DOI: 10.37863/umzh.v74i6.2357

UDC 512.5

M. Bibak (Payame Noor Univ., Tehran, Iran),
G. H. Rezaeezadeh (Univ. Shahrekord, Iran)

CHARACTERIZATION BY ORDER AND DEGREE PATTERN OF THE SIMPLE GROUPS $O_8^-(q)$ FOR CERTAIN q

ХАРАКТЕРИСТИКА ПОРЯДКІВ ТА СТЕПЕНІВ ПРОСТИХ ГРУП $O_8^-(q)$ ДЛЯ ЗАДАНОГО q

In this paper, it is demonstrated that every finite group G with the same order and degree pattern as $O_8^-(q)$ for certain q is necessarily isomorphic to the group $O_8^-(q)$.

Доведено, що будь-яка скінченна група G, яка має ті ж самі порядок та степінь, що й група $O_8^-(q)$ для деякого q, необхідно має збігатися з $O_8^-(q)$.

1. Introduction. Let G be a finite group, $\pi(G)$ the set of all prime divisors of its order and $\pi_e(G)$ the spectrum of G, that is, the set of its element orders. The Gruenberg – Kegel graph $\Gamma(G)$ or prime graph of G is a simple graph with vertex set $\pi(G)$, in which two distinct vertices p and q are adjacent by an edge if and only if $pq \in \pi_e(G)$.

For the first time the concept of degree pattern of prime graph was defined in [7]. Let G be a finite group and $\pi(G) = \{p_1, p_2, \ldots, p_k\}$ with $p_1 < p_2 < \ldots < p_k$. If $\deg(p)$ of a vertex $p \in \pi(G)$ is the number of edges incident to p, then the degree pattern of G is defined as $D(G) = = \big(\deg(p_1), \deg(p_2), \ldots, \deg(p_k)\big)$. A finite group G is called k-fold OD-characterizable if there are exactly k nonisomorphic groups H such that $|H| = |G|$ and $D(H) = D(G)$. Usually a 1-fold OD-characterizable group is called an OD-characterizable group.

A characterization of the finite group G by degree pattern was defined in [7], in which the authors proved that all the sporadic simple groups, the alternating groups $\mathbb{A}_p$, where p and $p-2$ are prime numbers, and some simple groups of Lie type are OD-characterizable, however the projective symplectic group $SP_6(3)$ is 2-fold OD-characterizable. In [6, 8, 12], it is shown that some projective special linear groups are OD-characterizable. In [15], it is proved that the automorphism groups of orthogonal groups $O_{10}^+(2)$ and $O_{10}^-(2)$ are OD-characterizable. Also, in a series of papers [4, 5, 9, 10], the characterization by order and degree pattern for some finite almost simple groups has been studied (recall that a group G is an almost simple group, if $S \leq G \leq \mathrm{Aut}(S)$, for some non-Abelian simple group S). In this paper, we prove that $O_8^-(q)$ where $q \in \{3-5, 8, 9, 13\}$ is OD-characterizable.

Throughout this paper, we use the following definition and notions related to $\Gamma(G)$: A set of vertices of a graph is called independent if its elements are pairwise nonadjacened. We denote by $t(G)$ the maximal number of vertices in independent sets of $\Gamma(G)$ and by $t(r, G)$ the maximal number of vertices in independent sets of $\Gamma(G)$ containing a prime r. Denote by $s(G)$ the number of connected components of $\Gamma(G)$ and by $\pi_i = \pi_i(G)$, $i = 1, 2, \ldots, s(G)$, the ith connected component of $\Gamma(G)$. If $2 \in \pi(G)$ we always suppose $2 \in \pi_1$.

Also, we use the following notations. For $p \in \pi(G)$, we denote by $\mathrm{Syl}_p(G)$ and G_p the set of all Sylow p-subgroups of G and a Sylow p-subgroup of G, respectively. If p is a prime and

m be a natural number, then we write $|m|_p$ for the p-part of m, i.e., the highest power of p that divides m. Given a prime p, we denote by $\mathfrak{S}_p$ the set of all finite non-Abelian simple groups G such that $\max\pi(G) = p$. Note that the full list of all finite non-Abelian simple groups S in $\mathfrak{S}_p$ for $5 \leq p \leq 997$, has been determined in [16]. In this paper, we deal with the finite non-Abelian simple groups in $\mathfrak{S}_p$, where $p \in \{41, 193, 241, 257, 313, 631\}$, and for convenience, we list them in Table 2. The other unexplained notations are standard and refer to [11].

2. Preliminaries. In this section, we list some basic and known results that will be used.

Definition 2.1. *A group G is a 2-Frobenius group if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively.*

The structure of finite groups with nonconnected prime graph is described in the following lemma.

