2} - g_n\|_{[-\pi,\pi]} \ge \|F_{q-2} - g_n\|_{[-b,b]} =$$

$$= \left\|\tilde{F}_{q-2} - \tilde{g}_n\right\|_{[-1,1]} = b^{2-q}\left\|F_{q-2} - b^{q-2}\tilde{g}_n\right\|_{[-1,1]} \ge b^{2-q}\,c,$$

which yields (1.2).

Theorem 1.2 is proved.

3. Approximation of $|x|$. Recall that

$$F_1(x) \equiv |x|.$$

In this section we prove, for trigonometric polynomials, an analog of Bernstein's estimate

$$\|F_1 - P_n\|_{[-b,b]} \ge c\,\frac{b}{n},$$

which is valid for every algebraic polynomial P_n of degree $\le n$ (for the exact constant c, see [8]).

To this end, we first extend to an arbitrary interval $[-b, b]$ the Bernstein – de la Vallée-Poussin inequality

$$\left\|T_n'\right\| \le n\|T_n\|, \tag{3.1}$$

which is valid for every $T_n \in \mathcal{T}_n$.

We begin with the following simple lemma.

Lemma 3.1. *If $f \in C[-a, a]$ is an even function and $g \in C[-a, a]$ is an odd function, then*

$$\|f\|_{[-a,a]} \le \|f + g\|_{[-a,a]} \quad \textit{and} \quad \|g\|_{[-a,a]} \le \|f + g\|_{[-a,a]}.$$

Proof. Let $M := \|f + g\|_{[-a,a]}$ and assume to the contrary, that there is a point $x \in [-a, a]$ such that $|f(x)| = K > M$. Then either $|f(x) + g(x)| \ge K$, or $|f(-x) + g(-x)| = |f(x) - g(x)| \ge K$, a contradiction. The proof for g is similar.

Lemma 3.1 is proved.

The following result is a special case of I. I. Privalov's theorem (see, e.g., [7, p. 96, 97]). However, we give another proof that provides sharp estimates.

Lemma 3.2. *For each $b \in (0, \pi]$ and every trigonometric polynomial $T_n \in \mathcal{T}_n$, there holds the inequality*

$$\left\|T_n'\right\|_{[-b/2,b/2]} \le \frac{n}{\sin\dfrac{b}{2}}\|T_n\|_{[-b,b]} \le \frac{\pi n}{b}\|T_n\|_{[-b,b]}. \tag{3.2}$$

Proof. Let $\tilde b \in (0, \pi/2]$. First we prove the inequality

$$\left|T_n'(0)\right| \le \frac{n}{\sin\tilde b}\,\|T_n\|_{[-\tilde b,\tilde b]}. \tag{3.3}$$

First we show, that (3.3) holds for any odd polynomial $T_n \in \mathcal{T}_n$. Indeed, denote by P_n the algebraic polynomial such that $P_n(\sin t) = T_n(t)$. Then, by Bernstein inequality for the algebraic polynomials,

$$\left|T_n'(0)\right| = \left|P_n'(0)\right| \le \frac{n}{\sin\tilde b}\,\|P_n\|_{\left[-\sin\tilde b,\sin\tilde b\right]} = \frac{n}{\sin\tilde b}\,\|T_n\|_{\left[-\tilde b,\tilde b\right]}.$$

Thus, (3.3) is proved for odd polynomials T_n. In order to prove (3.3) for an arbitrary polynomial $T_n \in \mathcal{T}_n$, we represent T_n, in the form $T_n := U_n + V_n$, where $U_n \in \mathcal{T}_n$ is an even polynomial, and $V_n \in \mathcal{T}_n$ is an odd polynomial. Then $T_n'(0) = V_n'(0)$ and by Lemma 3.1 $\|V_n\|_{\left[-\tilde b,\tilde b\right]} \le \|T_n\|_{\left[-\tilde b,\tilde b\right]}$. Hence (3.3) is valid for any $T_n \in \mathcal{T}_n$.

