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b-GENERALIZED DERIVATIONS ON PRIME RINGS*

b-УЗАГАЛЬНЕНІ ПОХІДНІ НА ПРОСТИХ КІЛЬЦЯХ

Let R be a prime ring with center $Z(R)$, right Martindale quotient ring Q and extended centroid C. By a b-generalized derivation we mean an additive mapping $g : R \to Q$ such that $g(xy) = g(x)y + bxd(y)$ for all $x, y \in R$, where $b \in Q$ and $d : R \to Q$ is an additive map. In this paper, we extend some well-known results concerning (generalized) derivations on prime rings to b-generalized derivations. Further, we investigate b-generalized derivation acting as a homomorphism or anti-homomorphism in a prime ring.

Нехай R — просте кільце з центром $Z(R)$, правим фактор-кільцем Мартіндейла Q та розширеним центроїдом C. Під b-узагальненою похідною ми розуміємо адитивне відображення $g : R \to Q$, для якого $g(xy) = g(x)y + bxd(y)$ для всіх $x, y \in R$, де $b \in Q$ і $d : R \to Q$ — адитивне відображення. У цій роботі деякі результати, які добре відомі для (узагальнених) похідних простих кілець, поширено на b-узагальнені похідні. Також вивчається те, як діє b-узагальнена похідна у простому кільці з точки зору гомоморфізму або антигомоморфізму.

1. Introduction. Throughout this paper, unless specially stated, R is an associative prime ring with center $Z(R)$, right Martindale quotient ring Q, extended centroid C and central closure $R_C = RC$. We refer the readers to the book [6] for definitions, axiomatic formulations and properties of these rings. I_d denotes the identity map from R to R (or to Q) defined by $I_d(x) = x$ for all $x \in R$. An additive mapping $d \colon R \to R$ is called a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A generalized derivation of R is defined as an additive mapping $F \colon R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, where d is a derivation of R. If $d = 0$, then we have $F(xy) = F(x)y$ for all $x, y \in R$, which is called a left multiplier mapping of R. In [14], Koşan and Lee propose the following new definition. An additive map $g \colon R \to Q$ is called a (left) b-generalized derivation of R with the associated map d if $g(xy) = g(x)y + bxd(y)$ for all $x, y \in R$, where $b \in Q$ and $d \colon R \to Q$ is an additive map. In this paper they aim to give a complete characterization of b-generalized derivations having nilpotent values with bounded index, also in the same paper Koşan and Lee proved that if R is a prime ring and $b \neq 0$, then the associated map d is a derivation. It is clear that a generalized derivation is a 1-generalized derivation. Conversely, let g is a b-generalized derivation of a prime ring R with the associated map d, if $b = 0$ or $d = 0$ in a definition of g, then g is a generalized derivation of R which is called a left multiplier mapping of R. Let α be an automorphism of R. An additive map $d \colon R \to R$ is called a skew derivation of R if $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in R$. An additive mapping $g \colon R \to R$ is said to be a generalized skew derivation of R if there exists skew derivation d of R with associated automorphism α such that $g(xy) = g(x)y + \alpha(x)d(y)$ for all $x, y \in R$, d is said to be an associated skew derivation of g and α is called an associated automorphism of g. Any mapping of R with form $g(x) = ax + \alpha(x)b$ for some $a, b \in R$ and $\alpha \in \mathrm{Aut}(R)$ is called inner generalized skew derivation. If g is a generalized skew derivation of R with the associated skew derivation d of R and α is X-inner with $\alpha(x) = qxq^{-1}$ for all $x \in R$, where $q \in Q$, then it is

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clear that g is a q-generalized derivation of R with the associated map $q^{-1}d$. From these facts we see that the definition of b-generalized derivation on a prime ring covers the concepts of generalized derivation and inner generalized skew derivation with an inner associated automorphism. Let S be a nonempty subset of a ring R. An additive mapping $F\colon R \to R$ is said to be a homomorphism (resp., anti-homomorphism) acting on S if $F(xy) = F(x)F(y)$ for all $x, y \in S$ (resp., $F(xy) = F(y)F(x)$ for all $x, y \in S$). Some well-known results concerning (generalized) derivations on prime rings have been obtained by a number of authors in literature. For example, in [2] the authors investigated the following identities and they characterized the structure of rings and maps: (i) $d\big([x, y]\big) = 0$, (ii) $d\big([x, y]\big) = [x, y]$, (iii) $d\big([x, y]\big) = -[x, y]$, (iv) $d(xy) = xy$ and (v) $d(xy) = -xy$ for all $x, y \in R$. Then, in [3], Argaç proved that if I is a nonzero ideal of a semiprime ring R and d be a derivation on R then one of the following conditions holds: (i) $d\big([x, y]\big) = [x, y]$, (ii) $d\big([x, y]\big) = -[x, y]$, (iii) either $d\big([x, y]\big) = [x, y]$ or $d\big([x, y]\big) = -[x, y]$ for all $x, y \in R$, then d is commuting on I. In recent years, a number of articles were discussed related problems in the context of prime and semiprime rings (see [1, 12, 17]). Moreover, in 1989, Bell and Kappe [7] showed that if a derivation of a prime ring R acts as a homomorphism or as an anti-homomorphism on a nonzero right ideal of R, then the derivation must be zero. Then Asma et al. [5] extended this result to Lie ideals of 2-torsion free prime rings. More precisely, they proved that if L is a noncentral Lie ideal of R, d is a derivation of R such that $u^2 \in L$, for all $u \in L$ and d acts as a homomorphism or as an anti-homomorphism on L, then $d = 0$. Further, Wang and You [18] eliminated the condition that $u^2 \in L$ for all $u \in L$ in their hypothesis and they proved that if R be a 2-torsion free prime ring, L is a nonzero Lie ideal of R and d is a derivation of R acts as a homomorphism or as an anti-homomorphism on L, then either $d = 0$ or $L \subseteq Z(R)$. Recently, Albaş and Argaç [2] extended Bell and Kappe's result to generalized derivations as follows:

Let R be a prime ring. Suppose $d\colon R \to R$ is a nonzero generalized derivation of R associated with a derivation α.