Lemma 2.1 (Gruenberg – Kegel theorem of [14])**.** *Let G be a finite group with $s(G) \geq 2$. Then one of the following statements holds*:

(a) *G is a Frobenius or a 2-Frobenius group*;

(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ where H is a nilpotent π_1-group, K/H is a non-Abelian simple group and G/K is a π_1-group such that $|G/K|$ divides $|\mathrm{Out}(K/H)|$. Moreover, each odd-order components of G is also an odd-order component of K/H.*

Lemma 2.2 (Corollary 3.8 of [1])**.** *Let G be a finite group with $n = |\pi(G)|$ and let $d_1 \leq \leq d_2 \leq \ldots \leq d_n$ be the degree sequence of $\Gamma(G)$. If $d_1 + d_{d_1+2} \leq n - 3$, then $t(G) \geq 3$.*

Lemma 2.3 (Lemma 2.8 of [6])**.** *Let $\Gamma(G)$ be the prime graph of G with exactly two vertices of degree 1. Then $t(G) \geq 3$, if one of the following statements holds*:

(1) *$|\pi(G)| = 6$ and $\Gamma(G)$ has at least two vertices of degree 2*;

(2) *$|\pi(G)| \geq 7$ and $\Gamma(G)$ has at least two vertices of degree 3.*

Lemma 2.4 [13]**.** *Let G be a finite group with $t(G) \geq 3$, $t(2, G) \geq 2$, and K be the maximal normal solvable subgroup of G. Then there exists a non-Abelian simple group S such that $S \leq \leq G/K \leq \mathrm{Aut}(S)$.*

Lemma 2.5 (Lemma 2.7 of [6])**.** *Let G be a finite group of even order with $t(G) \geq 3$. Then G is nonsolvable, and so it is not a 2-Frobenius group. If, moreover, $|G|_3 \neq 3$ or $|G|_5 \neq 5$, then G is not a Frobenius group.*

The following two lemmas give a complete description of the spectra of groups $O^-_{2n}(q)$ for all possible values q.

Lemma 2.6 (Corollaries 8 and 9 of [2])**.** *Let $O = O^{\varepsilon}_{2n}(q)$, where q be a power of an odd prime p, $n \geq 4$ and $\varepsilon \in \{+, -\}$. Moreover, assume that $d = (4, q^n - 1)$ and $c = \dfrac{d}{2}$. Then $\pi_e(O)$ consists of all divisors of the following numbers*:

(1) $\dfrac{q^n - \varepsilon}{d}$;

(2) $\dfrac{[q^{n_1} - \delta, q^{n_2} - \varepsilon\delta]}{e}$, *where $\delta \in \{+, -\}$, $n_1, n_2 > 0$, $n_1 + n_2 = n$; $e = 2$ if $(q^{n_1} - \delta)_2 = = (q^{n_2}\delta)_2$ and $e = 1$ otherwise*;

(3) $[q^{n_1} - \delta_1, q^{n_2} - \delta_2, \ldots, q^{n_s} - \delta_s]$, *where $s \geq 3$, $\delta_i \in \{+, -\}$, $n_i > 0$ for all $1 \leq i \leq s$, $n_1 + \ldots + n_s = n$ and $\delta_1\delta_2 \ldots \delta_s = \varepsilon$*;

(4) $p\left[q \pm 1, \dfrac{q^{n-2} + 1}{2}\right]$;

(5) $p[q \pm 1, q^{n_1} - \delta_1, q^{n_2} - \delta_2, \ldots, q^{n_s} - \delta_s]$, *where $s \geq 2$, $\delta_i \in \{+, -\}$, $n_i > 0$ for all $1 \leq i \leq s$, $n_1 + \ldots + n_s = n - 2$ and $\delta_1\delta_2 \ldots \delta_s = \varepsilon$*;

Table 1. *The order and degree pattern of simple groups $O_8^-(q)$ for certain q*

S	$\lvert S\rvert$	$D(S)$
$O_8^-(3)$	$2^{10}.3^{12}.5.7.13.41$	$(4,2,2,1,1,0)$
$O_8^-(4)$	$2^{24}.3^4.5^3.7.13.17.257$	$(3,5,5,2,3,2,0)$
$O_8^-(5)$	$2^{10}.3^4.5^{12}.7.13.31.313$	$(5,5,3,2,3,2,0)$
$O_8^-(8)$	$2^{36}.3^7.5.7^3.13.17.19.73.241$	$(2,6,3,6,3,1,2,2,1)$
$O_8^-(9)$	$2^{13}.3^{24}.5^3.7.13.17.41.73.193$	$(6,3,6,3,3,1,3,2,1)$
$O_8^-(43)$	$2^{10}.3^4.5^2.7^3.11^3.13.17.37.43^{12}.139.193.521.631$	$(9,9,6,9,9,9,5,2,6,5,2,2,4)$

(6) $p^l\dfrac{q^{n_1}\pm 1}{3}$, *where* $l>0$ *and* $p^{l-1}+3+2n_1=2n$;

(7) $p^L\big[q^{n_1}\pm 1,\ldots,q^{n_s}\pm 1\big]$, *where* $l>0$, $s\geq 2$ *and* $n_i>0$ *for all* $1\leq i\leq s$ *and* $p^{l-1}+3+2(n_1+n_2+\ldots+n_s)=2n$;

(8) p^l *if* $2n=p^{l-1}+3$ *for* $l>0$.

