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PROBABILISTIC WEAK SOLUTIONS FOR NONLINEAR STOCHASTIC EVOLUTION PROBLEMS INVOLVING PSEUDOMONOTONE OPERATORS*

ІМОВІРНІСНІ СЛАБКІ РОЗВ'ЯЗКИ НЕЛІНІЙНИХ СТОХАСТИЧНИХ ЕВОЛЮЦІЙНИХ ЗАДАЧ, ЩО МІСТЯТЬ ПСЕВДОМОНОТОННІ ОПЕРАТОРИ

We study an important class of stochastic nonlinear evolution problems with pseudomonotone elliptic parts and establish the existence of probabilistic weak (or martingale) solutions. No solvability theory has been developed so far for these equations despite numerous works involving various generalizations of the monotonicity condition. Key to our work is a sign result for the Itô differential of an approximate solution that we establish, as well as several compactness results of the analytic and probabilistic nature, and a characterization of pseudomonotone operators due to F. E. Browder.

Вивчається важливий клас стохастичних нелінійних еволюційних задач із псевдомонотонними еліптичними частинами. Встановлено існування ймовірнісних слабких (або мартингальних) розв'язків. На сьогодні не існує теорії розв'язності, розробленої для таких рівнянь, незважаючи на те, що є багато робіт, в яких вивчаються різні узагальнення умов монотонності. Ключем до нашої роботи є знаковий результат для диференціала Іто наближеного розв'язку, який ми встановлюємо, а також кілька результатів щодо компактності аналітичної та ймовірнісної природи і характеризація псевдомонотонних операторів по Ф. Е. Браудеру.

1. Introductory background. Stochastic partial differential equations (SPDEs) have become one of the main areas of research in mathematics and in applied sciences due to their crucial relevance in the modelling of important processes either subjected or generating random excitations or fluctuations such as turbulence in fluids, filtering theory, random media, finance; just to cite a few.

Their investigation may be traced back to the pioneering work of Bensoussan and Temam [4], [3] followed by the theses of Pardoux [27] and Viot [39]. These works generalized the deterministic results of Lions [25], Browder [12], Vishik [40] to their stochastic counterparts and had a huge influence on the field, as witnessed by the numerous important works that followed; for instance, [1, 20, 19, 31, 28]; just to cite a few. The monotonicity and compactness methods were key in the progress made. The weakening of the monotonicity condition by local monotonicity for SPDEs was undertaken in recent years in works by Liu, Röckner and their coworkers [22 – 24, 14]. It is not an exaggeration to say that these latest works have revived the interest for the investigation of existence of solutions for nonlinear SPDEs which can't be handled by the popular method of semigroup theory.

Despite these impressive advances made in the field of SPDEs, several important classes of equations have up to date not been studied by experts. Among them the fundamental class of evolution SPDEs involving pseudomonotone operators. Their deterministic counterparts have been the object of investigation by two of the most influential mathematicians of our time, namely Brezis

* Dedicated to the loving memory of Academician Igor Volodymyrovych Skrypnyk (on his 80th birthday).

and Browder in the seminal works [5, 9] (Chapt. 17) and [10]. Their results expanded the frontiers of study of many important classes of nonlinear partial differential equations, among them the so-called class of strongly nonlinear elliptic and parabolic equations with zeroth-order nonlinear perturbation terms introduced and studied in [6, 7, 13, 11]. The survey papers by Dubinskii and Skrypnik [16, 17, 36, 37] give authoritative accounts of further developments with extensive references. Due to the lack of a relevant theory for pseudomonotone SPDEs, the work on stochastic strongly nonlinear parabolic equations has so far been out of reach. To make a breakthrough in that direction of research, the genuine pseudomonotone case that is being investigated and settled in our work is unavoidable. It should be noted that a sign condition is key in the application of the main result on pseudomonotone operators. In the deterministic case, an efficient intermediary tool used to that effect is a sign condition involving the derivative of an approximation of the solution of the problem at hand (see, for instance, Lemma 7.4 in [30, p. 192], and [21]). The stochastic version of that result is subtle and its proof more delicate than in the deterministic case. The key to our work is the successful establishment of such a sign condition involving Itô's differential of an approximation of our required solution. Our notion of solution is that of martingale solution or weak probabilistic. One initial main challenge in this work in the fact that Galerkin's method seems hopeless, since the corresponding system of stochastic ordinary differential equations satisfied by the Fourier coefficients of the Galerkin approximation lacks the counterpart of Carathéodory existence theorem for deterministic ordinary differential equations which was key in the deterministic theory (see, for instance, [30]). We rely instead on a numerical scheme introduced by Gyöngy and Millet in [18] for the case of strongly monotone nonlinear stochastic parabolic equations. Our main result can be seen as a generalization of the corresponding results of [18, 20, 22 – 24, 27, 39] and many others to pseudomonotone stochastic parabolic equations. From the probabilistic methodological side, we rely on the still unavoidable fundamental compactness results of Prokhorov [29] and Skorokhod [35] which are crucial in establishing probabilistic weak solutions.