Now for the polynomial $T_n(x + t) \in \mathcal{T}_n$, $x \in \mathbb{R}$, it follows by (3.3) that

$$\left|T_n'(x)\right| \le \frac{n}{\sin \tilde{b}} \left\|T_n\right\|_{\left[x-\tilde{b}, x+\tilde{b}\right]}.$$

Hence, for $x \in [-b/2, b/2]$, we get

$$|T_n'(x)| \le \frac{n}{\sin \dfrac{b}{2}} \|T_n\|_{[x-b/2, x+b/2]} \le \frac{n}{\sin \dfrac{b}{2}} \|T_n\|_{[-b,b]},$$

which is (3.2).

Lemma 3.2 is proved.

We are ready to prove Lemma 3.3. We follow the arguments in [1, p. 434, 435].

Lemma 3.3. *For each* $b \in (0, \pi]$ *and polynomial* $T_n \in \mathcal{T}_n$, *we have*

$$\|F_1 - T_n\|_{[-b,b]} \ge \frac{c_1 b}{n}, \tag{3.4}$$

where $c_1 \ge (32\pi)^{-1} \approx 0.01$.

Proof. Let

$$c^* := \frac{1}{16\pi},$$

and assume to the contrary, that there is a polynomial $\tilde{T}_n \in \mathcal{T}_n$ such that

$$\left\|F_1 - \tilde{T}_n\right\|_{[-b,b]} < \frac{c^* b}{2n}. \tag{3.5}$$

Then there is an even polynomial $\hat{T}_n \in \mathcal{T}_n$ such that

$$\left\|F_1 - \hat{T}_n\right\|_{[-b,b]} \le \frac{c^* b}{n}$$

and

$$\hat{T}_n(0) = 0.$$

Hence $\hat{T}_n$ may be represented in the form

$$\hat{T}_n(t) = a_1(1 - \cos t) + \ldots + a_n(1 - \cos nt) = 2\sum_{k=1}^{n} a_k \sin^2\left(\frac{kt}{2}\right).$$

Thus, for $T_n(t) := \hat{T}_n(2t)$, we have

$$\|2F_1 - T_n\|_{[-b/2,b/2]} \le \frac{c^* b}{n}. \tag{3.6}$$

Denote

$$\tau_n(t) := \frac{T_n(t)}{\sin t} \quad \left(\tau_n(0) = T_n'(0)\right).$$

Then τ_n is an odd trigonometric polynomial of degree $< 2n$.

First we prove that

$$\|\tau_n\|_{[-b/2,b/2]} < 4. \tag{3.7}$$

Indeed, by virtue of (3.6), one has, for $b/8 \le |t| \le b/2$,

$$|T_n(t)| \le 2|t| + \frac{c^*b}{n} \le 2|t| + \frac{8c^*|t|}{n} < \left(2 + \frac{1}{2\pi n}\right)|t| < 2.2|t|.$$

Hence, if $b/8 \le |t| \le b/2$, then

$$|\tau_n(t)| < \frac{2.2|t|}{\sin t} \le \frac{1.1b}{\sin b/2} < \frac{1.1\pi}{\sin \pi/2} = 1.1\pi < 4.$$

Thus, assuming the contrary, that there is a point $t_0 \in [-b/2, b/2]$ such that

$$\|\tau_n\|_{[-b/2,b/2]} = |\tau_n(t_0)| = M \ge 4,$$

we conclude, that $t_0 \in [-b/8, b/8]$. Since Lemma 3.2 implies that

$$b\left\|\tau_n'\right\|_{[-b/4,b/4]} \le \frac{b}{\sin b/4}(2n-1)M < 2nM\frac{b}{\sin b/4} \le$$

$$\le 2nM\frac{\pi}{\sin \pi/4} = 2\sqrt{2}\pi n M,$$

we get, for $t \in I_n := \left[t_0 - \dfrac{c^*b}{n}, t_0 + \dfrac{c^*b}{n}\right] \subset \left(-\dfrac{b}{4}, \dfrac{b}{4}\right)$,

