DOI: 10.37863/umzh.v74i5.7045

UDC 517.5

S. O. Chaichenko (Donbas State Pedagog. Univ., Sloviansk, Donetsk region),
A. L. Shidlich (Inst. Math. Nat. Acad. Sci. Ukraine; Nat. Univ. Life and Environmental Sci. Ukraine, Kyiv),
T. V. Shulyk (Donbas State Pedagog. Univ., Sloviansk, Donetsk region)

DIRECT AND INVERSE APPROXIMATION THEOREMS IN THE BESICOVITCH – MUSIELAK – ORLICZ SPACES OF ALMOST PERIODIC FUNCTIONS

ПРЯМІ ТА ОБЕРНЕНІ ТЕОРЕМИ НАБЛИЖЕННЯ У ПРОСТОРАХ БЕЗИКОВИЧА – МУСЄЛАКА – ОРЛИЧА МАЙЖЕ ПЕРІОДИЧНИХ ФУНКЦІЙ

In terms of the best approximations of functions and generalized moduli of smoothness, direct and inverse approximation theorems are proved for Besicovitch almost periodic functions whose Fourier exponent sequences have a single limit point at infinity and their Orlicz norms are finite. Special attention is paid to the study of cases where the constants in these theorems are unimprovable.

У термінах найкращих наближень функцій та узагальнених модулів гладкості доведено прямі та обернені апроксимаційні теореми для майже періодичних за Безиковичем функцій, послідовності коефіцієнтів Фур'є яких мають єдину граничну точку в нескінченності, а їхні норми Орлича є скінченними. Особливу увагу приділено вивченню випадків, коли сталі у цих теоремах непокращувані.

1. Introduction. The establishment of connections between the difference and differential properties of the function being approximated and the value of the error of its approximation by some methods was originated in the well-known works of Jackson (1911) and Bernstein (1912), in which the first direct and inverse approximation theorems were obtained. Subsequently, similar studies were carried out by many authors for various functional classes and for various approximating aggregates, and their results constitute the classics of modern approximation theory. Moreover, the exact results (in particular, in the sense of unimprovable constants) deserve special attention. A fairly complete description of the results on obtaining direct and inverse approximation theorems is contained in the monographs [14, 28, 30, 31].

In spaces of almost periodic functions, direct approximation theorems were established in the papers [8, 12, 23, 24, 26]. In particular, Prytula [23] obtained direct approximation theorem for Besicovitch almost periodic functions of the order 2 (B^2-a.p. functions) in terms of the best approximations of functions and their moduli of continuity. In [24] and [8], such theorems were obtained, respectively, with moduli of smoothness of B^2-a.p. functions of arbitrary positive integer order and with generalized moduli of smoothness. In [26], direct and inverse approximation theorems were obtained in the Besicovitch – Stepanets spaces $B\mathcal{S}^p$. The main goal of this article is to obtain such theorems in the Besicovitch – Musielak – Orlicz spaces $B\mathcal{S}_{\mathrm{M}}$. These spaces are natural generalizations of the all spaces mentioned above, and the results obtained can be viewed as an extension of these results to the spaces $B\mathcal{S}_{\mathrm{M}}$.

2. Preliminaries. ***2.1. Definition of the spaces $B\mathcal{S}_{\mathrm{M}}$.*** Let B^s, $1 \le s < \infty$, be the space of all functions Lebesgue summable with the sth degrees in each finite interval of the real axis, in

which the distance is defined by the equality

$$D_{B^s}(f,g)=\left(\varlimsup_{T\to\infty}\frac{1}{2T}\int\limits_{-T}^{T}|f(x)-g(x)|^s dx\right)^{1/s}.$$

Further, let $\mathcal{T}$ be the set of all trigonometric sums of the form $\tau_N(x)=\sum_{k=1}^{N}a_k e^{\mathrm{i}\lambda_k x}$, $N\in\mathbb{N}$, where λ_k and a_k are arbitrary real and complex numbers ($\lambda_k\in\mathbb{R}$, $a_k\in\mathbb{C}$).

An arbitrary function f is called a Besicovitch almost periodic function of order s (or B^s-a.p. function) and is denoted by $f\in B^s$-a.p. [20] (Ch. 5, § 10), [10] (Ch. 2, § 7), if there exists a sequence of trigonometric sums $\tau_1,\tau_2,\ldots$ from the set $\mathcal{T}$ such that

$$\lim_{N\to\infty}D_{B^s}(f,\tau_N)=0.$$

If $s_1\geq s_2\geq 1$, then (see, for example, [12, 13]) B^{s_1}-a.p.$\subset B^{s_2}$-a.p.$\subset B$-a.p., where B-a.p.$:=B^1$-a.p. For any B-a.p. function f, there exists the average value

$$A\{f\}:=\lim_{T\to\infty}\frac{1}{T}\int\limits_{0}^{T}f(x)dx.$$

The value of the function $A\{f(\cdot)e^{-\mathrm{i}\lambda\cdot}\}$, $\lambda\in\mathbb{R}$, can be nonzero at most on a countable set. As a result of numbering the values of this set in an arbitrary order, we obtain a set $\mathcal{S}(f)=\{\lambda_k\}_{k\in\mathbb{N}}$ of Fourier exponents, which is called the spectrum of the function f. The numbers $A_{\lambda_k}=A_{\lambda_k}(f)=$ $=A\{f(\cdot)e^{-\mathrm{i}\lambda_k\cdot}\}$ are called the Fourier coefficients of the function f. To each function $f\in B$-a.p. with spectrum $\mathcal{S}(f)$ there corresponds a Fourier series of the form $\sum_k A_{\lambda_k}e^{\mathrm{i}\lambda_k x}$. If, in addition, $f\in B^2$-a.p., then the Parseval equality holds (see, for example, [10], Ch. 2, § 9)

$$A\{|f|^2\}=\sum_{k\in\mathbb{N}}|A_{\lambda_k}|^2.$$

Further, we will consider only those B-a.p. functions from the spaces $B\mathcal{S}^p$, the sequences of Fourier exponents of which have a single limit point at infinity. For such functions f, the Fourier series are written in the symmetric form

$$S[f](x)=\sum_{k\in\mathbb{Z}}A_k e^{\mathrm{i}\lambda_k x},\quad\text{where}\quad A_k=A_k(f)=A\{f(\cdot)e^{-\mathrm{i}\lambda_k\cdot}\},\tag{2.1}$$

$\lambda_0:=0$, $\lambda_{-k}=-\lambda_k$, $|A_k|+|A_{-k}|>0$, $\lambda_{k+1}>\lambda_k>0$ for $k>0$.

Let $\mathbf{M}=\{M_k(t)\}_{k\in\mathbb{Z}}$, $t\geq 0$, be a sequence of Orlicz functions. In other words, for every $k\in\mathbb{Z}$, the function $M_k(t)$ is a nondecreasing convex function for which $M_k(0)=0$ and $M_k(t)\to\infty$ as $t\to\infty$. Let $\mathbf{M}^*=\{M_k^*(v)\}_{k\in\mathbb{Z}}$ be the sequence of functions defined by the relations

$$M_k^*(v):=\sup\{uv-M_k(u):\ u\geq 0\},\quad k\in\mathbb{Z}.$$

Consider the set $\Gamma=\Gamma(\mathbf{M}^*)$ of sequences of positive numbers $\gamma=\{\gamma_k\}_{k\in\mathbb{Z}}$ such that $\sum_{k\in\mathbb{Z}}M_k^*(\gamma_k)\leq 1$. The modular space (or Musielak – Orlicz space) $B\mathcal{S}_{\mathbf{M}}$ is the space of all

functions f ($f \in B$-a.p.) such that the following quantity (which is also called the Orlicz norm of f) is finite:

$$\|f\|_{\mathbf{M}} := \|\{A_k\}_{k\in\mathbb{Z}}\|_{l_{\mathbf{M}}(\mathbb{Z})} := \sup\left\{\sum_{k\in\mathbb{Z}} \gamma_k |A_k(f)| : \quad \gamma \in \Gamma(\mathbf{M}^*)\right\}. \tag{2.2}$$

By definition, B-a.p. functions are considered identical in $B\mathcal{S}_{\mathbf{M}}$ if they have the same Fourier series.

The spaces $B\mathcal{S}_{\mathbf{M}}$ defined in this way are Banach spaces. Functional spaces of this type have been studied by mathematicians since the 1940s (see, for example, the monographs [21, 22, 25]). In particular, the subspaces $\mathcal{S}_{\mathbf{M}}$ of all 2π-periodic functions from $B\mathcal{S}_{\mathbf{M}}$ were considered in [3, 5]. If all the functions M_k are identical (namely, $M_k(t) \equiv M(t)$, $k \in \mathbb{Z}$), the spaces $\mathcal{S}_{\mathbf{M}}$ coincide with the ordinary Orlicz type spaces $\mathcal{S}_M$ [15]. If $M_k(t) = \mu_k t^{p_k}$, $p_k \geq 1$, $\mu_k \geq 0$, then $\mathcal{S}_{\mathbf{M}}$ coincide with the weighted spaces $\mathcal{S}_{\mathbf{p},\mu}$ with variable exponents [2].