(i) If d acts as a homomorphism on R and if $d \neq 0$, then $d = I_d$.

(ii) If d acts as an anti-homomorphism on R and if $d \neq 0$, then $d = I_d$.

More recently, Dhara [11] extended Albaş and Argaç's result to semiprime rings and he showed that if F is a nonzero generalized derivation of a semiprime ring R associated with a derivation d satisfying $F(xy) = F(x)F(y)$ for all $x, y \in I$, a nonzero ideal of R, then $d(I) = 0$ and F is a commuting left multiplier mapping on I. In particular, he proved that if R is a prime ring, then $d = 0$ and F is identity mapping of R. Moreover, he showed that if $F(xy) = F(y)F(x)$ for all $x, y \in I$, then $d(I) = 0$ or R contains a nonzero central ideal. In particular, he proved that if R is a prime ring, then R is commutative and F is left multiplier mapping of R.

In this paper, by motivating above results, we investigate a b-generalized derivation g on a prime ring R satisfying any one of the following identities: (i) $g\big([x, y]\big) = 0$, (ii) $g\big([x, y]\big) = \mp[x, y]$, (iii) $g(xy) = \mp xy$, (iv) $g(xy) = \mp yx$, (v) $g(xy) \mp xy \in Z(R)$, (vi) $g(xy) \mp yx \in Z(R)$, (vii) $g(xy) = g(x)g(y)$ and (viii) $g(xy) = g(y)g(x)$ for all $x, y \in R$.

We recall some well-known lemmas which will be used in the sequel. The following first lemma is obtained from [9, 15].

Lemma 1.1. *If R is a prime ring, U its maximal right quotient ring and I a two-sided ideal of R, then*

(i) *I, R and U satisfy the same generalized polynomial identity with coefficients in U* [9] (*Theorem* 2);

(ii) *I, R and U satisfy the same differential identities* [15] (*Theorem* 2).

Lemma 1.2 ([8], Proposition 8)**.** *Let R be a prime ring with right Martindale quotient ring Q and central closure $R_C = RC$. Suppose that*

$$\sum_{j=1}^{n} f_j(z)xa_j + \sum_{i=1}^{k} c_i z h_i(x) = 0 \quad \textit{for all} \quad x, z \in R,$$

where $a_j, c_i \in R$ and $f_j \colon R \to R_C$, $h_i \colon R \to R_C$ are any maps. If the sets $\{a_1, \ldots, a_n\}$ and $\{c_1, \ldots, c_k\}$ are C-independent, then there exist $q_{ij} \in Q(R_C)$, $i = 1, \ldots, k$, $j = 1, \ldots, n$, such that

$$f_j(z) = -\sum_{i=1}^{k} c_i z q_{ij}, \qquad h_i(x) = \sum_{j=1}^{n} q_{ij} x a_j$$

for all $x, z \in R$, $i = 1, \ldots, k$, $j = 1, \ldots, n$.

Lemma 1.3 ([13], Lemma 2)**.** *Let R be a prime ring with right Martindale quotient ring Q and central closure $R_C = RC$. If $f \colon R \to R_C$ be an additive map satisfying $f(xy) = f(x)y$ for all $x, y \in R$, then there exists $q \in Q(R_C)$ such that $f(x) = qx$ for all $x \in R$.*

Lemma 1.4 [6]**.** *Let R be a prime ring, $Z(R)$ the center of R and $a, b \in R$. If $a \in Z(R)$ and $ab \in Z(R)$, then either $a = 0$ or $b \in Z(R)$.*

2. Main results. Now we start by giving the necessary remark that will be used in the proof of some our main results. The proof of remark is obviously but to keep the integrity we prove the following.

***Remark* 2.1.** Let R be a prime ring with the right Martindale quotient ring Q and $b \in Q$. If g is a b-generalized derivation of R with the associated derivation d, then the map $f = -g \colon R \to Q$ defined by $(-g)(x) = -g(x)$ for all $x \in R$ is also a b-generalized derivation of R with the associated map $-d$.

Proof. By the hypothesis, we know that $g(xy) = g(x)y + bxd(y)$ for all $x, y \in R$. Moreover, it is clear that since d is a derivation we have $(-d)(xy) = -d(xy) = -d(x)y - xd(y) = (-d)(x)y + x(-d)(y)$ for all $x, y \in R$. This implies that $-d$ is also a derivation of R. So, we obtain $(-g)(xy) = -g(xy) = -g(x)y - bxd(y) = (-g)(x)y + bx(-d)(y)$ for all $x, y \in R$, which shows $f = -g$ is also b-generalized derivation of R with the associated derivation $-d$.

The following theorem is a generalization of Theorem 3.1 of [2].