$$|\tau_n(t)| \ge |\tau_n(t_0)| - |\tau_n(t) - \tau_n(t_0)| \ge$$

$$\ge |\tau_n(t_0)| - |t - t_0|\left\|\tau_n'\right\|_{I_n} \ge M - |t - t_0|\|\tau_n'\|_{[-b/4,b/4]} \ge$$

$$\ge M - c^*2\sqrt{2}\pi M = \left(1 - \sqrt{2}/8\right)M > 0.8M.$$

Hence, for $t \in I_n$, we have

$$\frac{|T_n(t)|}{|t|} \ge 0.8\frac{|\sin t|}{|t|}M \ge 0.8\frac{\sin\frac{\pi}{6}}{\frac{\pi}{6}}M = \frac{2.4}{\pi}M > \frac{3}{4}M,$$

which, in turn, implies

$$\|T_n - 2F_1\|_{I_n} \ge \left(\frac{3M}{4} - 2\right)\|F_1\|_{I_n} \ge \|F_1\|_{I_n} \ge \frac{c^*b}{n},$$

contradicting (3.6). Therefore, (3.7) is proved.

By virtue of Lemma 3.2 and (3.7),

$$\left\|\tau_n'\right\|_{[-b/4,b/4]} \le \frac{2\pi}{b}(2n-1)\|\tau_n\|_{[-b/2,b/2]} < \frac{16\pi}{b}n = \frac{n}{c^*b}.$$

Therefore, for $t \in (0, b/4]$,

$$|\tau_n(t)| = \left|\int_0^t \tau_n'(u)\,du\right| < \frac{tn}{c^*b},$$

whence

$$|T_n(t)| < \frac{tn}{c^*b}\sin t < \frac{t^2 n}{c^*b}.$$

Hence, for $t = \dfrac{c^*b}{n}$, we get

$$2t - T_n(t) > t\left(2 - \frac{tn}{c^*b}\right) = t = \frac{c^*b}{n},$$

contradicting (3.6) and, in turn, (3.5).

Lemma 3.3 is proved.

The following lemma is a consequence of Lemma 3.1.

Lemma 3.4. *For each $b \in (0,\pi]$, any linear function l and every trigonometric polynomial $T_n \in \mathcal{T}_n$, we have*

$$\|F_1 + l - T_n\|_{[-b,b]} \ge \frac{c_1 b}{n}. \tag{3.8}$$

Proof. We represent T_n in the form $T_n = T_e + T_o$, where T_e is an even polynomial, and T_o is an odd polynomial. Let $l(x) = ax + k =: l_o(x) + l_e$. Denote $\tilde{T}_e := T_e - l_e \in \mathcal{T}_n$, the even polynomial. By (3.4), $\|F_1 - \tilde{T}_e\| \ge c_1 b/n$. Since $l_o - T_o$ is an odd function, it follows by Lemma 3.1 that (3.8) is valid.

Lemma 3.4 is proved.

4. Proof of Theorem 1.3. The following result readily follows from [3, Lemma 3.1].

Lemma 4.1. *Given $q \ge 3$. If a function $f \in C^{q-2}[-2b,2b]$ has a convex $(q-2)$nd derivative $f^{(q-2)}$ on $[0,2b]$ and a concave $(q-2)$nd derivative $f^{(q-2)}$ on $[-2b,0]$, then*

$$b^{q-2}\left\|f^{(q-2)}\right\|_{[-b,b]} \le c_2\|f\|_{[-2b,2b]}. \tag{4.1}$$

Indeed, let $\|f^{(q-2)}\|_{[-b,b]} \ne 0$ and $x^* \in [-b,b]$ be such that $\left|f^{(q-2)}(x^*)\right| = \left\|f^{(q-2)}\right\|_{[-b,b]}$. If either $x^* = 0$ and $f^{(q-2)}(0) < 0$, or $x^* > 0$, then [3, (3.1)] yields,

$$b^{q-2}\left\|f^{(q-2)}\right\|_{[-b,b]} = b^{q-2}\left\|f^{(q-2)}\right\|_{[0,b]} \le c_2\|f\|_{[0,2b]} \le c_2\|f\|_{[-2b,2b]}.$$

Otherwise (4.1) follows from [3, (3.2)].