If all functions $M_k(u) = u^p\left(p^{-1/p}q^{-1/p'}\right)^p$, $p > 1$, $1/p + 1/p' = 1$, then $B\mathcal{S}_{\mathbf{M}}$ are the Besicovitch – Stepanets spaces $B\mathcal{S}^p$ [26] of functions $f \in B$-a.p. with the norm

$$\|f\|_{\mathbf{M}} = \|f\|_{B\mathcal{S}^p} = \|\{A_{\lambda_k}(f)\}_{k\in\mathbb{N}}\|_{l_p(\mathbb{N})} = \left(\sum_{k\in\mathbb{N}} |A_{\lambda_k}(f)|^p\right)^{1/p}. \tag{2.3}$$

The subspaces of all 2π-periodic Lebesgue summable functions from $B\mathcal{S}^p$ coincide with the well-known spaces $\mathcal{S}^p$ (see, for example, [28], Ch. XI). For $p = 2$, the sets $B\mathcal{S}^p = B\mathcal{S}^2$ coincide with the sets of B^2-a.p. functions and the spaces $\mathcal{S}^p$ with the ordinary Lebesgue spases of 2π-periodic square-summable functions, i.e., $\mathcal{S}^2 = L_2$.

By G_{λ_n} we denote the set of all B-a.p. functions whose Fourier exponents belong to the interval $(-\lambda_n, \lambda_n)$ and define the value of the best approximation of $f \in B\mathcal{S}_{\mathbf{M}}$ by the equality

$$E_{\lambda_n}(f)_{\mathbf{M}} = E_{\lambda_n}(f)_{B\mathcal{S}_{\mathbf{M}}} = \inf_{g\in G_{\lambda_n}} \|f - g\|_{\mathbf{M}}. \tag{2.4}$$

2.2. Generalized moduli of smoothness. Let Φ be the set of all continuous bounded nonnegative pair functions $\varphi(t)$ such that $\varphi(0) = 0$ and the Lebesgue measure of the set $\{t \in \mathbb{R} : \varphi(t) = 0\}$ is equal to zero. For an arbitrary fixed $\varphi \in \Phi$, consider the generalized modulus of smoothness of a function $f \in B\mathcal{S}_{\mathbf{M}}$

$$\omega_\varphi(f,\delta)_{\mathbf{M}} := \sup_{|h|\leq\delta} \sup\left\{\sum_{k\in\mathbb{Z}} \gamma_k \varphi(\lambda_k h)|A_k(f)| : \ \gamma \in \Gamma\right\}, \quad \delta \geq 0. \tag{2.5}$$

Consider the connection between the modulus (2.5) and some well-known moduli of smoothness. Let $\Theta = \{\theta_j\}_{j=0}^m$ be a nonzero collection of complex numbers such that $\sum_{j=0}^m \theta_j = 0$. We associate the collection Θ with the difference operator $\Delta_h^\Theta(f) = \Delta_h^\Theta(f,t) = \sum_{j=0}^m \theta_j f(t - jh)$ and the modulus of smoothness

$$\omega_\Theta(f,\delta)_{\mathbf{M}} := \sup_{|h|\leq\delta} \|\Delta_h^\Theta(f)\|_{\mathbf{M}}.$$

Note that the collection $\Theta(m) = \left\{\theta_j = (-1)^j\binom{m}{j},\ j = 0, 1, \ldots, m\right\}$, $m \in \mathbb{N}$, corresponds to the classical modulus of smoothness of order m, i.e.,

$$\omega_{\Theta(m)}(f,\delta)_{\mathbf{M}} = \omega_m(f,\delta)_{\mathbf{M}}.$$

For any $k \in \mathbb{Z}$, the Fourier coefficients of the function $\Delta_h^{\Theta}(f)$ satisfy the equality

$$|A_k(\Delta_h^{\Theta}(f))| = |A_k(f)| \left| \sum_{j=0}^{m} \theta_j e^{-\mathrm{i}\lambda_k j h} \right|.$$

Therefore, taking into account (2.2), we see that, for $\varphi_{\Theta}(t) = \left| \sum_{j=0}^{m} \theta_j e^{-\mathrm{i}jt} \right|$, $\omega_{\varphi_{\Theta}}(f,\delta)_{\mathbf{M}} = \omega_{\Theta}(f,\delta)_{\mathbf{M}}$. In particular, for $\varphi_m(t) = 2^m |\sin(t/2)|^m = 2^{\frac{m}{2}} (1-\cos t)^{\frac{m}{2}}$, $m \in \mathbb{N}$, we have $\omega_{\varphi_m}(f,\delta)_{\mathbf{M}} = \omega_m(f,\delta)_{\mathbf{M}}$.

Further, let

$$F_h(f,t) = f_h(x) := \frac{1}{2h} \int_{t-h}^{t+h} f(u)du$$

be the Steklov function of a function $f \in B\mathcal{S}_{\mathbf{M}}$. Define the differences as follows:

$$\widetilde{\Delta}_h^1(f) := \widetilde{\Delta}_h^1(f,t) = F_h(f,t) - f(t) = (F_h - \mathbb{I})(f,t),$$

$$\widetilde{\Delta}_h^m(f) := \widetilde{\Delta}_h^m(f,t) = \widetilde{\Delta}_h^1(\Delta_h^{m-1}(f),t) = (F_h - \mathbb{I})^m(f,t) = \sum_{k=0}^{m} k^{m-k} \binom{m}{k} F_{h,k}(f,t),$$

where $m = 2,3,\ldots$, $F_{h,0}(f) := f$, $F_{h,k}(f) := F_h(F_{h,k}(f))$ and $\mathbb{I}$ is the identity operator in $B\mathcal{S}_{\mathbf{M}}$. Consider the following smoothness characteristics

$$\widetilde{\omega}_m(f,\delta) := \sup_{0 \le h \le \delta} \|\widetilde{\Delta}_h^m(f)\|_{\mathbf{M}}, \quad \delta > 0.$$

It can be shown [6] that $\omega_{\tilde{\varphi}_m}(f,\delta)_{\mathbf{M}} = \widetilde{\omega}_m(f,\delta)_{\mathbf{M}}$ for for $\tilde{\varphi}_m(t) = (1 - \operatorname{sinc} t)^m$, $m \in \mathbb{N}$, where $\operatorname{sinc} t = \{\sin t/t, \text{ when } t \ne 0, \text{ and } 1, \text{when } t = 0\}$.

In the general case, moduli similar to (2.5) were studied in [3 – 5, 8, 11, 19, 26, 32, 34].

3. Main results*. ***3.1. Jackson-type inequalities.*** In this subsection, direct theorems are established for functions $f \in B\mathcal{S}_{\mathbf{M}}$ in terms of the best approximations and generalized moduli of smoothness. In particular, for functions $f \in B\mathcal{S}_{\mathbf{M}}$ with the Fourier series of the form (2.1), we prove Jackson-type inequalities of the kind as

$$E_{\lambda_n}(f)_{\mathbf{M}} \le K(\tau)\omega_{\varphi}\left(f, \frac{\tau}{\lambda_n}\right)_{\mathbf{M}}, \quad \tau > 0, \quad n \in \mathbb{N}.$$

Let $V(\tau)$, $\tau > 0$, be a set of bounded nondecreasing functions v that differ from a constant on $[0,\tau]$.

Theorem 3.1. *Assume that the function $f \in B\mathcal{S}_{\mathbf{M}}$ has the Fourier series of the form* (2.1). *Then, for any $\tau > 0$, $n \in \mathbb{N}$ and $\varphi \in \Phi$, the following inequality holds*:

$$E_{\lambda_n}(f)_{\mathbf{M}} \le K_{n,\varphi}(\tau)\omega_{\varphi}\left(f, \frac{\tau}{\lambda_n}\right)_{\mathbf{M}}, \tag{3.1}$$

* The results of this section was supported by the project "Innovative methods in the theory of differential equations, computational mathematics and mathematical modeling" (project number 0122U000670).

where

$$K_{n,\varphi}(\tau) := \inf_{v \in V(\tau)} \frac{v(\tau) - v(0)}{I_{n,\varphi}(\tau, v)} \tag{3.2}$$

and

$$I_{n,\varphi}(\tau, v) := \inf_{k \in \mathbb{N},\, k \geq n} \int_0^{\tau} \varphi\left(\frac{\lambda_k t}{\lambda_n}\right) dv(t). \tag{3.3}$$

Furthermore, there exists a function $v_* \in V(\tau)$ that realizes the greatest lower bound in (3.2).