Lemma 2.7 (Corollary 4 of [2]). *Let* $O=O_{2n}^{\varepsilon}(q)$, *where* q *is even,* $n\geq 4$ *and* $\varepsilon\in\{+,-\}$. *The set* $\pi_e(O)$ *consists of all divisors of the following numbers*:

(1) $\big[q^{n_1}\pm\tau_1,q^{n_2}\pm\tau_2,\ldots,q^{n_s}\pm\tau_s\big]$, *where* $s\geq 1$, $\tau_i\in\{+,-\}$, $n_i>0$ *for all* $1\leq i\leq s$, $n_1+\ldots+n_s=n$ *and* $\tau_1\tau_2\ldots\tau_s=\varepsilon$;

(4) $p\left[q\pm 1,\dfrac{q^{n-2}+1}{2}\right]$;

(5) $p\big[q\pm 1,q^{n_1}-\delta_1,q^{n_2}-\delta_2,\ldots,q^{n_s}-\delta_s\big]$, *where* $s\geq 2$, $\delta\in\{+,-\}$, $n_i>0$ *for all* $1\leq i\leq s$, $n_1+\ldots+n_s=n-2$ *and* $\delta_1\delta_2\ldots\delta_s=e$;

(6) $p^l\dfrac{q^{n_1}\pm 1}{3}$, *where* $l>0$ *and* $p^{l-1}+3+2n_1=2n$;

(7) $p^L\big[q^{n_1}\pm 1,\ldots,q^{n_s}\pm 1\big]$, *where* $l>0$, $s\geq 2$ *and* $n_i>0$ *for all* $1\leq i\leq s$ *and* $p^{l-1}+3+2(n_1+n_2+\ldots+n_s)=2n$;

(8) p^l *if* $2n=p^{l-1}+3$ *for* $l>0$.

By using Lemmas 2.6, 2.7 and [16], we contain some results which are listed in the Table 1.

Lemma 2.8 (Lemma 2.1 of [12]). *Let S be a finite non-Abelian simple group in* $\mathfrak{S}_p$ *where* $5\leq$ $\leq p\leq 997$. *Then* $\pi(\mathrm{Out}(S))\subseteq\{2,3,5,7,11\}$.

Lemma 2.9 (Lemma 2.12 of [3]). *Let G be a group and N be a normal subgroup of G with order* p^n, $n\geq 1$. *If* $(r,|\mathrm{Aut}(N)|)=1$, *where* $r\in\pi(G)$, *then G has an element of order* pr.

3. Main results. In this section, we study the characterization problem for the simple groups $O_8^-(q)$ with $q\in\{3,4,5,8,9,43\}$ by their orders and degree patterns.

Proposition 3.1. *The orthogonal group $O_8^-(3)$ is OD-characterizable.*

Proof. Assume that G be a finite group such that $|G|=|O_8^-(3)|=2^{10}.3^{12}.5.7.13.41$ and $D(G)=D(O_8^-(3))=(4,2,2,1,1,0)$. By Lemma 2.3, it follows that $t(G)\geq 3$. Furthermore, $t(2,G)\geq 2$ because $\deg(2)=4$ and $|\pi(G)|=6$. Consequently, from Lemma 2.4 we imply that there exists a finite non-Abelian simple group S such that $S\leq G/K\leq\mathrm{Aut}(S)$, where K is the maximal normal solvable subgroup of G.

We show that K is a $\{13,41\}'$-group. Assume that K is not a $\{13,41\}'$-group. Then either $13\in\pi(K)$ or $41\in\pi(K)$. Suppose that $\{r,s\}=\{13,41\}$, $r\in\pi(K)$ and R is a Sylow r-subgroup of K. Then $N_G(R)$ contains an element of order s, so G contains an element of order $r.s$, which is a contradiction. Therefore, K is a $\{13, 41\}'$-group.

Since $S \leq G/K \leq \mathrm{Aut}(S)$, it follows that $\{13, 41\} \subseteq \pi(G/K) \subseteq \pi(\mathrm{Aut}(S))$. On the other hand, $\pi(\mathrm{Aut(S)}/S) = \pi(\mathrm{Out}(S)) \cap \{13, 41\} = \varnothing$ by Lemma 2.8. Hence, $\{13.41\} \subseteq \pi(S)$ and so by using the collected results contained in Table 2, we conclude that S is isomorphic to $O_8^-(3)$. Therefore, $O_8^-(3) \leq G/K \leq \mathrm{Aut}(O_8^-(3))$, and since $|G| = |O_8^-(3)|$, we deduce that $|K| = 1$ and $G \cong O_8^-(3)$.