Specifically, we consider stochastic nonlinear evolution problem

$$(P)\begin{cases} du + A_t(u)dt = f(t)dt + G(t,u)dW(t), \\ u(0) = 0, \end{cases}$$

for $t \in [0,T]$, where $u = u(t)$ is the unknown process, the contributors f and G to the forcing are given, W is a d -dimensional Wiener process and $A_t = A(t,u)$ is a pseudomonotone operator acting from a reflexive and separable Banach space V to its dual V'. Namely for almost everywhere (a.e.) $t \in [0,T]$, A_t is bounded and if u_j converges to u weakly in V and $\overline{\lim}_{j\to\infty}\langle A_t(u_j), u_j - u\rangle_{V'}$, $V \leq 0$, then

$$A_t(u_j) \longrightarrow A_t(u) \quad \text{weakly in} \quad V' \quad \text{and} \quad \langle A_t(u_j), u_j\rangle_{V',V} \longrightarrow \langle A_t(u), u\rangle_{V',V}.$$

The theory of pseudomonotone operators was pioneered by Brezis in [5] and actively studied by Browder [9]. Our characterization of pseudomonotonicity is due to Browder [10] (see also [37], Chapt. 1, § 1).

This paper is organized as follows. In Section 2, we state the assumptions on the investigated problem and formulate our main result. In Section 3, we introduce a suitable numerical scheme for an approximation of the solution of our original problem and establish some crucial compactness results for a sequence of probability measures generated by the approximating solutions. Section 4 is devoted to the proof of convergence of the approximating solutions to the genuine one leading to the proof of the main result on the existence of a probabilistic weak (martingale) solution. In Section 5, we provide an example of application of the main result to a stochastic evolution problem involving higher-order nonlinear partial differential operators. The last section is devoted to some closing remarks on the comparison of our work with those of Liu and Röckner [22–24] dealing with local monotonicity.

2. Assumptions and formulation of the main result. Let V be a reflexive and separable Banach space compactly embedded into the separable Hilbert space H; H is identifiable with its dual H', and we denote by $\langle\cdot,\cdot\rangle$ the duality pairing of V' and V. Then we have the Gelfand triple

$$V \subset H \simeq H' \subset V'.$$

For $r \in [1,\infty]$, $T > 0$ and X a Banach space, $L^r(0,T;X)$ denotes the usual Lebesgue–Bochner space of functions defined on $[0,T]$ with values in X endowed with the usual norm. Given a complete probability space $\big(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in I},\mathbb{P}\big)$ with a filtration $(\mathcal{F}_t)_{t\in I}$ (I is the time interval $[0,T]$), and the numbers $1 \le r \le \infty, 1 \le q < \infty$, $L^q\big(\Omega, L^r(0,T;X)\big)$ denotes the space of progressively measurable processes endowed with the corresponding norm.

We now formulate assumptions on our problem (P). Let $2 \le p < \infty$.

(i) For a.e. $t \in [0,T]$, $A_t : V \to V'$ is pseudomonotone.

(ii) There exist a constant $c_1 > 0$ and a nonnegative function $h_1 \in L^1([0,T])$ such that

$$\langle A_t(u), u\rangle \ge c_1\big\|u\big\|_V^p - h_1(t) \quad \text{for all} \quad u \in V \quad \text{and a.e.} \quad t \in [0,T].$$

(iii) There exist a constant $c_2 > 0$ and a function $h_2 \in L^{p'}([0,T])$ such that

$$\big\|A_t(u)\big\|_{V'} \le c_1\big\|u\big\|_V^{p-1} + h_2(t) \quad \text{for all} \quad u \in V \quad \text{and a.e.} \quad t \in [0,T].$$

(iv) The nonlinear operator $G(t,u) : [0,T]\times H \to H^d$ is continuous in (t,u) and there exists a positive constant C such that

$$\big\|G(t,u)\big\|_{H^d} \le C\big(1 + \|u(t)\|_H\big);$$

H^d denotes the product of d copies of H.