In the spaces L_2 of 2π-periodic square-summable functions, for moduli of continuity $\omega_m(f;\delta)$ and $\tilde{\omega}_m(f;\delta)$, such result was obtained by Babenko [7], and Abilov and Abilova [6], respectively. In the spaces $\mathcal{S}^p$ of functions of one and several variables, this result for classical moduli of smoothness was obtained in [27] and [1], respectively. In the Musielak – Orlicz spaces $\mathcal{S}_{\mathbf{M}}$, similar result was obtained for generalized moduli of smoothness in [3].

In the Besicovitch – Stepanets spaces $B\mathcal{S}^p$, a similar theorem was proved in [26]. It was noted above that in the case when all functions $M_k(u) = u^p \left(p^{-1/p} q^{-1/p'}\right)^p$, $p > 1$, $1/p + 1/p' = 1$, we have $B\mathcal{S}_{\mathbf{M}} = B\mathcal{S}^p$ and $\|f\|_{\mathbf{M}} = \|f\|_{B\mathcal{S}^p}$. In the case $p = 1$, the similar equalities $B\mathcal{S}_{\mathbf{M}} = B\mathcal{S}^1$ and $\|f\|_{\mathbf{M}} = \|f\|_{B\mathcal{S}^1}$ obviously can be obtained if all $M_k(u) = u$, $k \in \mathbb{Z}$, and the set Γ is a set of all sequences of positive numbers $\gamma = \{\gamma_k\}_{k\in\mathbb{Z}}$ such that $\|\gamma\|_{l_\infty(\mathbb{Z})} = \sup_{k\in\mathbb{Z}} \gamma_k \leq 1$. Comparing estimate (3.1) with the corresponding result of Theorem 1 from [26], we see that in the case when $B\mathcal{S}_{\mathbf{M}} = B\mathcal{S}^1$, the inequality (3.1) is unimprovable on the set of all functions $f \in B\mathcal{S}^1$, $\|f - A_0(f)\|_{\mathbf{M}} \neq 0$. Furthermore, Theorem 1 [26] implies the existence of the function $v_* \in V(\tau)$ that realizes the greatest lower bound in (3.2).

Proof. In the proof of Theorem 3.1, we mainly use the ideas outlined in [7, 16, 17, 26, 27], taking into account the peculiarities of the spaces $B\mathcal{S}_{\mathbf{M}}$. From (2.2) and (2.4), it follows that for any $f \in B\mathcal{S}_{\mathbf{M}}$ with the Fourier series of the form (2.1), we have

$$E_{\lambda_n}(f)_{\mathbf{M}} = \|f - S_n(f)\|_{\mathbf{M}} = \sup\left\{ \sum_{|k|\geq n} \gamma_k |A_k(f)| :\ \gamma \in \Gamma \right\}, \tag{3.4}$$

where $S_n(f) := \sum_{|k|<n} A_k(f) e^{\mathrm{i}\lambda_k x}$.

By the definition of supremum, for arbitrary $\varepsilon > 0$ there exists a sequence $\tilde{\gamma} \in \Gamma$, $\tilde{\gamma} = \tilde{\gamma}(\varepsilon)$, such that the following relations holds:

$$\sum_{|k|\geq n} \tilde{\gamma}_k |A_k(f)| + \varepsilon \geq \sup\left\{ \sum_{|k|\geq n} \gamma_k |A_k(f)| :\ \gamma \in \Gamma \right\}.$$

For arbitrary $\varphi \in \Phi$ and $h \in \mathbb{R}$, consider the sequence of numbers $\{\varphi(\lambda_k h) A_k(f)\}_{k\in\mathbb{Z}}$. If there exists a function $\Delta_h^{\varphi}(f) \in B$-a.p. such that, for all $k \in \mathbb{Z}$,

$$A_k(\Delta_h^{\varphi}(f)) = \varphi(\lambda_k h) A_k(f), \tag{3.5}$$

then here and below we denote by $\|\Delta_h^{\varphi}(f)\|_{\mathbf{M}}$ the Orlicz norm (2.2) of the function $\Delta_h^{\varphi}(f)$. If such a B-a.p. function $\Delta_h^{\varphi}(f)$ does not exist, then to simplify notation we also use the notation

$\|\Delta_h^{\varphi}(f)\|_{\mathbf{M}}$, meaning by it the $l_{\mathbf{M}}$-norm of the sequence $\{\varphi(\lambda_k h)A_k(f)\}_{k\in\mathbb{Z}}$. In view of (2.2) and (3.5), we obtain

$$\|\Delta_h^{\varphi} f\|_{\mathbf{M}} \geq \sup\left\{\sum_{|k|\geq n}\gamma_k\varphi(\lambda_k h)|A_k(f)| : \gamma\in\Gamma\right\} \geq \sum_{|k|\geq n}\tilde{\gamma}_k\varphi(\lambda_k h)|A_k(f)| =$$

$$= \frac{I_{n,\varphi}(\tau,v)}{v(\tau)-v(0)}\sum_{|k|\geq n}\tilde{\gamma}_k|A_k(f)| + \sum_{|k|\geq n}\tilde{\gamma}_k|A_k(f)|\left(\varphi(\lambda_k h) - \frac{I_{n,\varphi}(\tau,v)}{v(\tau)-v(0)}\right).$$

For any $u\in[0,\tau]$, we get

$$\|\Delta_{\frac{u}{\lambda_n}}^{\varphi} f\|_{\mathbf{M}} \geq \frac{I_{n,\varphi}(\tau,v)}{v(\tau)-v(0)}\sum_{|k|\geq n}\tilde{\gamma}_k|A_k(f)| +$$

$$+\sum_{|k|\geq n}\tilde{\gamma}_k|A_k(f)|\left(\varphi\left(\frac{\lambda_k u}{\lambda_n}\right) - \frac{I_{n,\varphi}(\tau,v)}{v(\tau)-v(0)}\right). \tag{3.6}$$

The both sides of inequality (3.6) are nonnegative and, in view of the boundedness of the function φ, the series on its right-hand side is majorized on the entire real axis by the absolutely convergent series $\mathcal{K}(\varphi)\sum_{|k|\geq n}\tilde{\gamma}_k|A_k(f)|$, where $\mathcal{K}(\varphi) := \max_{u\in\mathbb{R}}\varphi(u)$. Then integrating this inequality with respect to $dv(u)$ from 0 to τ, we get

$$\int_0^{\tau}\|\Delta_{\frac{u}{\lambda_n}}^{\varphi} f\|_{\mathbf{M}}dv(u) \geq I_{n,\varphi}(\tau,v)\sum_{|k|\geq n}\tilde{\gamma}_k|A_k(f)| +$$

$$+\sum_{|k|\geq n}\tilde{\gamma}_k|A_k(f)|\left(\int_0^{\tau}\varphi\left(\frac{\lambda_k u}{\lambda_n}\right)dv(u) - I_{n,\varphi}(\tau,v)\right).$$

By virtue of the definition of $I_{n,\varphi}(\tau,v)$, we see that the second term on the right-hand side of the last relation is nonnegative. Therefore, for any function $v\in V(\tau)$, we have

$$\int_0^{\tau}\|\Delta_{\frac{u}{\lambda_n}}^{\varphi} f\|_{\mathbf{M}}dv(u) \geq I_{n,\varphi}(\tau,v)\sum_{|k|\geq n}\tilde{\gamma}_k|A_k(f)| \geq$$

$$\geq I_{n,\varphi}(\tau,v)\left(\sup\left\{\sum_{|k|\geq n}\gamma_k|A_k(f)| : \gamma\in\Gamma\right\} - \varepsilon\right),$$

wherefrom due to an arbitrariness of choice of the number ε, we conclude that

$$\int_0^{\tau}\|\Delta_{\frac{u}{\lambda_n}}^{\varphi} f\|_{\mathbf{M}}dv(u) \geq I_{n,\varphi}(\tau,v)E_{\lambda_n}(f)_{\mathbf{M}}.$$

Hence,

$$E_{\lambda_n}(f)_{\mathbf{M}} \le \frac{1}{I_{n,\varphi}(\tau, v)} \int_0^{\tau} \|\Delta^{\varphi}_{\frac{u}{\lambda_n}} f\|_{\mathbf{M}} dv(u) \le \frac{1}{I_{n,\varphi}(\tau, v)} \int_0^{\tau} \omega_{\varphi}\left(f, \frac{u}{\lambda_n}\right)_{\mathbf{M}} dv(u), \tag{3.7}$$

whence taking into account nondecreasing of the function ω_{φ}, we immediately obtain relation (3.1).

Theorem 3.1 is proved.