Theorem 2.1. *Let R be a noncommutative prime ring, Q the right Martindale quotient ring of R, C the extended centroid of R and $b \in Q$. Suppose that g is a b-generalized derivation of R such that $g\big([x, y]\big) = 0$ for all $x, y \in R$. Then $g = 0$.*

Proof. Since g is a b-generalized derivation of R, for all $x, y \in R$ and $b \in Q$, we have $g(xy) = g(x)y + bxd(y)$. We first assume that either $b = 0$ or $d = 0$, then we get $g(xy) = g(x)y$ for all $x, y \in R$. By Lemma 1.3, we obtain g is a generalized derivation of R. In view of [2] (Theorem 3.1), $g = 0$, as desired. Thus, from now on, we assume that both $b \neq 0$ and $d \neq 0$. By the hypothesis, $g\big([x, y]\big) = 0$ for all $x, y \in R$. Taking yx instead of y in the last relation and then using the hypothesis, we have $0 = g\big([x, yx]\big) = g([x, y]x) = g\big([x, y]\big)x + b[x, y]d(x) = b[x, y]d(x)$ for all $x, y \in R$. Hence, we obtain

$$b[x, y]d(x) = 0 \tag{2.1}$$

for all $x, y \in R$. Replacing y by yr in (2.1), we get

$$b[x, y]rd(x) + by[x, r]d(x) = 0 \tag{2.2}$$

for all $r, x, y \in R$. By Lemma 1.1 (ii), since R and Q satisfy the same differential identity, thus Q satisfies the relation (2.2). By taking b instead of y in (2.2), we have

$$b[x, b]rd(x) + b^2[x, r]d(x) = 0 \tag{2.3}$$

for all $r, x \in Q$. Comparing the relations (2.1) and (2.3), we arrive at

$$b[x, b]rd(x) = 0$$

for all $r, x \in Q$. By the primeness of Q, we have for each $x \in Q$ either $b[x, b] = 0$ or $d(x) = 0$. Let $H = \{x \in Q : b[x, b] = 0\}$ and $K = \{x \in Q : d(x) = 0\} = \mathrm{Ker}(d)$. It is clear that $(H, +)$ and $(K, +)$ are two additive subgroups of $(Q, +)$ such that $(Q, +) = (H, +) \bigcup (K, +)$. Since a group cannot be the union of two proper subgroups, we get either $Q = H$ or $Q = K$. Since $d \neq 0$, we arrive at the relation $b[x, b] = 0$ for all $x \in Q$. It follows from [16] (Lemma 1) that $b \in C$ since $b \neq 0$. If $b \in C$, the relation (2.3) gives us $b^2[x, r]d(x) = 0$ for all $r, x \in Q$ and the last relation implies that $[x, y]rd(x) = 0$ for all $r, x, y \in Q$. The primeness of Q yields that for each $x \in Q$ we have either $[x, y] = 0$ or $d(x) = 0$. Let $H' = \{x \in Q : x \in C\}$ and $K' = \{x \in Q : d(x) = 0\} = \mathrm{Ker}(d)$. By the same process as above, we get either $x \in C$ for all $x \in Q$ that is Q is commutative or $d(x) = 0$ for all $x \in Q$. But, by the hypothesis and the initial assumption that $d \neq 0$, both cases give us contradictions.

The following two theorems are motivated by [2] (Theorem 3.2 and Corollary 3.3).

Theorem 2.2. *Let R be a noncommutative prime ring, Q the right Martindale quotient ring of R and $b \in Q$. Suppose that g is a nonzero b-generalized derivation of R. If $g\big([x, y]\big) = [x, y]$ for all $x, y \in R$ or $g\big([x, y]\big) = -[x, y]$ for all $x, y \in R$, then $g = I_d$ or $g = -I_d$, respectively.*

Proof. By the hypothesis, assume first that $g\big([x, y]\big) = [x, y]$ for all $x, y \in R$. Since g is a b-generalized derivation of R, for all $x, y \in R$ and $b \in Q$, we have $g(xy) = g(x)y + bxd(y)$. Let either $b = 0$ or $d = 0$, then we see that g is of the form $g(xy) = g(x)y$ for all $x, y \in R$. By Lemma 1.3, we get g is a generalized derivation of R. Hence, this fact implies that $g = I_d$ by [2] (Theorem 3.2), as asserted. Now, we may assume that both $b \neq 0$ and $d \neq 0$. Replacing y by yx in the assumption and using this, we have $[x, y]x = [x, yx] = g\big([x, yx]\big) = g([x, y]x) = g\big([x, y]\big)x + b[x, y]d(x)$ for all $x, y \in R$. The last relation implies that $b[x, y]d(x) = 0$ for all $x, y \in R$. By the proof of Theorem 2.1, we know that if $b[x, y]d(x) = 0$ for all $x, y \in R$, then we can get easily $[x, y] = 0$ for all $x, y \in R$ since $d \neq 0$. But this forces R to be commutative, a contradiction.

Now assume next that $g\big([x, y]\big) = -[x, y]$ for all $x, y \in R$. If either $b = 0$ or $d = 0$, then we get $g(xy) = g(x)y$ for all $x, y \in R$. By Lemma 1.3, we have g is a generalized derivation of R. According to [2] (Theorem 3.2), we get $g = -I_d$, as desired. Now we consider only the cases that $b \neq 0$ and $d \neq 0$. Repeating the same arguments as in the proof of the first assumption, we arrive at R is commutative, a contradiction.