(v) $f(t)$ is a deterministic functional on V, measurable and there exists a positive constant C such that

$$\int\limits_0^T \big\|f(t)\big\|_{V'}^{p'} dt \le C.$$

Next, we define the concept of probabilistic weak or martingale solution for the problem (P).

Definition 1. *A probabilistic weak solution of the problem (P) is a system*

$$\big(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},\mathbb{P},W,u\big),$$

where

(1) $(\Omega,\mathcal{F},\mathbb{P})$ *is a probability space*, $(\mathcal{F}_t)$ *a filtration on it*,

(2) W *is a d-dimensional $\mathcal{F}_t$-standard Wiener process*,

(3) $(\omega, t) \to u(\omega, t)$ *is progressively measurable,*

(4) $u \in L^2(\Omega, L^\infty(0,T;H)) \cap L^p\big(\Omega, L^p(0,T;V)\big)$, *and, for all* $t \in [0,T]$, $u(t)$ *satisfies the integral identity*

$$\big(u(t), v\big) - \int\limits_0^t \langle A_s(u), v\rangle ds = \int\limits_0^t \langle f(s), v\rangle ds + \left(\int\limits_0^t G(s, u(s))dW(s), v\right) \quad \forall v \in V, \quad \mathbb{P}\text{-}a.s. \quad (1)$$

Note that the last equation implies that almost surely (a.s.)

$$u(\cdot) \in C(0,T;V')$$

and since $u(\cdot)$ is also bounded in H, then it is almost surely in $C\,(0,T;H \text{ weak})$, the space of H-valued weakly continuous functions on $[0,T]$; that, for any $v \in H$, the function

$$[0,T] \to [0,\infty) : t \mapsto \big(u(t), v\big)$$

is continuous. This follows by arguing as in [38] (Chapt. 3, § 3).

The definition means that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the Wiener process W are unknown alongside the process u.

The main result of this paper is the following theorem.

Theorem 1. *Assume that the conditions* (i) – (v) *are satisfied and* V *is compactly embedded into* H. *Then problem* (*P*) *has a probabilistic weak* (*martingale*) *solution in the sense of the above Definition* 1.

The remaining part of the paper is devoted to the proof of this theorem.

3. Numerical approximation of (P) and compactness results. In this section, we introduce a suitable numerical scheme for Problem (P) and derive needed compactness results for probability measures generated by the approximating solutions.

From the onset, the pseudomonotonicity of A_t limits the methodological options in the proof of Theorem 1, since due to lack of a convenient stochastic version of Carathéodory's deterministic theorem on the existence of solutions to ordinary differential equations, we are unable to use Galerkin's method. We rely instead on a semidiscretized version of (P), following [18]. We set $f = 0$, since the presence of f does not add any complication. Let $\{t_i\}$ be a regular partition of the interval $[0,T]$ given by $t_i = i\dfrac{T}{M}$, $i = 0, 1, \ldots, M$ and set $\tau = \dfrac{T}{M}$. On an intermediary probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ with a prescribed d-dimensional standard Wiener process $\bar{W}(t)$, we consider the approximation $\big(u^M(t)\big)$ of the presumed solution $u(t)$ of problem (P), which is required to satisfy the following conditions: $u^M(0) = 0$,

$$u^M(t_{i+1}) = u^M(t_i) - \tau A^M_{t_i}\big(u^M(t_{i+1})\big) + \tilde{G}^M_{t_i}\big(t_i, u^M(t_i)\big)\Delta\bar{W}_i, \quad (2)$$

where

$$A^M_{t_i}(\cdot) = \frac{1}{\tau}\int\limits_{t_i}^{t_{i+1}} A(s,\cdot)ds, \qquad \tilde{G}^M_{t_i}(t_i,\cdot) = \frac{1}{\tau}\int\limits_{t_i}^{t_{i+1}} G(s,\cdot)ds, \quad \Delta\bar{W}_i = \bar{W}(t_{i+1}) - \bar{W}(t_i),$$

and we define the piecewise functions $u^M(t)$ and $A^{M,\cdot}_t(\cdot)$ by setting

$$u^M(t) = u^M(t_{i+1}) \quad \text{for} \quad t \in (t_i, t_{i+1}], \quad i = 0, \ldots, M-1, \tag{3}$$

and

$$A_t^{M\cdot}(\cdot) = A_{t_{i+1}}^M(\cdot) \quad \text{for} \quad t \in (t_i, t_{i+1}], \quad i = 0, \ldots, M-1.$$