Recall that $F_r(x) = |x|x^{r-1}/r!$. We have the following lemma.

Lemma 4.2. *For every $b \in (0,\pi]$, every trigonometric polynomial $T_n \in \mathcal{T}_n$, satisfying*

$$tT_n^{(r+1)}(t) \ge 0 \quad \textit{for} \quad |t| \le b,$$

and any algebraic polynomial P_r of degree $\le r$, we have

$$n\|F_r + P_r - T_n\|_{[-b,b]} \ge c_3 b^r, \quad n \in \mathbb{N}. \tag{4.2}$$

Proof. Since $F_r^{(r-1)} = F_1$ and $P_r^{(r-1)}$ is linear, it follows by Lemma 3.4 that

$$\left\|T_n^{(r-1)} - F_r^{(r-1)} - P_r^{(r-1)}\right\|_{[-b/2,b/2]} \ge \frac{c_1 b}{2n}.$$

Now, $T_n^{(r-1)} - F_r^{(r-1)} - P_r^{(r-1)}$ is convex in $[0, b]$ and concave in $[-b, 0]$, so, by virtue of Lemma 4.1,

$$\|T_n - F_r - P_r\|_{[-b,b]} \geq \frac{1}{c_2}\left(\frac{b}{2}\right)^{r-1}\left\|T_n^{(r-1)} - F_r^{(r-1)} - P_r^{(r-1)}\right\|_{[-b/2,b/2]} \geq \frac{c_1}{c_2 n}\left(\frac{b}{2}\right)^r.$$

Hence, (4.2) follows with $c_3 \geq 2^{-r}c_1/c_2$.

Lemma 4.2 is proved.

***Proof of Theorem* 1.3.** Given $Y_s \in \mathbb{Y}_s$, again, let

$$b := \min_{j\in\mathbb{Z}}\{y_{i+1} - y_i\},$$

and by shifting the periodic function f, we may assume, without loss of generality, that $y_{2s} = -b$ and $y_{2s-1} = 0$. Then $f := \varepsilon_{q-1,b}$ is the desired function. Indeed, by (2.3) and (2.4), $\varepsilon_{q-1,b} \in \Delta^{(q)}(Y_s) \cap W^{q-1}$. So, we have to prove (1.3).

To this end, take an arbitrary polynomial $T_n \in \mathcal{T}_n \cap \Delta^{(q)}(Y_s)$. By (2.2),

$$\varepsilon_{q-1,b}(x) = F_{q-1}(x) + p_{q-1,b}(x), \quad x \in [-b, 2\pi - b],$$

where $p_{q-1,b}$ is an algebraic polynomial of degree $\leq q-1$. Therefore, Lemma 4.2 implies (1.3) with $C(q, Y_s) \geq c_3 b^{q-1}$.

Theorem 1.3 is proved.

5. Auxiliary results. Let $S \in C^\infty(\mathbb{R})$, be a monotone odd function such that $S(x) = \operatorname{sgn} x$, $|x| \geq 1$.

Put

$$s_j := \left\|S^{(j)}\right\|, \quad j \in \mathbb{N}_0.$$

Fix $d \in (0, \pi]$, and for $\lambda \in (0, d/3]$, let

$$\tilde{S}_{\lambda,d}(x) := \begin{cases} S\left(\dfrac{x-2\lambda}{\lambda}\right), & \text{if} \quad x \in [0, 2\pi - d], \\ -S\left(\dfrac{x-2\lambda+d}{\lambda}\right), & \text{if} \quad x \in [-d, 0]. \end{cases}$$

Finally, denote

$$S_{\lambda,d}(x) := \tilde{S}_{\lambda,d}(x) - \gamma_d, \quad x \in [-d, 2\pi - d],$$

where γ_d was defined in (2.1), extended periodically to $\mathbb{R}$.