Now we consider some realisations of Theorem 3.1. Setting $\varphi_{\alpha}(t) = 2^{\frac{\alpha}{2}}(1-\cos t)^{\frac{\alpha}{2}}$, $\alpha > 0$, $\omega_{\varphi_{\alpha}}(f,\delta)_{\mathbf{M}} =: \omega_{\alpha}(f,\delta)_{\mathbf{M}}$, $\tau = \pi$, and $v(u) = 1 - \cos u$, $u \in [0, \pi]$, we get the following assertion.

***Corollary* 3.1.** *For arbitrary numbers $n \in \mathbb{N}$ and $\alpha > 0$, and for any function $f \in B\mathcal{S}_{\mathbf{M}}$ with the Fourier series of the form* (2.1), *the following inequalities hold*:

$$E_{\lambda_n}(f)_{\mathbf{M}} \le \frac{1}{2^{\frac{\alpha}{2}} I_n(\frac{\alpha}{2})} \int_0^{\pi} \omega_{\alpha}\left(f, \frac{u}{\lambda_n}\right)_{\mathbf{M}} \sin u \, du, \tag{3.8}$$

where

$$I_n\left(\frac{\alpha}{2}\right) = \inf_{k\in\mathbb{N}, k\ge n} \int_0^{\pi} \left(1 - \cos\frac{\lambda_k u}{\lambda_n}\right)^{\frac{\alpha}{2}} \sin u \, du. \tag{3.9}$$

If, in addition, $\dfrac{\alpha}{2} \in \mathbb{N}$, *then*

$$I_n\left(\frac{\alpha}{2}\right) = \frac{2^{\frac{\alpha}{2}+1}}{\frac{\alpha}{2}+1}, \tag{3.10}$$

and the inequality (3.8) *cannot be improved for any $n \in \mathbb{N}$.*

Proof. Estimate (3.8) follows from (3.7). In [27] (relation (52)), it was shown that for any $\theta \ge 1$ and $s \in \mathbb{N}$ the following inequality holds:

$$\int_0^{\pi} (1 - \cos\theta t)^s \sin t dt \ge \frac{2^{s+1}}{s+1},$$

which turns into equality for $\theta = 1$. Therefore, setting $s = \dfrac{\alpha}{2}$ and $\theta = \dfrac{\lambda_{\nu}}{\lambda_n}$, $\nu = n, n+1, \ldots$, and the monotonicity of the sequence of Fourier exponents $\{\lambda_k\}_{k\in\mathbb{Z}}$, we see that for $\dfrac{\alpha}{2} \in \mathbb{N}$, indeed, the equality (3.10) holds.

Let us prove that in this case, the constant $\dfrac{\frac{\alpha}{2}+1}{2^{\alpha+1}}$ in inequality (3.8) is unimprovable for $\dfrac{\alpha}{2} \in \mathbb{N}$. It suffices to verify that the function

$$f^*(x) = \gamma + \beta e^{-\lambda_n x} + \delta e^{\lambda_n x}, \tag{3.11}$$

where γ, β and δ are arbitrary complex numbers, satisfies the equality

$$E_{\lambda_n}(f^*)_{\mathbf{M}} = \frac{\frac{\alpha}{2}+1}{2^{\alpha+1}} \int_0^{\pi} \omega_{\alpha}\left(f^*, \frac{t}{\lambda_n}\right)_{\mathbf{M}} \sin t \, dt, \quad \alpha > 0. \tag{3.12}$$

We have $E_{\lambda_n}(f^*)_{\mathbf{M}} = |\beta| + |\delta|$, the function $\|\Delta^{\varphi_\alpha}_{u/\lambda_n} f^*\|_{\mathbf{M}} = 2^{\frac{\alpha}{2}}(|\beta| + |\delta|)(1 - \cos u)^{\frac{\alpha}{2}}$ does not decrease with respect to u on $[0, \pi]$. Therefore, $\omega_\alpha\left(f^*, \dfrac{u}{\lambda_n}\right)_{\mathbf{M}} = \|\Delta^{\varphi_\alpha}_{u/\lambda_n} f^*\|_{\mathbf{M}}$ and

$$\frac{2^{\alpha+1}}{\frac{\alpha}{2}+1} E_{\lambda_n}(f^*)_{\mathbf{M}} - \int_0^\pi \omega_\alpha\left(f^*, \frac{t}{\lambda_n}\right)_{\mathbf{M}} \sin t\, dt =$$

$$= (|\beta| + |\delta|)\left(\frac{2^{\alpha+1}}{\frac{\alpha}{2}+1} - 2^{\frac{\alpha}{2}} \int_0^\pi (1 - \cos t)^{\frac{\alpha}{2}} \sin t\, dt\right) = 0.$$

Corollary 3.1 is proved.

It was shown in [27] that $I_n(s) \geq 2$ when $s \geq 1$ and $I_n(s) \geq 1 + 2^{s-1}$ when $s \in (0, 1)$. Combining these two estimates and (3.8), we obtain the following statement, which establishes a Jackson-type inequality with a constant uniformly bounded in the parameter $n \in \mathbb{N}$.

***Corollary* 3.2.** *Assume that the function $f \in B\mathcal{S}_{\mathbf{M}}$ has the Fourier series of the form* (2.1) *and $\|f - A_0(f)\|_{\mathbf{M}} \neq 0$. Then, for any $n \in \mathbb{N}$ and $\alpha > 0$,*

$$E_{\lambda_n}(f)_{\mathbf{M}} < c_\alpha \omega_\alpha\left(f, \frac{\pi}{\lambda_n}\right)_{\mathbf{M}}, \tag{3.13}$$

where $c_\alpha = 2^{-\alpha/2}$ for $\alpha \geq 2$ and $c_\alpha = 4 \cdot 2^{-\alpha/2}/3$ for $0 < \alpha < 2$. Furthermore, in the case where $\alpha = m \in \mathbb{N}$, the following more accurate estimate holds:

$$E_{\lambda_n}(f)_{\mathbf{M}} < \frac{4 - 2\sqrt{2}}{2^{m/2}} \omega_m\left(f, \frac{\pi}{\lambda_n}\right)_{\mathbf{M}}. \tag{3.14}$$

Proof. Relation (3.14) follows from the estimate $I_n\left(\dfrac{\alpha}{2}\right) \geq 1 + \dfrac{1}{\sqrt{2}}$, which is a consequence of the above estimates for the value of $I_n(s)$ in the case $\alpha = m \in \mathbb{N}$ [27].

If the weight function $v_2(t) = t$, then we obtain the following assertion.

***Corollary* 3.3.** *Assume that the function $f \in B\mathcal{S}_{\mathbf{M}}$ has the Fourier series of the form* (2.1) *and $\alpha \geq 1$. Then, for any $0 < \tau \leq \dfrac{3\pi}{4}$ and $n \in \mathbb{N}$,*

$$E_{\lambda_n}(f)_{\mathbf{M}} \leq \frac{1}{2^\alpha \displaystyle\int_0^\tau \sin^\alpha \frac{t}{2}\, dt} \int_0^\tau \omega_\alpha\left(f, \frac{t}{\lambda_n}\right)_{\mathbf{M}} dt. \tag{3.15}$$

Relation (3.15) *becomes equality for the function f^* of the form* (3.11).

Inequalities (3.8) and (3.15) can be considered as an extension of the corresponding results of Serdyuk and Shidlich [26] to the Besicovitch – Musielak spaces $B\mathcal{S}_{\mathbf{M}}$, and they coincide with them in the case $B\mathcal{S}_{\mathbf{M}} = B\mathcal{S}^1$. In the spaces $\mathcal{S}^p$ of functions of one and several variables, analogues of Theorem 3.1 and Corollaries 3.1 and 3.3 were proved in [27] and [1], respectively. The inequalities of this type were also investigated in [8, 17, 27, 32, 34].