Before proceeding further, let us estimate the H-norm of $\tilde{G}_{t_i}^M(t_i, \cdot)$. We have, by using Hölder's inequality and Fubini's theorem, that

$$\left\| \tilde{G}_{t_i}^M(t_i, \cdot) \right\|_H^2 \leq \frac{1}{\tau} \int\limits_{t_i}^{t_{i+1}} \left\| G(s, u^M(s)) \right\|_H^2 ds. \tag{4}$$

We deal next with the a priori estimates of the sequence (u^M).

3.1. Estimates for $\boldsymbol{u^M}$. For sufficiently small τ, denoting by Id the identity operator, it is known that the operator $Id + \tau A_{t_i}^M$ is pseudomonotone (see [30, p. 203]). Therefore owing to the results of [5] (see also [37] (Chapt. 1, § 4), we have that for almost all $\bar{\omega} \in \bar{\Omega}$, (2) has at least a weak solution $u^M(t_{i+1}) \in V$, given $u^M(t_i) \in V$.

From relation (2), we obtain

$$\begin{gathered} \left\| u^M(t_{i+1}) \right\|_H^2 - \left\| u^M(t_i) \right\|_H^2 = \\ = \tau^2 \left\| A_{t_i}^M\big(u^M(t_{i+1})\big) \right\|_H^2 + \left\| \tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big) \Delta W_i \right\|_H^2 + \\ +2 \left(u^M(t_i), \tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big) \Delta W_i \right) - 2\tau \left\langle A_{t_i}^M\big(u^M(t_{i+1})\big), u^M(t_i) \right\rangle - \\ -2\tau \left\langle A_{t_i}^M\big(u^M(t_{i+1})\big), \tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big) \Delta W_i \right\rangle. \end{gathered}$$

At this point, we substitute $u^M(t_i)$ by $u^M(t_{i+1}) + \tau A_{t_i}^M\big(u^M(t_{i+1})\big) - \tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big)\Delta W_i$ in the pairing of $A_{t_i}^M$ and $u^M(t_i)$ and get after some cancellations

$$\begin{gathered} \left\| u^M(t_{i+1}) \right\|_H^2 = \left\| u^M(t_i) \right\|_H^2 - \tau^2 \left\| A_{t_i}^M\big(u^M(t_{i+1})\big) \right\|_H^2 + \\ + \left\| \tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big) \Delta W_i \right\|_H^2 - 2\tau \left\langle A_{t_i}^M\big(u^M(t_{i+1})\big), u^M(t_{i+1}) \right\rangle + \\ +2 \left(u^M(t_i), \tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big) \Delta W_i \right). \end{gathered}$$

We deduce that

$$\begin{gathered} \left\| u^M(t_{i+1}) \right\|_H^2 + 2\tau \left\langle A_{t_i}^M\big(u^M(t_{i+1})\big), u^M(t_{i+1}) \right\rangle \leq \\ \leq \left\| u^M(t_i) \right\|_H^2 + \left\| \tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big) \Delta W_i \right\|_H^2 + 2 \left(u^M(t_i), \tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big) \Delta W_i \right). \end{gathered} \tag{5}$$

In particular,

$$\left\| u^M(t_{i+1}) \right\|_H^2 \leq \left\| u^M(t_i) \right\|_H^2 + \left\| \tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big) \Delta W_i \right\|_H^2 +$$

$$+2\left(u^M(t_i),\tilde{G}^M_{t_i}\left(t_i,u^M(t_i)\right)\Delta W_i\right)+2\int\limits_{t_i}^{t_{i+1}}|h_1(s)|ds \tag{6}$$

in view of condition (ii) applied to the second term in the left-hand side of (5).