Note that

$$\left\|S_{\lambda,d}^{(j)}\right\| = \lambda^{-j}s_j, \quad j \in \mathbb{N}, \tag{5.1}$$

and

$$\int_{-\pi}^{\pi} S_{\lambda,d}(x)dx = 0.$$

Definition 5.1. *For each* $\lambda \in (0, d/3]$ *and* $r \in \mathbb{N}$ *denote by* $\varepsilon_{r,d,\lambda}$ *the* 2π*-periodic function* $\varepsilon_{r,d,\lambda} \in C^\infty(\mathbb{R})$ *such that*

1) $\int_{-\pi}^{\pi} \varepsilon_{r,d,\lambda}(x)dx = 0$

and

2) $\varepsilon^{(r)}_{r,d,\lambda} = S_{\lambda,d}(x),\ x \in [-d, 2\pi - d].$

Note that, for each $j \in \mathbb{N}$, we have

$$[-d, 2\pi - d] \cap \operatorname{supp} \varepsilon^{(r+j)}_{r,d,\lambda} = [-d + \lambda, -d + 3\lambda] \cup [\lambda, 3\lambda], \tag{5.2}$$

and that (5.1) implies

$$\left\| \varepsilon^{(r+j)}_{r,d,\lambda} \right\| = \lambda^{-j} s_j, \quad j \in \mathbb{N}. \tag{5.3}$$

Also,

$$\left\| \varepsilon^{(j)}_{r,d,\lambda} \right\| < c_4, \quad j = 0, \ldots, r, \quad \text{in particular,} \quad \left\| \varepsilon^{(r)}_{r,d,\lambda} \right\| < 2. \tag{5.4}$$

Lemma 5.1. *We have*

$$\|\varepsilon_{r,d,\lambda} - \varepsilon_{r,d}\| \le c_5 \lambda. \tag{5.5}$$

Proof. Put $\varepsilon_j := \varepsilon_{j,d} - \varepsilon_{j,d,\lambda},\ j = 1, \ldots, r$. Since $\int_{-\pi}^{\pi} \varepsilon_j(x)dx = 0$, it follows that for any $1 \le j \le r$ there is an $x_j \in [-\pi, \pi]$ such that $\varepsilon_j(x_j) = 0$. Hence, we first conclude that

$$\|\varepsilon_1\| \le \int\limits_{-d}^{2\pi - d} \left| \operatorname{sgn} x - \tilde{S}_{\lambda,d}(x) \right| dx = 8\lambda.$$

Assume by induction that $\|\varepsilon_j\| \le c\lambda$ for some $j < r$, and note that $\varepsilon'_{j+1} = \varepsilon_j$. Thus, for $x \in$ $\in [x_{j+1} - \pi, x_{j+1} + \pi]$,

$$|\varepsilon_{j+1}(x)| = |\varepsilon_{j+1}(x) - \varepsilon_{j+1}(x_{j+1})| = \left| \int\limits_{x_{j+1}}^{x} \varepsilon_j(t)\, dt \right| \le \pi c \lambda.$$

Lemma 5.1 is proved.

Lemma 5.2. *Let* $0 < b \le d$ *and* $r \in \mathbb{N}$ *be given. Let* $n \in \mathbb{N}$ *and* $T_n \in \mathcal{T}_n$ *be such that* $tT_n^{(r+1)}(t) \ge 0$ *for* $|t| \le b$. *For any algebraic polynomial* P_r *of degree* $\le r$, *if*

$$0 < \lambda \le \min\left\{ \frac{c_3 b^r}{2nc_5}, \frac{d}{3} \right\} =: \min\left\{ c_6 \frac{b^r}{n}, \frac{d}{3} \right\},$$