***Corollary* 2.1.** *Let R be a noncommutative prime ring, Q the right Martindale quotient ring of R and $b \in Q$. Suppose that g is a nonzero b-generalized derivation of R. If either $g\big([x,y]\big) = [x,y]$ or $g\big([x,y]\big) = -[x,y]$ holds for all $x, y \in R$, then either $g = I_d$ or $g = -I_d$, respectively.*

Proof. By the hypothesis, we may assume that R admits a b-generalized derivation g such that $g\big([x,y]\big) = [x,y]$ or $g\big([x,y]\big) = -[x,y]$ for all $x, y \in R$. Therefore, for each $y \in R$ we consider two subsets $H_y = \big\{x \in R \colon g\big([x,y]\big) = [x,y]\big\}$ and $K_y = \big\{x \in R \colon g\big([x,y]\big) = -[x,y]\big\}$ of R. Then H_y and K_y are two additive subgroups of $(R,+)$ such that $(R,+) = (H_y,+) \bigcup (K_y,+)$ and since a group cannot be the union of two proper subgroups, we get for each $y \in R$ either $R = H_y$ or $R = K_y$. By the same process as above, we have either $R = \big\{y \in R \colon R = H_y\big\}$ or $R = \big\{y \in R \colon R = K_y\big\}$ and by using Theorem 2.2, we obtain required conclusion.

Theorem 2.3. *Let R be a noncommutative prime ring, Q the right Martindale quotient ring of R and $b \in Q$. Suppose that g is a nonzero b-generalized derivation of R. If one of the following conditions holds:*

(i) *$g(xy) = xy$ for all $x, y \in R$,*

(ii) *$g(xy) = -xy$ for all $x, y \in R$,*

(iii) *either $g(xy) = xy$ or $g(xy) = -xy$ for all $x, y \in R$,*

then either $g = I_d$ or $g = -I_d$.

Proof. (i) Assume that g is a nonzero b-generalized derivation of R satisfying that $g(xy) = xy$ for all $x, y \in R$. Moreover, by the hypothesis we have also $g(yx) = yx$ for all $x, y \in R$. Comparing the last relations gives us $g\big([x,y]\big) = [x,y]$ for all $x, y \in R$ and by using Theorem 2.2, we get $g = I_d$.

(ii) Assume next that g is a nonzero b-generalized derivation of R satisfying that $g(xy) = -xy$ for all $x, y \in R$. Using the same arguments in (i) and by using Theorem 2.2, we obtain $g = -I_d$ and so we have desired result.

(iii) For each $x \in R$, we set $H_x = \big\{y \in R \colon g(xy) = xy\big\}$ and $K_x = \big\{y \in R \colon g(xy) = -xy\big\}$. In the same way as in the proof of Corollary 2.1 and using (i), (ii), we have either $g = I_d$ or $g = -I_d$.

Theorem 2.4. *Let R be a prime ring, Q the right Martindale quotient ring of R and $b \in Q$. Suppose that g is a nonzero b-generalized derivation of R. If $g(xy) = \mp yx$ for all $x, y \in R$, then R is commutative and either $g = I_d$ or $g = -I_d$, respectively.*

Proof. Assume first that $g(xy) = yx$ for all $x, y \in R$. Replacing y by yz in the last relation, we have $g((xy)z) = z(xy)$ for all $x, y, z \in R$. On the other hand, we get $g(x(yz)) = (yz)x$ for all $x, y, z \in R$. Comparing the last two relations yields that $[zx, y] = 0$ for all $x, y, z \in R$. By Lemma 1.1 (i) yields that the last relation is also satisfied by Q. Replacing z by 1 in the last relation implies that $[x,y] = 0$ for all $x, y \in Q$ and this gives us Q is commutative, so is R.

Now we may assume that $g(xy) = -yx$ for all $x, y \in R$. By Remark 2.1, we know that if g is a b-generalized derivation of R with the associated derivation d, then $f = -g$ is also b-generalized derivation of R with the associated derivation $-d$. Thus, we have $f(xy) = (-g)(xy) = -g(xy) = -(-yx) = yx$ for all $x, y \in R$ and by the first assumption as above, we obtain R is commutative.

So, in both cases, we can easily see that R is commutative. If one can assume that $g(xy) = yx$ for all $x, y \in R$, since R is commutative, then we have $g(xy) = xy$ for all $x, y \in R$. By the definition of g, the last relation gives us $(g(x) - x)y + bxd(y) = 0$ for all $x, y \in R$. Replacing y by yz in the last relation and using this, we obtain $(g(x) - x)yz + bxd(y)z + bxyd(z) = 0$ for all $x, y, z \in R$. Comparing the last relations, we have $bxyd(z) = 0$ for all $x, y, z \in R$. By Lemma 1.1 (ii) yields that the last relation is also satisfied by Q. By the primeness of Q, we have either $b = 0$

or $d = 0$. In both cases, we get $(g(x) - x)y = 0$ for all $x, y \in R$, and since R is prime ring, we obtain $g = I_d$.

Now we assume that $g(xy) = -yx$ for all $x, y \in R$, since R is commutative, then we have $g(xy) = -xy$ for all $x, y \in R$. By the same process as above, we get $g = -I_d$.

The following theorem is a generalization of Theorem 2.1 of [4].