Due to the vanishing of the expectation of the last term in (5), the conditions on A_t and the relation (4), we deduce from (5) that

$$\bar{\mathbb{E}}\left\|u^M(t_{i+1})\right\|_H^2+\bar{\mathbb{E}}\int\limits_{t_i}^{t_{i+1}}\left\|u^M(s)\right\|_V^p ds\le$$

$$\le\bar{\mathbb{E}}\left\|u^M(t_i)\right\|_H^2+\bar{\mathbb{E}}\int\limits_{t_i}^{t_{i+1}}\left\|G\left(s,u^M(s)\right)\right\|_H^2 ds+$$

$$+2\int\limits_{t_i}^{t_{i+1}}|h_1(s)|ds \quad \text{for} \quad i=0,1,\ldots,M-1. \tag{7}$$

Of interest for our purpose are higher order moments for $\left\|u^M(t_{i+1})\right\|_H$; the fourth moment will do. For that, we square both sides of (6) and get

$$\bar{\mathbb{E}}\left\|u^M(t_{i+1})\right\|_H^4\le\bar{\mathbb{E}}\left\|u^M(t_i)\right\|_H^4+\sum_{l=1}^{5}I_l, \tag{8}$$

where

$$I_1=\bar{\mathbb{E}}\left\|\tilde{G}^M_{t_i}\left(t_i,u^M(t_i)\right)\Delta W_i\right\|_H^4,\ I_2=4\bar{\mathbb{E}}\left(u^M(t_i),\tilde{G}^M_{t_i}\left(t_i,u^M(t_i)\right)\Delta W_i\right)^2,$$

$$I_3=2\bar{\mathbb{E}}\left[\left\|u^M(t_i)\right\|_H^2\left\|\tilde{G}^M_{t_i}\left(t_i,u^M(t_i)\right)\Delta W_i\right\|_H^2\right],$$

$$I_4=4\bar{\mathbb{E}}\left[\left\|u^M(t_i)\right\|_H^2\left(u^M(t_i),\tilde{G}^M_{t_i}\left(t_i,u^M(t_i)\right)\Delta W_i\right)\right],$$

$$I_5=4\bar{\mathbb{E}}\left[\left\|\tilde{G}^M_{t_i}\left(t_i,u^M(t_i)\right)\Delta W_i\right\|_H^2\left(u^M(t_i),\tilde{G}^M_{t_i}\left(t_i,u^M(t_i)\right)\Delta W_i\right)\right].$$

We now estimate each term in the right-hand side of (8). We have

$$I_1\le C\bar{\mathbb{E}}\left(\left\|\Delta W_i\right\|^4\left\|\tilde{G}^M_{t_i}\left(t_i,u^M(t_i)\right)\right\|_H^4\right).$$

But $\tilde{G}^M_{t_i}$ being $\bar{\mathcal{F}}_{t_i}$-measurable, it is independent of ΔW_i, and hence

$$I_1\le C\bar{\mathbb{E}}\left(\left\|\Delta W_i\right\|^4\right)C\bar{\mathbb{E}}\left(\left\|\tilde{G}^M_{t_i}\left(t_i,u^M(t_i)\right)\right\|_H^4\right)\le$$

$$\leq C\tau^2\bar{\mathbb{E}}\left(\left\|\tilde{G}^M_{t_i}\big(t_i,u^M(t_i)\big)\right\|^4_H\right). \tag{9}$$

Noting that $\left(u^M(t_i),\tilde{G}^M_{t_i}\big(t_i,u^M(t_i)\big)\right)$ is independent of ΔW_i and using Young's inequality, we get

$$I_2\leq C\bar{\mathbb{E}}\big\|\Delta W_i\big\|^2\bar{\mathbb{E}}\left(u^M(t_i),\tilde{G}^M_{t_i}\big(t_i,u^M(t_i)\big)\right)\leq$$

$$\leq C\tau\left(\bar{\mathbb{E}}\left\|u^M(t_i)\right\|^4_H+\bar{\mathbb{E}}\left\|\tilde{G}^M_{t_i}\left(t_i,u^M(t_i)\right)\right\|^4_H\right). \tag{10}$$

Similarly

$$I_3\leq C\tau\left(\bar{\mathbb{E}}\left\|u^M(t_i)\right\|^4_H+\bar{\mathbb{E}}\left\|\tilde{G}^M_{t_i}\left(t_i,u^M(t_i)\right)\right\|^4_H\right). \tag{11}$$