then

$$2n\|\varepsilon_{r,d,\lambda} + P_r - T_n\|_{[-b,b]} \ge c_3 b^r. \tag{5.6}$$

Proof. Inequalities (4.2) and (5.5) imply

$$2n\|\varepsilon_{r,d,\lambda} + P_r - T_n\|_{[-b,b]} \ge 2n\|\varepsilon_{r,d} + P_r - T_n\|_{[-b,b]} - 2n\|\varepsilon_{r,d,\lambda} - \varepsilon_{r,d}\| \ge$$

$$\ge 2c_3 b^r - 2nc_5\lambda \ge c_3 b^r.$$

Fix $r \ge 2$ and $m \in \mathbb{N}$, and let $q := r+1$ and

$$c_7 := c_6^m s_m^{-1}.$$

For $0 < b \le d$ and each $n \ge 3c_6 b^r$, denote

$$\lambda_{n,b} := c_6 \frac{b^r}{n}$$

and

$$f_{n,b} := c_7 \frac{b^{rm}}{n^m} \varepsilon_{r,d,\lambda_{n,b}}.$$

Then we have the following lemma.

Lemma 5.3. *We get*

$$\left\|f_{n,b}^{(r+m)}\right\| \le 1, \tag{5.7}$$

$$\left\|f_{n,b}^{(r+j)}\right\| \le c_8 n^{j-m}, \quad j = 0,\ldots,m, \tag{5.8}$$

and

$$\left\|f_{n,b}^{(j)}\right\| \le \frac{c_9}{n^m}, \quad j = 0,\ldots,r. \tag{5.9}$$

For each collection Y_s such that $y_{2s} = -d$, $y_{2s-1} = 0$ and $d = \min_{1\le j\le 2s}\{y_{j-1} - y_j\}$, we have

$$f_{n,b} \in \Delta^{(q)}(Y_s), \tag{5.10}$$

and, for every polynomial $T_n \in \mathcal{T}_n$, satisfying $tT_n^{(q)}(t) \ge 0$ for $|t| \le b$ and any algebraic polynomial P_r of degree $\le r$, we obtain

$$n^{m+1}\|f_{n,b} + P_r - T_n\|_{[-b,b]} \ge c_{10} b^{r(m+1)}. \tag{5.11}$$

Proof. First, (5.9) and (5.10) are clear from the definition of $\varepsilon_{r,d,\lambda_{n,b}}$ and (5.4), respectively.

We prove (5.7) and (5.8) together. By virtue of (5.3), we have, for $j = 0,\ldots,m$,

$$\left\|f_{n,b}^{(r+j)}\right\| = c_7 \frac{b^{rm}}{n^m}\left(c_6 \frac{b^r}{n}\right)^{-j} s_j = c_6^m s_m^{-1} \frac{b^{rm}}{n^m}\left(c_6 \frac{b^r}{n}\right)^{-j} s_j = c_6^{m-j} n^{j-m} b^{r(m-j)} \frac{s_j}{s_m},$$

that is, (5.7) and (5.8).

Finally, we prove (5.11). Let $\tilde{P}_r := \left(c_7 \dfrac{b^{rm}}{n^m}\right)^{-1} P_r$, $\tilde{T}_r := \left(c_7 \dfrac{b^{rm}}{n^m}\right)^{-1} T_r$, apply Lemma 5.2 and get

$$n^{m+1}\|f_{n,b} + P_r - T_n\|_{[-b,b]} = n^{m+1} c_7 \frac{b^{rm}}{n^m}\left\|\varepsilon_{r,b,\lambda} + \tilde{P}_r - \tilde{T}_n\right\|_{[-b,b]} \ge$$

$$\ge n^{m+1} c_7 \frac{b^{rm}}{n^m}\frac{c_3 b^r}{2n} =: c_{10} b^{r(m+1)}.$$

Lemma 5.3 is proved.