Proposition 3.2. *The orthogonal group $O_8^-(4)$ is OD-characterizable.*

Proof. Suppose that G be a finite group such that $|G| = |O_8^-(4)| = 2^{24}.3^4.5^3.7.13.17.257$ and $D(G) = D(O_8^-(4)) = (3, 5, 5, 2, 2, 3, 0)$. Then the prime graph of G has the following form:

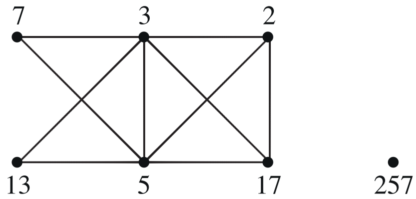

Fig. 1

Since $\{13, 17, 257\}$ is an independent set in $\Gamma(G)$, it follows that $t(G) \geq 3$. By Lemma 2.5, G is neither a Frobenius group nor a 2-Frobenius group, and hence Lemma 2.1 implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K/H is a non-Abelian simple group and G/K is a π_1-group such that $|G/K| \,\big|\, |\mathrm{Out}(K/H)|$. Moreover, each odd-order components of G is also an odd-order component of K/H. Thus 257 is an isolated vertex of prime graph of K/H. Now, according to the results collected in Table 2, we deduce that K/H is isomorphic to one of the following groups: $L_2(2^8)$ or $O_8^-(4)$.

If K/H is isomorphic to $L_2(2^8)$, then $(|G/K|, 13) = 1$ by $|\mathrm{Out}(K/H)| = 16$ and so the Sylow 13-subgroup of H is of order 13 and is normal in G. Since $(257, |\mathrm{Aut}(H_{13})|) = 1$, it follows that G has an element of order 257.13 by Lemma 2.9, which contradicts our assumption $\deg(257) = 0$.

Therefore, K/H is isomorphic to $O_8^-(4)$, and since $|G| = |O_8^-(4)|$, we obtain $|H| = 1$ and $G \cong O_8^-(4)$.

Proposition 3.3. *The orthogonal group $O_8^-(5)$ is OD-characterizable.*

Proof. Assume that G be a finite group such that $|G| = |O_8^-(5)| = 2^{10}.3^4.5^{12}.7.13.31.313$ and $D(G) = D(O_8^-(5)) = (5, 5, 3, 2, 3, 2, 0)$. According to these conditions on G, we conclude that $\Gamma(G)$ has the following form:

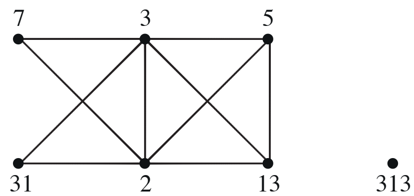

Fig. 2

From the structure of the prime graph of G, as shown in Fig. 2, we deduce that $t(G) \geq 3$. Hence, by Lemma 2.5 implies that G is neither a Frobenius group nor a 2-Frobenius group. So, it follows by Lemma 2.1 that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K/H is a non-Abelian simple group and G/K is a π_1-group such that $|G/K| \,\big|\, \big|\mathrm{Out}(K/H)\big|$. Moreover, $\{313\}$ is a prime component of K/H. By using Table 2, one can easily obtain that $K/H \cong L_2(5^4)$ or $O_8^-(5)$.

If $K/H \cong L_2(5^4)$, then $(|G/K|, 31) = 1$ by $\left|\mathrm{Out}(L_2(5^4))\right| = 8$. Hence, the Sylow 31-subgroup of H is of order 31 and is normal in G. Since $(313, |\mathrm{Aut}(H_{31})|) = 1$, we deduce that G has an element of order 31.313 by Lemma 2.9, which is a contradiction.

Therefore, we have $K/H \cong O_8^-(5)$. Because $|G| = |O_8^-(5)|$, we can get that $|H| = 1$, and, thus, $G \cong O_8^-(5)$.

Proposition 3.4. *The orthogonal group $O_8^-(8)$ is OD-characterizable.*

Proof. Suppose that G be a finite group such that $|G| = |O_8^-(8)| = 2^{36}.3^7.5.7^3.13.17.19.73.241$ and $D(G) = (2,6,3,6,3,1,2,2,1)$. By Lemma 2.3, $t(G) \geq 3$. Since $\deg(2) = 2$ and $|\pi(G)| = 9$, it follows that $t(2,G) \geq 2$. Consequently, from Lemma 2.4 we implies that there exists a finite non-Abelian simple group S such that $S \leq G/K \leq \mathrm{Aut}(S)$, where K is the maximal normal solvable subgroup of G.