Theorem 2.5. *Let R be a prime ring, Q the right Martindale quotient ring of R, $Z(R)$ the center of R, C the extended centroid of R and $b \in Q$. Suppose that g is a nonzero b-generalized derivation of R. If $g(xy) \mp xy \in Z(R)$ for all $x, y \in R$, then R is commutative or either $g = I_d$ or $g = -I_d$, respectively.*

Proof. Let us consider that the b-generalized derivation g satisfies the condition $g(xy) - xy \in \in Z(R)$ for all $x, y \in R$. If we take either $b = 0$ or $d = 0$ in the definition of g, then we get $g(xy) = = g(x)y$ for all $x, y \in R$. Replacing y by yz in the hypothesis, we have $g((xy)z) - xyz \in Z(R)$ for all $x, y, z \in R$ and in this relation, using the fact that $g(xy) = g(x)y$ gives us $(g(xy) - xy)z \in Z(R)$ for all $x, y, z \in R$. By Lemma 1.4, we have either $g(xy) = xy$ for all $x, y \in R$ or $z \in Z(R)$ for all $z \in R$. If $z \in Z(R)$ for all $z \in R$, then R must be commutative and this is our desired result. So, we may assume that $g(xy) = xy$ for all $x, y \in R$. In this case, by using Theorem 2.3, we get $g = I_d$.

Now, we may assume that both $b \neq 0$ and $d \neq 0$. Replacing y by yz in the hypothesis, we obtain $g(xyz) - xyz \in Z(R)$ for all $x, y, z \in R$. The last relation yields that $(g(xy) - xy)z + bxyd(z) \in \in Z(R)$ for all $x, y, z \in R$. Commuting the last relation with z gives us that

$$0 = \big[z, bxyd(z)\big] \tag{2.4}$$

for all $x, y, z \in R$. Suppose first that $b \in C$. In this case by the relation (2.4), we have

$$0 = \big[z, xyd(z)\big] = xy[z, d(z)] + x[z, y]d(z) + [z, x]yd(z) \tag{2.5}$$

for all $x, y, z \in R$, since $b \neq 0$. Replacing x by rx in (2.5), we obtain $r(xy[z, d(z)] + x[z, y]d(z) + + [z, x]yd(z)) + [z, r]xyd(z) = 0$ for all $r, x, y, z \in R$. Combining the last relation with (2.5), we get $[z, r]xyd(z) = 0$ for all $r, x, y, z \in R$. By Lemma 1.1 (ii), we know that the last relation is also satisfied by Q. So, replacing y by 1 in this relation, we have $[z, r]xd(z) = 0$ for all $r, x, z \in Q$. By the primeness of Q, we get for each $z \in Q$ either $[r, z] = 0$ or $d(z) = 0$. For any $z \in Q$, we set $H = \{z \in Q : z \in C\}$ and $K = \{z \in Q : d(z) = 0\}$. Using the same process as above, since d is nonzero, we get $z \in C$ for all $z \in Q$, that is, $Q = C$ implying that Q is commutative, so is R, as asserted.

Now we can assume that $b \notin C$. Moreover, Lemma 1.1 (i) implies that the relation (2.4) is satisfied by Q. Replacing x by bx in (2.4) and using (2.4) yield that $0 = [z, b^2xyd(z)] = b[z, bxyd(z)] + + [z, b]bxyd(z)$ for all $x, y, z \in Q$. Comparing the last relation with (2.4), we obtain $[z, b]bxyd(z) = = 0$ for all $x, y, z \in Q$. Taking 1 instead of y in the last relation implies that $[z, b]bxd(z) = 0$ for all $x, z \in Q$. By the primeness of Q, we have for each $z \in Q$ either $[z, b]b = 0$ or $d(z) = 0$. Repeating the same process as above, we conclude that $[z, b]b = 0$ for all $z \in Q$ since $d \neq 0$. Further, replacing z by zr in the last relation gives us $[z, b]rb + z[r, b]b = 0$ for all $r, z \in Q$. Comparing the last two relations, we have $[z, b]rb = 0$ for all $r, z \in Q$. The primeness of Q implies that either $b \in C$ or $b = 0$, but both cases give a contradiction with the assumption of $b \notin C$.

Now, let us next consider that the b-generalized derivation g satisfies the condition $g(xy) + xy \in \in Z(R)$ for all $x, y \in R$ and this relation implies that $-(g(xy) + xy) = -g(xy) - xy \in Z(R)$. By the definition of $(-g)(x) = -g(x)$, we have $(-g)(xy) - xy \in Z(R)$ and by Remark 2.1, we know that when g is a b-generalized derivation of R with the associated derivation d, then $f = -g$ is also b-generalized derivation of R with the associated derivation $-d$. In this case the b-generalized derivation $f = -g$ satisfies the condition $f(xy) - xy \in Z(R)$ for all $x, y \in R$ and by the first part in the proof, we reach either R is commutative or $f = I_d$. By the definition of f, we have either R is commutative or $g = -I_d$ which proves the theorem.

Theorem 2.6. *Let R be a prime ring, Q the right Martindale quotient ring of R, $Z(R)$ the center of R, C the extended centroid of R and $b \in Q$. Suppose that g is a nonzero b-generalized derivation of R. If $g(xy) \mp yx \in Z(R)$ for all $x, y \in R$, then R is commutative.*