6. Proof of Theorem 1.4. Set $r := q-1$ and $m := p-r$. Given $Y_s \in \mathbb{Y}_s$, let

$$d := \min_{1\le j\le 2s}\{y_{j-1}-y_j\},$$

and by shifting the periodic function f, we may assume, without loss of generality, that $y_{2s} = -d$ and $y_{2s-1} = 0$. Obviously, it follows that $y_{2s-2} \ge d$.

We will prove, that the desired function f may be taken in the form

$$f(x) := \sum_{k=1}^{\infty} f_{n_{k+1},b_k},$$

where integers n_k and numbers b_k are chosen as follows. We put $n_1 := \lceil 3c_6 d^r\rceil$ and $b_1 := d/4$. Then let n_2 be such that $b_2 := \lambda_{n_2,b_1} < b_1/3$. Assume that n_k and b_k have been chosen. Then we take $n_{k+1} \ge 2n_k$, to be such that

$$3\lambda_{n_{k+1},b_k} < b_k, \tag{6.1}$$

$$\varepsilon_{n_{k+1}} c_{10} b_k^{r(m+1)} \ge k, \tag{6.2}$$

and

$$\frac{c_9}{n_{k+1}^m} \le \frac{c_{10} b_{k-1}^{r(m+1)}}{10 n_k^{m+1}}. \tag{6.3}$$

Denote

$$b_{k+1} := \lambda_{n_{k+1},b_k}. \tag{6.4}$$

It follows by (5.2) and (6.4) that, for any $j \in \mathbb{N}$,

$$[-d, 2\pi - d] \cap \operatorname{supp} f^{(r+j)}_{n_{k+1},b_k} = [-d + b_{k+1}, -d + 3b_{k+1}] \cup [b_{k+1}, 3b_{k+1}]. \tag{6.5}$$

Hence by (6.1), for any $j \in \mathbb{N}$,

$$\operatorname{supp} f^{(r+j)}_{n_{k+1},b_k} \cap \operatorname{supp} f^{(r+j)}_{n_k,b_{k-1}} = \varnothing. \tag{6.6}$$

We divide the proof of Theorem 1.4 into two lemmas.

Lemma 6.1. *We have*

$$f \in W^p \cap \Delta^{(q)}(Y_s). \tag{6.7}$$

Proof. Inequalities (5.8) and (5.9) imply, for all $j = 0, \ldots, p-1$,

$$\left\| f^{(j)}_{n_{k+1},b_k} \right\| \le \frac{c}{n_{k+1}}, \quad k \in \mathbb{N}.$$

Hence, for each $j = 0, \ldots, p-1$,

$$\sum_{k=1}^{\infty} \left\| f^{(j)}_{n_{k+1},b_k} \right\| \le c \sum_{k=1}^{\infty} \frac{1}{n_{k+1}} \le \frac{c}{n_2} \sum_{k=1}^{\infty} \frac{1}{2^j} = c,$$

so that f is well defined on $\mathbb{R}$, it is periodic, $f \in C^{p-1}$, for each $j = 0, \ldots, p-1$,

$$f^{(j)}(x) \equiv \sum_{k=1}^{\infty} f^{(j)}_{n_{k+1},b_k}(x),$$

which, combined with (5.10), implies that $f \in \Delta^{(q)}(Y_s)$.

Then (6.6) means, that for each point $x \in (-d, 0) \cup (0, 2\pi - d)$ there is neighbourhood, where the sum in $f^{(r+j)}$ consists of at most one term not identically zero. Hence, $f \in C^{\infty}((-d, 0) \cup (0, 2\pi - d))$ and, in particular, $f \in C^{p}((-d, 0) \cup (0, 2\pi - d))$. Combining with (5.7), we have $\left\| f^{(p)} \right\| \le 1$.

Lemma 6.1 is proved.

Lemma 6.2. *For each $k > 2$, we have*

$$n_k^{m+1} \varepsilon_{n_k} E^{(q)}_{n_k}(f, Y_s) \ge k/2. \tag{6.8}$$

Proof. Fix $k > 1$. Then by (6.1) and (6.4), for every $1 \le j \le k-1$,

$$f^{(r+1)}_{n_{j+1},b_j}(x) = 0, \quad \text{if} \quad |x| \le b_k.$$

Hence,

$$P_r(x) := \sum_{j=1}^{k-1} f_{n_{j+1},b_j}(x), \quad |x| \le b_k, \tag{6.9}$$

is an algebraic polynomial of degree $\le r$.