We show that K is a p'-group, where $p \in \{73, 241\}$. Assume to the contrary that $p \in \pi(K)$. Let $r \in \{13, 17, 19\}$ and $r\big||K|$, then a Hall $\{p, r\}$-subgroup of K is a cyclic group of order $p.r$, and, hence, p is adjacent to r for all $r \in \{13, 17, 19\}$, which is a contradiction. Now, we may assume that $r \notin \pi(K)$. Let $K_p \in \mathrm{Syl}_p(K)$, then $N_G(K_p)$ contains an element of order r, so G contains an element of order pr for all $r \in \{13, 17, 19\}$, which is again a contradiction. Therefore, K is a $\{73, 241\}'$-group.

By Lemma 2.8, $\pi(\mathrm{Out}(S)) \cap \{13, 41\} = \varnothing$. On the other hand, since K is a $\{73, 241\}'$-group and $S \leq G/K \leq \mathrm{Aut}(S)$, it follows that the order of S is divisible by 73.241. According to the results in Table 2, we obtain the only possibility for S is $O_8^-(8)$. Therefore, $O_8^-(8) \leq G/K \leq \mathrm{Aut}(O_8^-(8))$, and since $|G| = |O_8^-(8)|$, we conclude that $|K| = 1$ and $G \cong O_8^-(8)$.

Proposition 3.5. *The orthogonal group $O_8^-(9)$ is OD-characterizable.*

Proof. Let G be a finite group such that $|G| = |O_8^-(9)| = 2^{13}.3^{24}.5^3.7.13.17.41.73.193$ and $D(G) = (6,3,6,3,3,1,3,2,1)$. By Lemma 2.3, we have $t(G) \geq 3$. Furthermore, $t(2,G) \geq 2$ because of $|\pi(G)| = 9$ and $\deg(2) = 6$. Therefore, Lemma 2.4 implies that there is a finite non-Abelian simple group S such that $S \leq G/K \leq \mathrm{Aut}(S)$, where K is the maximal normal solvable subgroup of G.

We show that K is a p'-group, where $p \in \{73, 193\}$. By way of contradiction, let $p \in \pi(K)$. If $r \in \{13, 17, 41\}$, then, by using the same technique as in the proof of Propositions 3.4, we derive that G has an element of order pr for all $r \in \{13, 17, 19\}$, which is impossible because $\deg(73) = 2$ and $\deg(193) = 1$. Therefore, K is a $\{73, 193\}'$-group.

From Lemma 2.8, we know that $\pi(\mathrm{Out}(S)) \cap \{73, 193\} = \varnothing$. Since K is a $\{73, 193\}'$-group and $S \leq G/K \leq \mathrm{Aut}(S)$, it follows that the order of S is divisible by 73.193. Now, Table 2 shows us that S is isomorphic to $O_8^-(9)$. Since $O_8^-(9) \leq G/K \leq \mathrm{Aut}(O_8^-(9))$ and $|G| = |O_8^-(9)|$, we conclude that $|K| = 1$ and $G \cong O_8^-(9)$.

Proposition 3.6. *The orthogonal group $O_8^-(43)$ is OD-characterizable.*

Proof. Let G be a finite group with $|G| = |O_8^-(43)| = 2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^3 \cdot 11^3 \cdot 13 \cdot 17 \cdot$ $\cdot\, 37 \cdot 43^{12} \cdot 139 \cdot 193 \cdot 521 \cdot 631$ and $D(G) = (9,9,6,9,9,9,5,2,6,5,2,2,4)$. Since $d_1 = 2$ and $d_4 \leq |\pi(G)| - 5$, then Lemma 2.2 implies that $t(G) \geq 3$. Moreover, $t(2,G) \geq 2$ because $|\pi(G)| = 12$ and $\deg(2) = 9$. Thus, by Lemma 2.4, there exists a finite non-Abelian simple group S such that $S \leq G/K \leq \mathrm{Aut}(S)$, where K is the maximal normal solvable subgroup of G.

We show that K is a p'-group, where $p \in \{521, 631\}$. Assume to the contrary that $|K|$ is divisible by p. If $r \in \{13, 17, 37, 139, 193\}$, then, by using a similar arguments as in the proof of Proposition 3.4, we can show that G has an element of order pr for all $r \in \{13, 17, 37, 139, 193\}$, which is contradiction because $\deg(631) = 4$ and $\deg(521) = 2$. Therefore, K is a $\{521, 631\}'$-group.

By Lemma 2.8, $\mathrm{Out}(S)$ is a $\{521, 631\}'$-group. Since $S \leq G/K \leq \mathrm{Aut}(S)$ and K is a $\{521, 631\}'$-group, it follows that S is a simple group with $\{521.631\} \subseteq \pi(S)$. Therefore, by using Table 2 implies that S is isomorphic to $O_8^-(43)$ and so $O_8^-(43) \leq G/K \leq \mathrm{Aut}(O_8^-(43))$. As $|G| = |O_8^-(43)|$, we deduce that $|K| = 1$ and so $G \cong O_8^-(43)$.