Proof. We assume first that $g(xy) - yx \in Z(R)$ for all $x, y \in R$. If we take either $b = 0$ or $d = 0$ in the definition of g, then we obtain $g(xy) = g(x)y$ for all $x, y \in R$ and by Lemma 1.3, there exists $a \in Q(R_C)$ such that $g(x) = ax$ for all $x \in R$. Using this form in the assumption, we have

$$axy - yx \in Z(R) \tag{2.6}$$

for all $x, y \in R$. Replacing x by xz in (2.6), we obtain that

$$axzy - yxz \in Z(R) \tag{2.7}$$

for all $x, y, z \in R$. Moreover, by (2.6) we have also $azy - yz \in Z(R)$ for all $y, z \in R$. Replacing y by yx in the last relation, we get

$$azyx - yxz \in Z(R) \tag{2.8}$$

for all $x, y, z \in R$. Comparing the relations (2.7) and (2.8) gives us

$$a[x, zy] \in Z(R) \tag{2.9}$$

for all $x, y, z \in R$. So, we have $a[x, z]y + az[x, y] \in Z(R)$ for all $x, y, z \in R$. Replacing y by yr in the last relation and using this, we arrive at $\big(a[x, z]y + az[x, y]\big)r + azy[x, r] \in Z(R)$ and commuting this relation with r, we have

$$\big[r, azy[x, r]\big] = 0 \tag{2.10}$$

for all $r, x, y, z \in R$. Lemma 1.1 (i) implies that the last relation is also satisfied by Q, so taking 1 instead of z in the last relation gives us that

$$\big[r, ay[x, r]\big] = 0 \tag{2.11}$$

for all $r, x, y \in Q$. Replacing y by yas in (2.11) and using this implies that $[r, ay]as[x, r] = 0$ for all $r, s, x, y \in Q$. By the primeness of Q, we get for each $r \in Q$ either $[r, ay]a = 0$ or $[x, r] = 0$. For any $r \in Q$, we set $H = \big\{r \in Q : [r, ay]a = 0\big\}$ and $K = \big\{r \in Q : [x, r] = 0\big\}$. Using the same process as above, we have either $Q = H$ or $Q = K$. If $Q = K$, then Q is commutative, so is R, as desired. Now we may assume that $Q = H$. In this case we get $[r, ay]a = 0$ for all $r, y \in Q$ and taking 1 instead of y in the last relation yields that

$$[r, a]a = 0 \tag{2.12}$$

for all $r \in Q$. Replacing r by rs in (2.12) and using this implies that

$$[r,a]sa = 0 \tag{2.13}$$

for all $r, s \in Q$ and taking sr instead of s in (2.13)

$$[r,a]sra = 0 \tag{2.14}$$

for all $r, s \in Q$. On the other hand, by the right multiplying (2.13) by r, we get

$$[r,a]sar = 0 \tag{2.15}$$

for all $r, s \in Q$. Comparing the relations (2.14) and (2.15) gives us that $[r,a]s[r,a] = 0$ for all $r, s \in Q$. By the primeness of Q, we have $a \in C$. By Lemma 1.4 and (2.9), we have either $a = 0$ or $[x, zy] \in Z(R)$ for all $x, y, z \in R$. If $a = 0$, from (2.6), we can easily see that R is commutative, desired result. If $[x, zy] \in Z(R)$ for all $x, y, z \in R$, then it is clear that R is commutative.

Now, we may assume that both $b \neq 0$ and $d \neq 0$. Replacing y by yz in the assumption, we get $g((xy)z) - yzx \in Z(R)$ for all $x, y, z \in R$. Using the definition of g in the last relation implies that $(g(xy) - yx)z + bxyd(z) + y[x, z] \in Z(R)$ for all $x, y, z \in R$. Commuting the last relation with z and using the assumption gives us that $[z, bxyd(z) + y[x, z]] = 0$ for all $x, y, z \in R$ and by Lemma 1.1 (ii) the last relation is also satisfied by Q, so taking 1 instead of x in the last relation yields that

$$\big[z, byd(z)\big] = 0 \tag{2.16}$$

for all $y, z \in Q$. Replacing y by ybr in (2.16) and using this implies that $[z, by]brd(z) = 0$ for all $r, y, z \in Q$. Taking 1 instead of y in the last relation, we have $[z, b]brd(z) = 0$ for all $r, z \in Q$. By the primeness of Q, we get for each $z \in Q$ either $[z, b]b = 0$ or $d(z) = 0$. For any $z \in Q$, we set $H = \{z \in Q \colon [z, b]b = 0\}$ and $K = \{z \in Q \colon d(z) = 0\}$. Using the same process as above, we have either $Q = H$ or $Q = K$. Since $d \neq 0$, we get $[z, b]b = 0$ for all $z \in Q$. Replacing z by zr in the last relation and using this yields that $[z, b]rb = 0$ for all $r, z \in Q$. By the primeness of Q and since $b \neq 0$, we obtain that $b \in C$. Using this fact in (2.16), we have $[z, yd(z)] = 0$ for all $y, z \in Q$. Replacing y by ys in the last relation and using this gives us that $[z, y]sd(z) = 0$ for all $s, y, z \in Q$. By the primeness of Q, we get for each $z \in Q$ either $[z, y] = 0$ or $d(z) = 0$. By the same process as above, since $d \neq 0$, we can easily see that R is commutative.

Now, let us next consider that the b-generalized derivation g satisfies the condition $g(xy) + yx \in Z(R)$ for all $x, y \in R$ and this relation implies that $-(g(xy) + yx) = -g(xy) - yx \in Z(R)$. By the definition of $(-g)(x) = -g(x)$, we have $(-g)(xy) - yx \in Z(R)$ and by Remark 2.1, we know that when g is a b-generalized derivation of R with the associated derivation d, then $f = -g$ is also b-generalized derivation of R with the associated derivation $-d$. In this case the b-generalized derivation $f = -g$ satisfies the condition $f(xy) - yx \in Z(R)$ for all $x, y \in R$ and by the first part in the proof, we have R is commutative, as asserted.

In addition, the following proof will be given as an alternative to the proof of Theorem 2.6. In this proof, it is shown that the chosen map is sufficient to be only an additive map and in the case when this map is not a b-generalized derivation, if the identity is hold, then R must be a commutative ring.