Now, by (5.9) and (6.3),

$$\sum_{j=k+1}^{\infty} \|f_{n_{j+1},b_j}\| \le c_9 \sum_{j=k+1}^{\infty} \frac{1}{n^m_{j+1}} \le \frac{c_9}{n^m_{k+2}} \sum_{j=0}^{\infty} \frac{1}{2^{jm}} = \frac{2c_9}{n^m_{k+2}} \le \frac{c_{10} b_k^{r(m+1)}}{5 n^{m+1}_{k+1}}. \tag{6.10}$$

Finally, we take an arbitrary polynomial $T_{n_{k+1}} \in \mathcal{T}_{n_{k+1}} \cap \Delta^{(q)}(Y_s)$ and note, that $tT^{(q)}_{n_{k+1}} \ge 0$ for $|t| \le b_k \le d$. Therefore (6.9), (6.10) and (5.11), imply

$$\|f - T_{n_{k+1}}\| \ge \|f - T_{n_{k+1}}\|_{[-b_k,b_k]} = \left\| P_r + \sum_{j=k}^{\infty} f_{n_{j+1},b_j} - T_{n_{k+1}} \right\|_{[-b_k,b_k]} =$$

$$= \left\| \left(P_r + f_{n_{k+1},b_k} - T_{n_{k+1}}\right) + \sum_{j=k+1}^{\infty} f_{n_{j+1},b_j} \right\|_{[-b_k,b_k]} \ge$$

$$\ge \left\| P_r + f_{n_{k+1},b_k} - T_{n_{k+1}} \right\|_{[-b_k,b_k]} - \left\| \sum_{j=k+1}^{\infty} f_{n_{j+1},b_j} \right\| \ge$$

$$\ge \frac{c_{10} b_k^{r(m+1)}}{n^{m+1}_{k+1}} - \frac{c_{10} b_k^{r(m+1)}}{5 n^{m+1}_{k+1}} = \frac{4 c_{10} b_k^{r(m+1)}}{5 n^{m+1}_{k+1}}.$$

Combining with (6.2), we obtain (6.8).

Lemma 6.2 is proved.

References

1. V. K. Dzyadyk, I. A. Shevchuk, *Theory of uniform approximation of functions by polynomials*, Walter de Gruyer, Berlin, New York (2008).
2. G. A. Dzyubenko, *Comonotone approximation of twice differentiable periodic functions*, Ukr. Math. J., **61**, № 4, 519 – 540 (2009).
3. D. Leviatan, I. A. Shevchuk, *Jackson type estimates for piecewise q-monotone approximation, $q \geq 3$, are not valid*, Pure and Appl. Funct. Anal., **1**, 85 – 96 (2016).
4. G. G. Lorentz, K. L. Zeller, *Degree of approximation by monotone polynomials I*, J. Approx. Theory, **1**, 501 – 504 (1968).
5. M. G. Pleshakov, *Comonotone Jacksons inequality*, J. Approx. Theory, **99**, 409 – 421 (1999).
6. P. A. Popov, *An analog of the Jackson inequality for coconvex approximation of periodic functions*, Ukr. Math. J., **53**, № 7, 1093 – 1105 (2001).
7. A. A. Privalov, *Theory of interpolation of functions*, Book 1, Saratov Univ. Publ. House, Saratov (1990) (in Russian).
8. R. S. Varga, A. J. Carpenter, *On the Bernstein conjecture in approximation theory*, Constr. Approx., **1**, 333 – 348 (1985).
9. V. D. Zalizko, *Coconvex approximation of periodic functions*, Ukr. Math. J., **59**, № 1, 28 – 44 (2007).

Received 31.12.21