Table 2. The orders of finite simple groups $S \in \mathfrak{S}_p$ except alternating groups

S	$\|S\|$
	$p = 41$
$L_2(3^4)$	$2^4 \cdot 3^4 \cdot 5 \cdot 41$
$S_4(9)$	$2^8 \cdot 3^8 \cdot 5^2 \cdot 17$
$Sz(32)$	$2^{10} \cdot 5^2 \cdot 31 \cdot 41$
$L_2(41)$	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 41$
$O_8^-(3)$	$2^{10} \cdot 3^{12} \cdot 5 \cdot 7 \cdot 13 \cdot 41$
$L_4(9)$	$2^{10} \cdot^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41$
$O_9(3)$	$2^{14} \cdot 3^{16} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41$
$S_8(3)$	$2^{14} \cdot 3^{16} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41$
$L_2(41^2)$	$2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 29^2 \cdot 41^2$
$S_4(41)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 29^2 \cdot 41^4$
$L_2(2^{10})$	$2^{10} \cdot 3 \cdot 5^2 \cdot 11 \cdot 31 \cdot 41$
$S_4(32)$	$2^{20} \cdot 3^2 \cdot 5^2 \cdot 11^2 \cdot 31^2 \cdot 41$
$U_5(4)$	$2^{20} \cdot 3^2 \cdot 5^4 \cdot 13 \cdot 17 \cdot 41$
$O_{10}^+(3)$	$2^{15} \cdot 3^{20} \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 41$
$U_6(4)$	$2^{30} \cdot 3^4 \cdot 5^6 \cdot 7 \cdot 13^2 \cdot 17 \cdot 41$
	$p = 193$
$L_2(3^8)$	$2^5 \cdot 3^8 \cdot 5 \cdot 17 \cdot 41 \cdot 193$
$S_4(3^4)$	$2^{10} \cdot 3^{16} \cdot 5^2 \cdot 17 \cdot 41^2 \cdot 193$
$L_2(193)$	$2^3 \cdot 3 \cdot 5^3 \cdot 97 \cdot 149 \cdot 193^2$
$S_4(193)$	$2^{14} \cdot 3^2 \cdot 5^3 \cdot 97^2 \cdot 149 \cdot 149 \cdot 193^4$
$U_3(109)$	$2^4 \cdot 3^3 \cdot 5^2 \cdot 11^2 \cdot 61 \cdot 109^3 \cdot 193$
$O_8^-(9)$	$2^{13} \cdot 3^{24} \cdot 5^3 \cdot 7 \cdot 13 \cdot 17 \cdot 41 \cdot 73 \cdot 193$
$L_4(3^4)$	$2^{13} \cdot 3^{24} \cdot 5^3 \cdot 7 \cdot 13 \cdot 17 \cdot 41^2 \cdot 73 \cdot 193$
$S_8(9)$	$2^{18} \cdot 3^{32} \cdot 5^4 \cdot 7 \cdot 13 \cdot 17 \cdot 41^2 \cdot 73 \cdot 193$
$O_9(9)$	$2^{18} \cdot 3^{32} \cdot 5^4 \cdot 7 \cdot 13 \cdot 17 \cdot 41^2 \cdot 73 \cdot 193$
$O_{10}^+(9)$	$2^{20} \cdot 3^{60} \cdot 5^4 \cdot 7 \cdot 11^2 \cdot 13 \cdot 17 \cdot 41^2 \cdot 67 \cdot 73 \cdot 193$

Table 2 (*continued*)