Alternative proof. Let us consider that the b-generalized derivation g satisfies the condition $g(xy) - yx \in Z(R)$ for all $x, y \in R$. Replacing y by yz in the last relation, we have $g(x(yz)) - (yz)x \in Z(R)$ for all $x, y, z \in R$. On the other hand, by the assumption we get $g(xz) - zx \in Z(R)$ for all $x, z \in R$. Replacing x by xy in the last relation, we obtain $g((xy)z) - z(xy) \in Z(R)$ for all $x, y, z \in R$. Comparing the last two relation yields that $[zx, y] \in Z(R)$ for all $x, y, z \in R$ and by Lemma 1.1 (i) gives us that $[zx, y] \in Z(Q) = C$ for all $x, y, z \in Q$. Replacing z by 1 in the last relation implies that $[x, y] \in C$ for all $x, y \in Q$. For each $x \in Q$, we can consider the last relation as $d_x(y) = [x, y] \in C$ for all $y \in Q$. By [16] (Theorem 2), we get either for each $x \in Q$, $d_x(y) = [x, y] = 0$ for all $y \in Q$ or Q is commutative. It is clear that two cases yield that Q is commutative, so is R, as desired.

Now, let us next consider that the b-generalized derivation g satisfies the condition $g(xy) + yx \in \in Z(R)$ for all $x, y \in R$ and this relation implies that $-(g(xy) + yx) = -g(xy) - yx \in Z(R)$. By the definition of $(-g)(x) = -g(x)$, we have $(-g)(xy) - yx \in Z(R)$ and by Remark 2.1, we know that when g is a b-generalized derivation of R with the associated derivation d, then $f = -g$ is also b-generalized derivation of R with the associated derivation $-d$. In this case the b-generalized derivation $f = -g$ satisfies the condition $f(xy) - yx \in Z(R)$ for all $x, y \in R$ and by the first part of the proof, we have that R is commutative, as asserted.

The following theorems may be considered as a generalization of Theorem 3.4 of [2].

Theorem 2.7. *Let R be a prime ring, Q the right Martindale quotient ring of R, C the extended centroid of R and $b \in Q$. Suppose that g is a nonzero b-generalized derivation of R such that $g(xy) = g(x)g(y)$ for all $x, y \in R$. Then either there exists $q \in Q(R_C)$ such that $g(x) = -bxq$, $d(x) = [q, x]$ for all $x \in R$ with $qb = -1$ or g is the identity map of R.*

Proof. Since g is a b-generalized derivation of R, for all $x, y \in R$ and $b \in Q$, we have $g(xy) = g(x)y + bxd(y)$. Let either $b = 0$ or $d = 0$, then we get $g(xy) = g(x)y$ for all $x, y \in R$. By Lemma 1.3, we get g is a generalized derivation of R. According to [2] (Theorem 3.4), we obtain g is the identity map of R, as desired. Now, we can assume that both $b \neq 0$ and $d \neq 0$. Replacing y by yz in the hypothesis yields that

$$g(xyz) = g\big((xy)z\big) = g(xy)z + bxyd(z) = g(x)g(y)z + bxyd(z) \tag{2.17}$$

for all $x, y, z \in R$. On the other hand,

$$g(xyz) = g\big((xy)z\big) = g(x)g(yz) = g(x)g(y)z + g(x)byd(z) \tag{2.18}$$

for all $x, y, z \in R$. Comparing the relations (2.17) and (2.18) gives us

$$(g(x)b - bx)Rd(z) = 0 \tag{2.19}$$

for all $x, z \in R$ and by Lemma 1.1 (ii) implies that Q also satisfies (2.19). So, by the primeness of Q and by assuming that $d \neq 0$, we get

$$g(x)b = bx \tag{2.20}$$

for all $x \in Q$. If $b \in C$, then by the fact that $b \neq 0$, the relation (2.20) implies that g is the identity map on R, as desired. So, we may assume that $b \notin C$. Replacing x by xy in (2.20) yields

$$g(x)yb + bx(d(y)b - y) = 0 \tag{2.21}$$

for all $x, y \in Q$. Setting

$$F_1(x) = g(x), \qquad H_1(y) = d(y)b - y, \qquad a_1 = b, \qquad c_1 = b,$$

we see that (2.21) is of the form

$$\sum_{j=1}^{1} F_j(x)ya_j + \sum_{i=1}^{1} c_i x H_i(y) = 0.$$

Considering the set $\{b\}$ is linearly C-independent and using Lemma 1.2 insure us the existence of element $q_{ij} \in Q(R_C)$, $i, j = 1$, such that

$$F_j(x) = -\sum_{i=1}^{1} c_i x q_{ij}, \qquad H_i(y) = \sum_{j=1}^{1} q_{ij} y a_j.$$

Then we have

$$g(x) = -bxq_{11} \tag{2.22}$$

and

$$d(y)b - y = q_{11}yb \tag{2.23}$$

for all $x, y \in Q$. Setting $q_{11} = q \in Q(R_C)$ and replacing x by xy in (2.22), we have $g(x)y + bxd(y) = -bxyq$ for all $x, y \in Q$. Using the relation (2.22) in the last relation, we get $bx(d(y) + yq - qy) = 0$ for all $x, y \in Q$. By the primeness of Q and by the fact that $b \neq 0$, we obtain $d(y) = [q, y]$ for all $y \in Q$. Using the last relation in (2.23), we have $y(qb + 1) = 0$ for all $y \in Q$ and again by the primeness of Q, we conclude that $qb = -1$. Consequently, there exists $q \in Q(R_C)$ such that $g(x) = -bxq$, $d(x) = [q, x]$ for all $x \in R$ with $qb = -1$, this completes the proof.