Proof. From inequality (3.7), it follows that

$$E_{\lambda_n}(f)_{\mathbf{M}} \le \frac{1}{2^{\frac{\alpha}{2}} I_n^*\left(\frac{\alpha}{2}\right)} \int_0^{\tau} \omega_\alpha\left(f, \frac{t}{\lambda_n}\right) dt,$$

where

$$I_n^*\left(\frac{\alpha}{2}\right) := \inf_{k\in\mathbb{N},\, k\ge n} \int_0^{\tau} \left(1 - \cos\frac{\lambda_k t}{\lambda_n}\right)^{\frac{\alpha}{2}} dt, \quad \alpha > 0, \quad n \in \mathbb{N}.$$

In [35], it is shown that for the function $F_\alpha(x) := \dfrac{1}{x}\displaystyle\int_0^x |\sin t|^\alpha\, dt$, any $h \in \left(0, \dfrac{3\pi}{4}\right)$ and $\alpha \ge 1$, the following relation is true:

$$\inf_{x\ge h/2} F_\alpha(x) = F_\alpha(h/2). \tag{3.16}$$

Since for $h = \dfrac{\lambda_k}{\lambda_n} \ge 1$ $(k \ge n)$

$$\int_0^{\tau} \left(1 - \cos\frac{\lambda_k t}{\lambda_n}\right)^{\frac{\alpha}{2}} dt = 2^{\frac{\alpha}{2}} \int_0^{\tau} \left|\sin\frac{\lambda_k t}{2\lambda_n}\right|^\alpha dt = 2^{\frac{\alpha}{2}} \tau F_\alpha\left(\frac{\lambda_k \tau}{2\lambda_n}\right),$$

from (3.16) $\left(\text{with } \tau \in \left(0, \dfrac{3\pi}{4}\right] \text{ and } \alpha \ge 1\right)$ we obtain

$$I_n^*\left(\frac{\alpha}{2}\right) = \inf_{k\in\mathbb{N}:k\ge n} \int_0^{\tau} \left(1 - \cos\frac{\lambda_k t}{\lambda_n}\right)^{\frac{\alpha}{2}} dt = \inf_{k\in\mathbb{N}:k\ge n} 2^{\frac{\alpha}{2}} \int_0^{\tau} \left|\sin\frac{\lambda_k t}{2\lambda_n}\right|^\alpha dt = 2^{\frac{\alpha}{2}} \int_0^{\tau} \sin^\alpha\frac{t}{2}\, dt.$$

For the functions f^* of the form (3.11), the equality

$$E_{\lambda_n}(f^*)_{\mathbf{M}} = \frac{1}{2^\alpha \displaystyle\int_0^{\tau} \sin^\alpha\frac{t}{2}\, dt} \int_0^{\tau} \omega_\alpha\left(f^*, \frac{t}{\lambda_n}\right)_{\mathbf{M}} dt$$

is verified similarly to the proof of equality (3.12).

Corollary 3.2 is proved.

In the case $\varphi(t) = \tilde{\varphi}_m(t) = (1 - \operatorname{sinc} t)^m$, $m \in \mathbb{N}$, where, by definition, $\operatorname{sinc} t = \{\sin t/t,\ \text{if } t \ne 0,\ \text{and } 1,\ \text{if } t = 0\}$, for $\tau = \pi$ and $v(u) = 1 - \cos u$, $u \in [0; \pi]$, from relation (3.7) we get

$$E_{\lambda_n}(f)_{\mathbf{M}} \le \frac{1}{\tilde{I}_n(m)} \int_0^{\pi} \tilde{\omega}_m\left(f, \frac{u}{\lambda_n}\right)_{\mathbf{M}} \sin u\, du,$$

where

$$\tilde{I}_n(m) = \inf_{k\in\mathbb{N},k\ge n} \int_0^{\pi} \left(1 - \operatorname{sinc}\frac{\lambda_k u}{\lambda_n}\right)^m \sin u\, du.$$

Taking into account the estimation [33]

$$1-\operatorname{sinc}\left(\frac{\lambda_k u}{\lambda_n}\right)\geq 1-\frac{\sin u}{u}\geq\left(\frac{u}{\pi}\right)^2,\quad k\geq n,\quad u\in[0;\pi],$$

we have

$$\tilde{I}_n(m)\geq\int\limits_0^\pi(1-\operatorname{sinc}\,u)^m\sin u\,du\geq\frac{1}{\pi^{2m}}\int\limits_0^\pi u^{2m}\sin u\,du=$$

$$=\frac{2m!}{\pi^{2m}}\left(\sum_{j=0}^{m}(-1)^j\frac{\pi^{2m-2j}}{(2m-2j)!}+\frac{\pi^{2m}}{2m!}(-1)^m\right):=\frac{2m!}{\pi^{2m}}\mathcal{K}(m).$$

Thereby, the following corollary follows from Theorem 3.1.

***Corollary* 3.4.** *For arbitrary numbers $n\in\mathbb{N}$ and $m>0$, and for any function $f\in B\mathcal{S}_{\mathbf{M}}$, with the Fourier series of the form* (2.1) *the following inequalities hold:*

$$E_{\lambda_n}(f)_{\mathbf{M}}\leq\frac{\pi^{2m}}{2m!\cdot\mathcal{K}(m)}\int\limits_0^\pi\tilde{\omega}_m\left(f,\frac{u}{\lambda_n}\right)_{\mathbf{M}}\sin u\,du,$$

where

$$\mathcal{K}(m)=\sum_{j=0}^{m}(-1)^j\frac{\pi^{2m-2j}}{(2m-2j)!}+\frac{\pi^{2m}}{2m!}(-1)^m.$$

In the case $m=1$, we have $2\mathcal{K}(1)=\pi^2-4$ and

$$E_{\lambda_n}(f)_{\mathbf{M}}\leq\frac{\pi^2}{\pi^2-4}\int\limits_0^\pi\tilde{\omega}_1\left(f,\frac{u}{\lambda_n}\right)_{\mathbf{M}}\sin u\,du\leq\frac{\pi^2\lambda_n}{\pi^2-4}\int\limits_0^{\frac{\pi}{\lambda_n}}\tilde{\omega}_1\left(f,u\right)_{\mathbf{M}}\sin\lambda_n u\,du.$$

If the weight function $v_2(t)=u^{m+1}$, then we obtain the following assertion.

***Corollary* 3.5.** *Assume that the function $f\in B\mathcal{S}_{\mathbf{M}}$ has the Fourier series of the form* (2.1) *and $m\geq 1$. Then, for any $0<\tau\leq\pi$ and $n\in\mathbb{N}$,*

$$E_{\lambda_n}(f)_{\mathbf{M}}\leq\pi^{m-1}\left(\frac{2\lambda_n}{\pi^2-4}\right)^m\lambda_n\int\limits_0^{\tau/\lambda_n}\tilde{\omega}_m(f,t)_{\mathbf{M}}t^m\,dt.\tag{3.17}$$

Ideed, applying Holder's inequality, we find

$$\int\limits_0^\pi\left(1-\operatorname{sinc}\,\frac{\lambda_k u}{\lambda_n}\right)^m du^{m+1}\geq(m+1)\int\limits_0^\pi\left(1-\frac{\sin u}{u}\right)^m u^m\,du=$$

$$=(m+1)\int\limits_0^\pi(u-\sin u)^m\,du\geq\frac{m+1}{\pi^{m-1}}\left(\int\limits_0^\pi(u-\sin u)\,du\right)^m=\frac{m+1}{\pi^{m-1}}\left(\frac{\pi^2-4}{2}\right)^m.$$

In the spaces L_2 of 2π-periodic square-summable functions, for moduli of smoothness $\tilde{\omega}_m(f;\delta)$, the results of this kind were obtained by Abilov and Abilova [6], and Vakarchuk [32]. Note that in the case $f\in B\mathcal{S}_{\mathbf{M}}=L_2$ the inequality (3.17) follows from the result of [6] (see Theorem 1). For $m=1$ and $f\in L_2$, the statements of Corollary 3.5 and Theorem 1 from [6] are identical, and the constant in the right-hand side of (3.17) cannot be reduced for every fixed n.

4. Inverse approximation theorem.

Theorem 4.1. *Assume that $f \in B\mathcal{S}_{\mathbf{M}}$ has the Fourier series of the form* (2.1), *the function $\varphi \in \Phi$ is nondecreasing on the interval $[0, \tau]$, $\tau > 0$, and $\varphi(\tau) = \max\{\varphi(t) : t \in \mathbb{R}\}$. Then, for any $n \in \mathbb{N}$, the following inequality holds*:

$$\omega_\varphi\left(f, \frac{\tau}{\lambda_n}\right)_{\mathbf{M}} \le \sum_{\nu=1}^{n}\left(\varphi\left(\frac{\tau\lambda_\nu}{\lambda_n}\right) - \varphi\left(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\right)\right) E_{\lambda_\nu}(f)_{\mathbf{M}}. \tag{4.1}$$

Proof. Let us use the proof scheme from [27] and [3], modifying it taking into account the peculiarities of the spaces $B\mathcal{S}_{\mathbf{M}}$ and the definition of the modulus ω_φ.