S	$\lvert S\rvert$
	$p = 241$
$U_3(16)$	$2^4.3.5.11^2.241$
$S_8(8)$	$2^{48}.3^9.5^2.7^4.13^3.17.19.241$
$L_2(2^{12})$	$2^{48}.3^9.5^2.7^4.13^3.17.19.241$
$O_8^-(8)$	$2^{36}.3^7.5.7^3.13.17.19.73.241$
$L_4(64)$	$2^{36}.3^7.5^2.7^3.13^2.17.19.73.241$
${}^3D_4(4)$	$2^{36}.3^7.5^2.7^3.13^2.17.19.73.241$
$G_2(16)$	$2^{36}.3^4.5^3.7^2.13^3.17.37.109.241$
$U_4(64)$	$2^{36}.3^4.5^3.7^2.13^3.17.37.109.241$
$S_6(64)$	$2^{54}.3^6.5^3.7^3.13^3.17.19.37.109.241$
$F_4(8)$	$2^{72}.3^{10}.5^2.7^4.13^2.17.37.73^2.109.241$
$L_3(2^{12})$	$2^{36}.3^5.5^2.7^2.13^2.17.19.37.73.109.241$
$O_8^+(64)$	$2^{72}.3^7.5^3.7^4.13^4.17^2.37.73.109.241^2$
$S_4(64)$	$2^{60}.3^9.5^2.7^5.13^2.17^2.19.31.73.151.241$
$O_{10}^+(8)$	$2^{60}.3^9.5^2.7^5.13^2.17^2.19.31.73.151.241$
	$p = 257$
$L_2(257)$	$2^8.3.43.257$
$L_2(2^8)$	$2^8.3.5.17.257$
$S_4(16)$	$2^{16}.3^2.5^2.17^2.257$
$U_4(16)$	$2^{24}.3^2.5^2.17^3.241.257$
$O_8^-(4)$	$2^{24}.3^4.5^3.7.13.17.257$
$S_8(4)$	$2^{32}.3^5.5^4.7.13.17^2.257$
$L_2(241^2)$	$2^5.3.5.7^3.11^2.113.241^2.257$
$S_4(241)$	$2^{10}.3^2.5^2.11^4.113.241^2.257$
$U_3(257)$	$2^{11}.3^2.7.13.43.241.257^3$
$O_{10}^-(4)$	$2^{40}.3^5.5^6.7.13.17^2.41.257$
$L_3(2^8)$	$2^{24}.3^2.5^2.7.13.17^2.241.257$
$S_6(16)$	$2^{36}.3^4.5^3.7.13.17^3.241.257$
$O_8^+(16)$	$2^{48}.3^5.5^4.7.13.17^4.241.257$
$F_4(4)$	$2^{48}.3^6.5^4.7^2.13^2.17^2.241.257$
$O_{10}^+(4)$	$2^{40}.3^6.5^4.7.11.13.17^2.31.257$
$L_5(16)$	$2^{40}.3^5.5^4.7.11.13.17^2.31.41.257$
$S_{10}(4)$	$2^{50}.3^6.5^6.7.11.13.17^2.31.41.257$
$S_{20}(2)$	$2^{100}.3^{14}.5^6.7^3.11^2.13.17^2.19.31^2.41.43.73.127.257$

Table 2 (*continued*)

S	$\lvert S\rvert$
$U_{10}(4)$	$2^{90}.3^6.5^{10}.7.11.13^3.17^2.29.31.37.41^2.109.113.257$
$L_7(16)$	$2^{84}.3^8.5^7.7^2.11.13^2.17^3.29.31.41.43.113.127.241.257$
$S_{14}(4)$	$2^{98}.3^7.5^6.7^2.11.13^2.17^3.29.31.41.43.113.127.241.257$
$O_{16}^+(4)$	$2^{112}.3^8.5^7.7^2.11.13^2.17^4.29.31.41.43.113.127.241.257^2$
$O_{22}^+(2)$	$2^{110}.3^{14}.5^6.7^3.11^2.13.17^2.19.23.31^2.41.43.73.89.127.257$
$E_7(4)$	$2^{126}.3^{11}.5^8.7^3.11.13^3.17^2.19.29.31.37.41.43.73.109.113.127.241.257$
	$p = 313$
$L_2(5^4)$	$2^6 \cdot 3 \cdot 5^4.13.313$
$S_4(25)$	$2^9 \cdot 3^2 \cdot 5^8 \cdot 13^2 \cdot 313$
$O_8^-(5)$	$2^{10}.3^4.5^{12}.7.13.31.313$
$O_9(5)$	$2^{15} \cdot 3^5 \cdot 5^4.7 \cdot 13^2.31.313$
$S_8(5)$	$2^{15} \cdot 3^5 \cdot 5^4.7 \cdot 13^2.31.313$
$L_4(25)$	$2^9 \cdot 3^4 \cdot 5^{12}.7.13^2.31.313$
$L_3(313)$	$2^7 \cdot 3^4 \cdot 13^2 \cdot 157.181^2.313^3$
$L_2(313^2)$	$2^6 \cdot 3 \cdot 5 \cdot 13.97.101.157.313^2$
$S_4(313)$	$2^9 \cdot 3^2.5 \cdot 13 \cdot 97.101.157^2.313^4$
$L_4(313)$	$2^{13} \cdot 3^4 \cdot 5 \cdot 13^3.97.101.157^2.181^2.313^6$
${}^3D_4(29)$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2.13^2.29^{12}.37.61.67^2.271^2.313$
	$p = 631$
$L_3(43)$	$2^4 \cdot 3^2 \cdot 7^2 \cdot 11 \cdot 43^3 \cdot 631$
$L_2(43)$	$2^2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 43^3 \cdot 139 \cdot 631$
$L_3(587)$	$2^4 \cdot 3 \cdot 7^2 \cdot 293^2 \cdot 547 \cdot 587^3 \cdot 631$
$L_3(631)$	$2^5 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 79 \cdot 307 \cdot 433 \cdot 631$
$L_4(43)$	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7^3 \cdot 11^2 \cdot 37 \cdot 43^6 \cdot 631$
$G_2(43)$	$2^6 \cdot 3^4 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 43^6 \cdot 139 \cdot 631$
$O_8^+(43$	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7^3 \cdot 11^3 \cdot 37 \cdot 43^{12} \cdot 139 \cdot 631$
$S_6(43)$	$2^9 \cdot 3^4 \cdot 5^2 \cdot 7^3 \cdot 11^3 \cdot 37 \cdot 43^9 \cdot 139 \cdot 631$
$O_7(43)$	$2^9 \cdot 3^4 \cdot 5^2 \cdot 7^3 \cdot 11^3 \cdot 37 \cdot 43^9 \cdot 139 \cdot 631$
$L_3(43^2)$	$2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 37 \cdot 43^6 \cdot 139 \cdot 631$
$L_4(43)$	$2^9 \cdot 3^4 \cdot 5^2 \cdot 7^3 \cdot 11^3 \cdot 13 \cdot 37 \cdot 43^6 \cdot 139 \cdot 631$
$O_8^-(43)$	$2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^3 \cdot 11^3 \cdot 13 \cdot 17 \cdot 37 \cdot 43^{12} \cdot 139 \cdot 193 \cdot 521 \cdot 631$
$S_8(43)$	$2^{14} \cdot 3^5 \cdot 5^4 \cdot 7^4 \cdot 11^4 \cdot 13 \cdot 37^2 \cdot 139 \cdot 193 \cdot 521 \cdot 631$
$O_9(43)$	$2^{14} \cdot 3^5 \cdot 5^4 \cdot 7^4 \cdot 11^4 \cdot 13 \cdot 37^2 \cdot 139 \cdot 193 \cdot 521 \cdot 631$