Theorem 2.8. *Let R be a prime ring of characteristic different from 2, Q the right Martindale quotient ring of R, C the extended centroid of R and $b \in Q$. Suppose that g is a nonzero b-generalized derivation of R such that $g(xy) = g(y)g(x)$ for all $x, y \in R$. Then R is commutative and g is the identity map of R.*

Proof. Since g is a b-generalized derivation of R, for all $x, y \in R$ and $b \in Q$, we have $g(xy) = g(x)y + bxd(y)$. Firstly, if either $b = 0$ or $d = 0$, then we get $g(xy) = g(x)y$ for all $x, y \in R$. By Lemma 1.3, we obtain that g is a generalized derivation of R. By [2] (Theorem 3.4), we have that g is the identity map of R and, using this fact in the hypothesis, we obtain $xy - yx = 0$ for all $x, y \in R$, which implies that R is commutative, this proves the theorem.

So, we can assume that both $b \neq 0$ and $d \neq 0$. Expanding $g(xyy)$ by using the definition of b-generalized derivation and then using the hypothesis we have $g(xyy) = g((xy)y) = g(xy)y + bxyd(y) = g(y)g(x)y + bxyd(y)$ for all $x, y \in R$. On the other hand, we also get $g(xyy) = g((xy)y) = g(y)g(xy) = g(y)g(x)y + g(y)bxd(y)$ for all $x, y \in R$. Comparing the last two relations, we see that

$$bxyd(y) = g(y)bxd(y) \tag{2.24}$$

for all $x, y \in R$. If $b \in C$, then, by the relation (2.24), we obtain

$$xyd(y) = g(y)xd(y) \tag{2.25}$$

for all $x, y \in R$ since $b \neq 0$. Replacing x by tx in (2.25), we arrive at $txyd(y) = g(y)txd(y)$ for all $t, x, y \in R$. Also left multiplying (2.25) by t, we get $txyd(y) = tg(y)xd(y)$ for all $t, x, y \in R$. Comparing the last two relations, we conclude that Q satisfies $[g(y), t]xd(y) = 0$, by Lemma 1.1 (ii). By the primeness of Q, we get for each $y \in Q$ either $[g(y), t] = 0$ or $d(y) = 0$. Let $H=\{y \in Q : g(y) \in C\}$ and $K=\{y \in Q : d(y) = 0\}$=Ker(d). It is clear that $(H, +)$ and $(K, +)$ are two additive subgroups of $(Q, +)$ such that $(Q, +) = (H, +) \bigcup (K, +)$. By the same process as above, we conclude that $g(y) \in C$ for all $y \in Q$ since $d \neq 0$. Replacing y by xy in the last relation, we get

$$g(xy) = g(x)y + bxd(y) \in C \tag{2.26}$$

for all $x, y \in Q$. Also replacing y by yz in (2.26), we obtain

$$(g(x)y + bxd(y))z + bxyd(z) \in C \tag{2.27}$$

for all $x, y, z \in Q$. Commuting the relation (2.27) with z and using (2.26) gives us

$$\big[z, bxyd(z)\big] = 0$$

for all $x, y, z \in Q$. By the proof of Theorem 2.5, we know that if $\big[z, bxyd(z)\big] = 0$ for all $x, y, z \in Q$, then we can get easily Q is commutative since $d \neq 0$ and this insures us the commutativity of R. In this case by Theorem 2.7, we have either there exists $q \in Q(R_C)$ such that $g(x) = -bxq$ for all $x \in R$ with $qb = -1$ or g is the identity map of R. If the second case occurs there is nothing to be proved. So, we may assume that the first case occurs. In this case since R is commutative and $qb = -1$, we have $g(x) = -bxq = -qbx = x$ for all $x \in R$, so, R is commutative and g is the identity map of R.

Now, we may assume that $b \notin C$. Replacing x by $xd(y)$ in (2.24), we have $bxd(y)yd(y) = g(y)bxd(y)d(y) = bxyd(y)d(y)$ for all $x, y \in R$ and Lemma 1.1 (ii) implies that the last relation is also satisfied by Q. Using (2.24) in the last relation yields that $bx(d(y)yd(y) - yd(y)d(y)) = 0$ for all $x, y \in Q$. The fact of the primeness of Q and $b \neq 0$ force us to conclude that $[y, d(y)]d(y) = 0$ for all $y \in Q$. By [10] (Theorem 2) we get either Q is commutative or there exists $\lambda \in C$ such that $d(x) = \lambda x$ for all $x \in Q$. If Q is commutative, by Theorem 2.7, we get the required result. So, we may assume that $d(x) = \lambda x$ for all $x \in Q$, but in this case, it is clear that if the derivation d is of the form $d(x) = \lambda x$ for all $x \in Q$, then it must be zero, but this contradicts with $d \neq 0$.

***Example* 2.1.** Consider F is a field with characteristic 2.

Suppose that $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}$. Define $g\colon R \to R$ as $g(x) = e_{11}x + e_{11}xe_{22}$ for all $x \in R$. One can be easily shown that g is a nonzero b-generalized derivation on R. It is easy to verify that g satisfies $g(xy) = g(y)g(x)$ for all $x, y \in R$, but neither R is commutative, nor g is the identity map of R. In this example we see that the primeness and characteristic hypotheses are essential in Theorem 2.8.

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