Let $f \in B\mathcal{S}_{\mathbf{M}}$. For any $\varepsilon > 0$ there exists a number $N_0 = N_0(\varepsilon) \in \mathbb{N}$, $N_0 > n$, such that, for any $N > N_0$, we have

$$E_{\lambda_N}(f)_{\mathbf{M}} = \|f - S_{N-1}(f)\|_{\mathbf{M}} < \varepsilon/\varphi(\tau).$$

Let us set $f_0 := S_{N_0}(f)$. Then in view of (3.5), we see that

$$\|\Delta_h^\varphi(f)\|_{\mathbf{M}} \le \|\Delta_h^\varphi(f_0)\|_{\mathbf{M}} + \|\Delta_h^\varphi(f - f_0)\|_{\mathbf{M}} \le$$

$$\le \|\Delta_h^\varphi(f_0)\|_{\mathbf{M}} + \varphi(\tau)E_{\lambda_{N_0+1}}(f)_{\mathbf{M}} < \|\Delta_h^\varphi(f_0)\|_{\mathbf{M}} + \varepsilon. \tag{4.2}$$

Further, let $S_{n-1} := S_{n-1}(f_0)$ be the Fourier sum of f_0. Then by virtue of (3.5), for $|h| \le \tau/\lambda_n$, we have

$$\|\Delta_h^\varphi(f_0)\|_{\mathbf{M}} = \|\Delta_h^\varphi(f_0 - S_{n-1}) + \Delta_h^\varphi S_{n-1}\|_{\mathbf{M}} \le \left\|\varphi(\tau)(f_0 - S_{n-1}) + \right.$$

$$\left. + \sum_{|k|\le n-1} \varphi(\lambda_k h)|A_k(f)|\right\|_{\mathbf{M}} \le \left\|\varphi(\tau)\sum_{\nu=n}^{N_0} H_\nu + \sum_{\nu=1}^{n-1}\varphi\left(\frac{\tau\lambda_\nu}{\lambda_n}\right)H_\nu\right\|_{\mathbf{M}}, \tag{4.3}$$

where $H_\nu(x) := |A_\nu(f)| + |A_{-\nu}(f)|$, $\nu = 1, 2, \ldots$.

Now we use the following assertion from [27].

Lemma 4.1 [27]**.** *Let $\{c_\nu\}_{\nu=1}^{\infty}$ and $\{a_\nu\}_{\nu=1}^{\infty}$ be arbitrary numerical sequences. Then the following equality holds for all natural N_1, N_2 and N $N_1 \le N_2 < N$*:

$$\sum_{\nu=N_1}^{N_2} a_\nu c_\nu = a_{N_1}\sum_{\nu=N_1}^{N} c_\nu + \sum_{\nu=N_1+1}^{N_2}(a_\nu - a_{\nu-1})\sum_{i=\nu}^{N} c_i - a_{N_2}\sum_{\nu=N_2+1}^{N} c_\nu. \tag{4.4}$$

Setting $a_\nu = \varphi\left(\frac{\tau\lambda_\nu}{\lambda_n}\right)$, $c_\nu = H_\nu(x)$, $N_1 = 1$, $N_2 = n-1$ and $N = N_0$ in (4.4), we get

$$\sum_{\nu=1}^{n-1}\varphi\left(\frac{\tau\lambda_\nu}{\lambda_n}\right)H_\nu(x) = \varphi\left(\frac{\tau\lambda_1}{\lambda_n}\right)\sum_{\nu=1}^{N_0} H_\nu(x) +$$

$$+\sum_{\nu=2}^{n-1}\left(\varphi\left(\frac{\tau\lambda_\nu}{\lambda_n}\right) - \varphi\left(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\right)\right)\sum_{i=\nu}^{N_0} H_i(x) - \varphi\left(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\right)\sum_{\nu=n}^{N_0} H_\nu(x).$$

Therefore,

$$\left\|\varphi(\tau)\sum_{\nu=n}^{N_0}H_\nu+\sum_{\nu=1}^{n-1}\varphi\left(\frac{\tau\lambda_\nu}{\lambda_n}\right)H_\nu\right\|_{\mathbf{M}}\le$$

$$\le\left\|\varphi(\tau)\sum_{\nu=n}^{N_0}H_\nu+\sum_{\nu=1}^{n-1}\left(\varphi\left(\frac{\tau\lambda_\nu}{\lambda_n}\right)-\varphi\left(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\right)\right)\sum_{i=\nu}^{N_0}H_i-\varphi\left(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\right)\sum_{\nu=n}^{N_0}H_\nu\right\|_{\mathbf{M}}\le$$

$$\le\left\|\sum_{\nu=1}^{n}\left(\varphi\left(\frac{\tau\lambda_\nu}{\lambda_n}\right)-\varphi\left(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\right)\right)\sum_{i=\nu}^{N_0}H_i\right\|_{\mathbf{M}}\le$$

$$\le\sum_{\nu=1}^{n}\left(\varphi\left(\frac{\tau\lambda_\nu}{\lambda_n}\right)-\varphi\left(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\right)\right)E_{\lambda_\nu}(f_0)_{\mathbf{M}}. \tag{4.5}$$

Combining relations (4.2), (4.3) and (4.5) and taking into account the definition of the function f_0, we see that, for $|h|\le\tau/\lambda_n$, the following inequality holds:

$$\|\Delta_h^\varphi(f)\|_{\mathbf{M}}\le\sum_{\nu=1}^{n}\left(\varphi\left(\frac{\tau\lambda_\nu}{\lambda_n}\right)-\varphi\left(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\right)\right)E_{\lambda_\nu}(f)_{\mathbf{M}}+\varepsilon$$

which, in view of arbitrariness of ε, gives us (4.1).

Theorem 4.1 is proved.

Consider an important special case when $\varphi(t)=\varphi_\alpha(t)=2^{\frac{\alpha}{2}}(1-\cos t)^{\frac{\alpha}{2}}=2^\alpha|\sin(t/2)|^\alpha$, $\alpha>0$. In this case, the function φ satisfies the conditions of Theorem 4.1 with $\tau=\pi$. Then for $\alpha\ge1$ using the inequality $x^\alpha-y^\alpha\le\alpha x^{\alpha-1}(x-y)$, $x>0,y>0$ (see, for example, [18], Ch. 1), and the usual trigonometric formulas, for $\nu=1,2,\ldots,n$, we have

$$\varphi\left(\frac{\tau\lambda_\nu}{\lambda_n}\right)-\varphi\left(\frac{\tau\lambda_{\nu-1}}{\lambda_n}\right)=2^\alpha\left(\left|\sin\frac{\pi\lambda_\nu}{\lambda_n}\right|^\alpha-\left|\sin\frac{\pi\lambda_{\nu-1}}{\lambda_n}\right|^\alpha\right)\le$$

$$\le2^\alpha\alpha\left|\sin\frac{\pi\lambda_\nu}{\lambda_n}\right|^{\alpha-1}\left|\sin\frac{\pi\lambda_\nu}{\lambda_n}-\sin\frac{\pi\lambda_{\nu-1}}{\lambda_n}\right|\le\alpha\left(\frac{2\pi}{\lambda_n}\right)^\alpha\lambda_\nu^{\alpha-1}(\lambda_\nu-\lambda_{\nu-1}).$$

If $0<\alpha<1$, then the similar estimate can be obtained using the inequality $x^\alpha-y^\alpha\le\alpha y^{\alpha-1}(x-y)$, which holds for any $x>0,y>0$ [18] (Ch. 1). Hence, for any $f\in B\mathcal{S}_{\mathbf{M}}$, we get the following estimate:

$$\omega_\alpha\left(f,\frac{\pi}{\lambda_n}\right)_{\mathbf{M}}\le\alpha\left(\frac{2\pi}{\lambda_n}\right)^\alpha\sum_{\nu=1}^{n}\lambda_\nu^{\alpha-1}(\lambda_\nu-\lambda_{\nu-1})E_{\lambda_\nu}(f)_{\mathbf{M}}\quad\alpha>0. \tag{4.6}$$

It should be noted that the constant in this estimate can be improved as follows.

Theorem 4.2. *Assume that $f\in B\mathcal{S}_{\mathbf{M}}$ has the Fourier series of the form* (2.1). *Then, for any $n\in\mathbb{N}$ and $\alpha>0$,*

$$\omega_\alpha\left(f,\frac{\tau}{\lambda_n}\right)_{\mathbf{M}}\le\left(\frac{\pi}{\lambda_n}\right)^\alpha\sum_{\nu=1}^{n}(\lambda_\nu^\alpha-\lambda_{\nu-1}^\alpha)E_{\lambda_\nu}(f)_{\mathbf{M}}. \tag{4.7}$$

Proof. We prove this theorem similarly to the proof of Theorem 4.1. For any $\varepsilon > 0$, denote by $N_0 = N_0(\varepsilon) \in \mathbb{N}$, $N_0 > n$, a number such that, for any $N > N_0$,

$$E_{\lambda_N}(f)_{\mathbf{M}} = \|f - S_{N-1}(f)\|_{\mathbf{M}} < \varepsilon.$$

Let us set $f_0 := S_{N_0}(f)$, $S_{n-1} := S_{n-1}(f_0)$ and $\|\Delta_h^\alpha(f)\|_{\mathbf{M}} := \|\Delta_h^{\varphi_\alpha}(f)\|_{\mathbf{M}}$, and use relations (4.2) and (4.3). We obtain

$$\|\Delta_h^\alpha(f)\|_{\mathbf{M}} < \|\Delta_h^\alpha(f_0)\|_{\mathbf{M}} + \varepsilon \tag{4.8}$$

and

$$\|\Delta_h^\alpha(f_0)\|_{\mathbf{M}} \le \left\| 2^\alpha \sum_{\nu=n}^{N_0} H_\nu + 2^\alpha \sum_{\nu=1}^{n-1} \left| \sin \frac{\pi\lambda_\nu}{2\lambda_n} \right|^\alpha H_\nu \right\|_{\mathbf{M}} \le$$

$$\le \left(\frac{\pi}{\lambda_n}\right)^\alpha \left\| \lambda_n^\alpha \sum_{\nu=n}^{N_0} H_\nu + \sum_{\nu=1}^{n-1} \lambda_\nu^\alpha H_\nu \right\|_{\mathbf{M}}, \tag{4.9}$$

where $|h| \le \pi/\lambda_n$ and $H_\nu(x) = |A_\nu(f)| + |A_{-\nu}(f)|$, $\nu = 1, 2, \ldots$.