References

1. B. Akbari, A. R. Moghaddamfar, *Recognizing by order and degree pattern of some projective special linear groups*, Int. J. Algebra Comput., **22**, № 6, 1 – 22 (2012).
2. A. A. Buturlakin, *Spectra of finite symplectic and orthogonal groups*, Siberian Adv. Math., **21**, № 3, 176 – 210 (2011).
3. Y. H. Chen, G. Y. Chen, J. B. Li, *Recognizing simple K_4-groups by few special conjugacy class sizes*, Bull. Malays. Math. Sci. Soc., **38**, № 1, 51 – 72 (2015).
4. M. R. Darafsheh, G. R. Rezaeezadeh, M. Bibak, M. Sajjadi, *OD-characterization of almost simple groups related to* ${}^2E_6(2)$, Adv. Algebra, **6**, 45 – 54 (2013).
5. M. R. Darafsheh, G. R. Rezaeezadeh, M. Sajjadi, M. Bibak, *OD-characterization of almost simple groups related to* $U_3(17)$, Quasigroups and Related Systems, **21**, 49 – 58 (2013).
6. A. Daneshkhah, Y. Jalilian, *A characterization of some projective special linear groups*, Ital. J. Pure and Appl. Math., **38**, 32 – 44 (2017).
7. A. R. Moghaddamfar, A. R. Zokayi, M. R. Darafsheh, *A characterization of finite simple groups by the degree of vertices of their prime graphs*, Algebra Colloq., **12**, № 3, 431 – 442 (2005).
8. G. R. Rezaeezadeh, M. Bibak, M. Sajjad, *Characterization of projective special linear groups in dimension three by their orders and degree patterns*, Bull. Iranian Math. Soc., **41**, № 3, 551 – 580 (2015).
9. G. R. Rezaeezadeh, M. R. Darafsheh, M. Sajjadi, M. Bibak, *OD-characterization of almost simple groups related to* $L_3(25)$, Bull. Iranian Math. Soc., **40**, № 3, 765 – 790 (2014).
10. G. R. Rezaeezadeh, M. R. Darafsheh, M. Bibak, M. Sajjadi, *OD-characterization of almost simple groups related to* $D_4(4)$, Iran. J. Math. Sci. and Inform., **10**, № 1, 23 – 43 (2015).
11. D. J. S. Robinson, *A course in the theory of groups*, 2nd ed., Springer-Verlag, New York etc. (2003).
12. M. Sajjadi, M. Bibak, G. R. Rezaeezadeh, *Characterization of some projective special linear groups in dimension four by their orders and degree patterns*, Bull. Iranian Math. Soc., **42**, № 1, 27 – 36 (2016).
13. A. V. Vasil'ev, *On connection between the structure of a finite group and the properties of its prime graph*, Sib. Math. J., **46**, № 3, 396 – 404 (2005).
14. J. S. Williams, *Prime graph components of finite group*, J. Algebra, **69**, 487 – 513 (1981).
15. Y. Yan, G. Chen, L. Wang, *OD-characterization of the automorphism groups of* $O_{10}^{\pm}(2)$, Indian J. Pure and Appl. Math., **43**, № 3, 183 – 195 (2012).
16. A. Zavarnitsine, *Finite simple groups with narrow prime spectrum*, Sib. Elektron. Mat. Izv., **6**, 1 – 12 (2009).

Received 04.02.20