By virtue of (4.4), for $a_\nu = \lambda_\nu^\alpha$, $c_\nu = H_\nu(x)$, $N_1 = 1$, $N_2 = n-1$ and $N = N_0$,

$$\sum_{\nu=1}^{n-1} \lambda_\nu^\alpha H_\nu(x) = \lambda_1^\alpha \sum_{\nu=1}^{N_0} H_\nu(x) + \sum_{\nu=2}^{n-1} \left(\lambda_\nu^\alpha - \lambda_{\nu-1}^\alpha\right) \sum_{i=\nu}^{N_0} H_i(x) - \lambda_{\nu-1}^\alpha \sum_{\nu=n}^{N_0} H_\nu(x).$$

Therefore,

$$\left\| \lambda_n^\alpha \sum_{\nu=n}^{N_0} H_\nu + \sum_{\nu=1}^{n-1} \lambda_\nu^\alpha H_\nu \right\|_{\mathbf{M}} = \left\| \sum_{\nu=1}^{n} \left(\lambda_\nu^\alpha - \lambda_{\nu-1}^\alpha\right) \sum_{i=\nu}^{N_0} H_i \right\|_{\mathbf{M}} \le$$

$$\le \sum_{\nu=1}^{n} \left(\lambda_\nu^\alpha - \lambda_{\nu-1}^\alpha\right) E_{\lambda_\nu}(f_0)_{\mathbf{M}}. \tag{4.10}$$

Combining relations (4.8), (4.9) and (4.10) and taking into account the definition of the function f_0, we see that, for $|h| \le \tau/\lambda_n$, the following inequality holds:

$$\|\Delta_h^\alpha(f)\|_{\mathbf{M}} \le \left(\frac{\pi}{\lambda_n}\right)^\alpha \sum_{\nu=1}^{n} \left(\lambda_\nu^\alpha - \lambda_{\nu-1}^\alpha\right) E_{\lambda_\nu}(f)_{\mathbf{M}} + \varepsilon$$

which, in view of arbitrariness of ε, gives us (4.7).

Theorem 4.2 is proved.

In (4.1), the constant π^α is exact in the sense that for any $\varepsilon > 0$, there exists a function $f^* \in B\mathcal{S}_{\mathbf{M}}$ such that, for all n greater than a certain number n_0, we have

$$\omega_\alpha\left(f^*, \frac{\pi}{\lambda_n}\right)_{\mathbf{M}} > \frac{\pi^\alpha - \varepsilon}{\lambda_n^\alpha} \sum_{\nu=1}^{n} \left(\lambda_\nu^\alpha - \lambda_{\nu-1}^\alpha\right) E_{\lambda_\nu}(f^*)_{\mathbf{M}}. \tag{4.11}$$

Consider the function $f^*(x) = e^{\mathrm{i}\lambda_{k_0}x}$, where k_0 is an arbitrary positive integer. Then $E_{\lambda_\nu}(f^*)_{\mathbf{M}} = 1$ for $\nu = 1, 2, \ldots, k_0$, $E_{\lambda_\nu}(f^*)_{\mathbf{M}} = 0$ for $\nu > k_0$ and

$$\omega_\alpha\left(f^*, \frac{\pi}{\lambda_n}\right)_{\mathbf{M}} \geq \|\Delta^\alpha_{\frac{\pi}{\lambda_n}} f^*\|_{\mathbf{M}} \geq 2^\alpha \left|\sin\frac{\lambda_{k_0}\pi}{2\lambda_n}\right|^\alpha.$$

Since $\sin t/t$ tends to 1 as $t \to 0$, then, for all n greater than a certain number n_0, the inequality $2^\alpha|\sin\lambda_{k_0}\pi/(2\lambda_n)|^\alpha > (\pi^\alpha - \varepsilon)\lambda_{k_0}^\alpha/{\lambda_n}^\alpha$ holds, which yields (4.11).

***Corollary* 4.1.** *Suppose that $f \in B\mathcal{S}_{\mathbf{M}}$ has the Fourier series of the form* (2.1). *Then, for any $n \in \mathbb{N}$ and $\alpha > 0$,*

$$\omega_\alpha\left(f, \frac{\pi}{\lambda_n}\right)_{\mathbf{M}} \leq \alpha\left(\frac{\pi}{\lambda_n}\right)^\alpha \sum_{\nu=1}^{n} \lambda_\nu^{\alpha-1}(\lambda_\nu - \lambda_{\nu-1})E_{\lambda_\nu}(f)_{\mathbf{M}}. \tag{4.12}$$

If, in addition, the Fourier exponents λ_ν, $\nu \in \mathbb{N}$, satisfy the condition

$$\lambda_{\nu+1} - \lambda_\nu \leq C, \quad \nu = 1, 2, \ldots, \tag{4.13}$$

with an absolute constant $C > 0$, then

$$\omega_\alpha\left(f, \frac{\pi}{\lambda_n}\right)_{\mathbf{M}} \leq C\alpha\left(\frac{\pi}{\lambda_n}\right)^\alpha \sum_{\nu=1}^{n} \lambda_\nu^{\alpha-1}E_{\lambda_\nu}(f)_{\mathbf{M}}. \tag{4.14}$$

5. Constructive characteristics of the classes of functions defined by the generalized moduli of smoothness. Let ω be the function (majorant) given on $[0, 1]$. For a fixed $\alpha > 0$, we set

$$B\mathcal{S}_{\mathbf{M}}H^\omega_\alpha = \left\{f \in B\mathcal{S}_{\mathbf{M}} : \quad \omega_\alpha(f, \delta)_{\mathbf{M}} = \mathcal{O}(\omega(\delta)), \quad \delta \to 0+\right\}. \tag{5.1}$$

Further, we consider the majorants $\omega(\delta)$, $\delta \in [0, 1]$, which satisfy the following conditions: 1) $\omega(\delta)$ is continuous on $[0, 1]$; 2) $\omega(\delta) \uparrow$; 3) $\omega(\delta) \neq 0$ for $\delta \in (0, 1]$; 4) $\omega(\delta) \to 0$ for $\delta \to 0$; as well as the condition

$$\sum_{v=1}^{n} \lambda_v^{s-1}\omega\left(\frac{1}{\lambda_v}\right) = \mathcal{O}\Big[\lambda_n^s\omega\left(\frac{1}{\lambda_n}\right)\Big], \tag{5.2}$$

where $s > 0$, and λ_ν, $\nu \in \mathbb{N}$, is a increasing sequence of positive numbers. In the case where $\lambda_\nu = \nu$, the condition (5.2) is the known Bari condition $(\mathcal{B}_s)$ (see, e.g., [9]).

Theorem 5.1. *Assume that the function $f \in B\mathcal{S}_{\mathbf{M}}$ has the Fourier series of the form* (2.1), *$\alpha > 0$ and the majorant ω satisfies the conditions* 1 – 4.

1. *If $f \in B\mathcal{S}_{\mathbf{M}}H^\omega_\alpha$, then the following relation is true*:

$$E_{\lambda_n}(f)_{\mathbf{M}} = \mathcal{O}\left[\omega\left(\frac{1}{\lambda_n}\right)\right]. \tag{5.3}$$

2. *If the numbers λ_ν, $\nu \in \mathbb{N}$ satisfy condition* (4.13) *and the function ω satisfies condition* (5.2) *with $s = \alpha$, then relation* (5.3) *yields the inclusion $f \in B\mathcal{S}_{\mathbf{M}}H^\omega_\alpha$.*

Proof. Let $f \in B\mathcal{S}_{\mathbf{M}}H^{\omega}_{\alpha}$. Then relation (5.3) follows from (5.1) and (3.13).

On the other hand, if $f \in B\mathcal{S}_{\mathbf{M}}$, the numbers λ_{ν}, $\nu \in \mathbb{N}$ satisfy condition (4.13) and the function ω satisfies condition (5.2) with $s = \alpha$, and relation (5.3) holds, then, by (4.14), we get

$$\omega_{\alpha}\left(f, \frac{1}{\lambda_n}\right)_{\mathbf{M}} \le \frac{C_1}{\lambda_n^{\alpha}} \sum_{\nu=1}^{n} \lambda_{\nu}^{\alpha-1} E_{\lambda_{\nu}}(f) \le \frac{C_1}{\lambda_n^{\alpha}} \sum_{\nu=1}^{n} \lambda_{\nu}^{\alpha-1} \omega\left(\frac{1}{\lambda_{\nu}}\right) = \mathcal{O}\Big[\omega\left(\frac{1}{\lambda_n}\right)\Big],$$

where $C_1 = \alpha(2\pi)^{\alpha} \cdot C$. Hence, the function f belongs to the set $B\mathcal{S}_{\mathbf{M}}H^{\omega}_{\alpha}$.

Theorem 5.1 is proved.

The function t^r, $0 < r \le \alpha$, satisfies condition (5.2) with $s = \alpha$. Hence, denoting by $B\mathcal{S}_{\mathbf{M}}H^{r}_{\alpha}$ the class $B\mathcal{S}_{\mathbf{M}}H^{\omega}_{\alpha}$ for $\omega(t) = t^r$ we establish the following statement.

***Corollary* 5.1.** *Let $f \in B\mathcal{S}_{\mathbf{M}}$ have the Fourier series of the form* (2.1), *$\alpha > 0$, $0 < r \le \alpha$ and condition* (4.13) *holds. The function f belongs to the set $B\mathcal{S}_{\mathbf{M}}H^{r}_{\alpha}$, iff the following relation is true*:

$$E_{\lambda_n}(f)_{\mathbf{M}} = \mathcal{O}(\lambda_n^{-r}).$$

In the spaces $\mathcal{S}^p$, for classical moduli of smoothness ω_m, Theorems 4.1 and 5.1 were proved in [27] and [1]. In the spaces $\mathcal{S}^p$, inequalities of the form (4.14) were also obtained in [29]. In spaces L_p of 2π-periodic Lebesgue summable with the pth degree functions, inequalities of the kind as (4.14) were obtained by M. Timan (see, for example, [30], Ch. 6, [31], Ch. 2). In the Musielak – Orlicz type spaces, inequalities of the kind as (4.1) were proved in [3].

References

1. F. G. Abdullayev, P. Özkartepe, V. V. Savchuk, A. L. Shidlich, *Exact constants in direct and inverse approximation theorems for functions of several variables in the spaces $\mathcal{S}^p$*, Filomat, **33**, № 5, 1471 – 1484 (2019).
2. F. Abdullayev, S. Chaichenko, M. Imash kyzy, A. Shidlich, *Direct and inverse approximation theorems in the weighted Orlicz-type spaces with a variable exponent*, Turkish J. Math., **44**, 284 – 299 (2020).
3. F. Abdullayev, S. Chaichenko, A. Shidlich, *Direct and inverse approximation theorems of functions in the Musielak – Orlicz type spaces*, Math. Inequal. Appl., **24**, № 2, 323 – 336 (2021).
4. F. Abdullayev, A. Serdyuk, A. Shidlich, *Widths of functional classes defined by majorants of generalized moduli of smoothness in the spaces $\mathcal{S}^p$*, Ukr. Math. J., **73**, № 6, 841 – 858 (2021).
5. F. Abdullayev, S. Chaichenko, M. Imashkyzy, A. Shidlich, *Jackson-type inequalities and widths of functional classes in the Musielak – Orlicz type spaces*, Rocky Mountain J. Math., **51**, № 4, 1143 – 1155 (2021).
6. V. A. Abilov, F. V. Abilova, *Problems in the approximation of 2π-periodic functions by Fourier sums in the space $L_2(2\pi)$*, Math. Notes, **76**, № 6, 749 – 757 (2004).
7. A. G. Babenko, *On exact constant in the Jackson inequality in L_2*, Math. Notes, **39**, № 5, 355 – 363 (1986).
8. V. F. Babenko, S. V. Savela, *Jackson – Stechkin-type inequalities for almost periodic functions*, Visn. Dnipropetrovsk Univ., **20**, № 6/1, 60 – 66 (2012).
9. N. K. Bari, S. B. Stechkin, *Best approximations and differential properties of two conjugate functions*, Trudy Mosk. Mat. Obshch., **5**, 483 – 522 (1956) (in Russian).
10. A. S. Besicovitch, *Almost periodic functions*, Dover Publ., Inc., New York (1955).
11. J. Boman, *Equivalence of generalized moduli of continuity*, Ark. Mat., **18**, 73 – 100 (1980).
12. E. A. Bredikhina, *Absolute convergence of Fourier series of almost periodic functions*, Dokl. Akad. Nauk SSSR, **179**, № 5, 1023 – 1026 (1968) (in Russian).
13. E. A. Bredikhina, *Almost periodic functions*, Encyclopedia Math., **4**, 543 – 545 (1984) (in Russian).
14. P. Butzer, R. Nessel, *Fourier analysis and approximation*, One-Dimensional Theory, Birkhäuser, Basel (1971).
15. S. Chaichenko, A. Shidlich, F. Abdullayev, *Direct and inverse approximation theorems of functions in the Orlicz type spaces $\mathcal{S}_M$*, Math. Slovaca, **69**, № 6, 1367 – 1380 (2019).
16. N. I. Chernykh, *On the Jackson inequality in L_2*, Proc. Steklov Inst. Math., **88**, 75 – 78 (1967) (in Russian).

17. N. I. Chernykh, *On the best approximation of periodic functions by trigonometric polynomials in L_2*, Mat. Zametki, **20**, № 3, 513 – 522 (1967) (in Russian).
18. G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge Univ. Press (1934).
19. A. I. Kozko, A. V. Rozhdestvenskii, *On Jackson's inequality for a generalized modulus of continuity in L_2*, Sb. Math., **195**, № 8, 1073 – 1115 (2004).
20. B. M. Levitan, *Almost periodic functions*, Gosudarstv., Izdat. Techn.-Teor. Lit., Moscow (1953) (in Russian).
21. J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces I: Sequence spaces*, Berlin (1977).
22. J. Musielak, *Orlicz spaces and modular spaces*, Springer, Berlin (1983).
23. Ya. G. Pritula, *Jackson's inequality for B^2-almost periodic functions*, Izv. Vyssh. Uchebn. Zaved. Mat., **8**, 90 – 93 (1972) (in Russian).
24. Ya. G. Pritula, M. M. Yatsymirskyi, *Estimates of approximations of B^2-almost periodic functions*, Visn. L'viv. Univ. Ser. Mekh.-Mat., **21**, 3 – 7 (1983) (in Ukrainian).
25. M. M. Rao, Z. D. Ren, *Applications of Orlicz spaces*, Marcel Dekker Inc., New York, Basel (2002).
26. A. S. Serdyuk, A. L. Shidlich, *Direct and inverse theorems on the approximation of almost periodic functions in Besicovitch – Stepanets spaces*, Carpathian Math. Publ., **13**, № 3, 687 – 700 (2021) (see also arXiv preprint, arXiv: 2105.06796).
27. A. I. Stepanets, A. S. Serdyuk, *Direct and inverse theorems in the theory of the approximation of functions in the space $\mathcal{S}^p$*, Ukr. Math. J., **54**, № 1, 126 – 148 (2002).
28. A. I. Stepanets, *Methods of approximation theory*, VSP, Leiden, Boston (2005).
29. M. D. Sterlin, *Exact constants in inverse theorems of approximation theory*, Sov. Math. Dokl., **13**, 160 – 163 (1972).
30. A. F. Timan, *Theory of approximation of functions of a real variable*, Fizmatgiz, Moscow (1960) (in Russian).
31. M. F. Timan, *Approximation and properties of periodic functions*, Nauk. dumka, Kiev (2009) (in Russian).
32. S. B. Vakarchuk, *Jackson-type inequalities with generalized modulus of continuity and exact values of the n-widths for the classes of (ψ, β)-differentiable functions in L_2. I*, Ukr. Math. J., **68**, № 6, 823 – 848 (2016).
33. S. B. Vakarchuk, V. I. Zabutnaya, *Inequalities between best polynomial approximations and some smoothness characteristics in the space L_2 and widths of classes of functions*, Math. Notes, **99**, № 2, 222 – 242 (2016)
34. S. N. Vasil'ev, *The Jackson – Stechkin inequality in $L_2[-\pi, \pi]$*, Proc. Steklov Inst. Math., Suppl., **1**, S243-S253 (2001) (in Russian).
35. V. R. Voitsekhivs'kyj, *Jackson type inequalities in approximation of functions from the space $\mathcal{S}^p$*, Approx. Theory and Relat. Top., Proc. Inst. Math. Nat. Acad. Sci. Ukraine, **35**, 33 – 46 (2002) (in Ukrainian).

Received 14.12.21