It is clear that

$$I_4=0. \tag{12}$$

Using Hölder's inequality, we easily show that

$$I_5\leq C\tau\left(\bar{\mathbb{E}}\left\|u^M(t_i)\right\|^4_H+\bar{\mathbb{E}}\left\|\tilde{G}^M_{t_i}\left(t_i,u^M(t_i)\right)\right\|^4_H\right). \tag{13}$$

Combining the estimates (8) – (13) and (4), it follows that

$$\bar{\mathbb{E}}\left\|u^M(t_{i+1})\right\|^4_H\leq\bar{\mathbb{E}}\left\|u^M(t_i)\right\|^4_H+C\bar{\mathbb{E}}\int\limits_{t_i}^{t_{i+1}}\left\|u^M(s)\right\|^4_H ds+$$

$$+\bar{\mathbb{E}}\int\limits_{t_i}^{t_{i+1}}\left\|G\big(s,u^M(s)\big)\right\|^4_H ds\quad\text{for}\quad i=0,\ldots,M-1. \tag{14}$$

Summing up the estimates (7) for $i=0,\ldots,l-1,\ l=1,\ldots,M,$ and using assumption (iv) on the nonlinear function G, we have

$$\bar{\mathbb{E}}\left\|u^M(t_l)\right\|^2_H+\bar{\mathbb{E}}\int\limits_0^{t_l}\left\|u^M(s)\right\|^p_V ds\leq$$

$$\leq\bar{\mathbb{E}}\left\|u_0\right\|^2_H+\bar{\mathbb{E}}\int\limits_0^{t_l}\left(1+\left\|u^M(s)\right\|^2_H\right)ds,\quad l=1,\ldots,M,$$

from which we deduce, owing to Gronwall's lemma, that

$$\sup_M\sup_{l=1,\ldots,M}\bar{\mathbb{E}}\left\|u^M(t_l)\right\|^2_H\leq C,\qquad\sup_M\sup_{l=1,\ldots,M}\bar{\mathbb{E}}\int\limits_0^{t_l}\left\|u^M(s)\right\|^p_V ds\leq C,$$

and subsequently, we get

$$\sup_M \sup_{t\in[0,T]} \bar{\mathbb{E}}\left\|u^M(t)\right\|_H^2 \le C, \qquad \sup_M \bar{\mathbb{E}}\int_0^T \left\|u^M(s)\right\|_V^p ds \le C. \tag{15}$$

We analogously also have

$$\sup_M \sup_{t\in[0,T]} \bar{\mathbb{E}}\left\|u^M(t)\right\|_H^4 \le C. \tag{16}$$

From condition (iii) on our data, we easily show that

$$\left\|A_s\big(u^M(t_{i+1})\big)\right\|_{V'} \le$$

$$\le C\left(\left\|u^M(t_{i+1})\right\|_V^{\frac{p}{p'}} + \|h_2(s)\|_{L^{p'}(D)}\right), \quad i=0,\ldots,M-1, \tag{17}$$

and thanks to the second estimate in (15) and the condition on h_2, we deduce that

$$E\int_0^T \left\|A_s\big(u^M(s)\big)\right\|_{V'}^{p'} ds < \infty. \tag{18}$$

Similarly

$$E\int_0^T \left\|G_s\big(u^M(s)\big)\right\|_H^4 ds < \infty. \tag{19}$$

Our next task is to estimate the incremental variation of u^M on the interval $[t_i, t_{i+1}]$ in the dual space V'. This estimate will be crucial for the proof of needed compactness results.

From the relations (2), we obtain

$$\left\|u^M(t_{i+1}) - u^M(t_i)\right\|_{V'}^{p'} \le C\tau^{p'}\left\|A_{t_i}^M\big(u^M(t_{i+1})\big)\right\|_{V'}^{p'} +$$

$$+C\left\|\tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big)\Delta W_i\right\|_{V'}^{p'}. \tag{20}$$

Since $p \ge 2$, we have that $p' \le 2$. Thus,

$$\bar{\mathbb{E}}\left\|\tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big)\Delta W_i\right\|_{V'}^{p'} =$$

$$= \bar{\mathbb{E}}\sup_{\varphi\in V', \|\varphi\|_{V'}=1}\left\langle \int_{t_i}^{t_{i+1}} \tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big)dW, \varphi\right\rangle^{p'} =$$

$$= \bar{\mathbb{E}}\sup_{\varphi\in V', \|\varphi\|_{V'}=1}\left(\int_{t_i}^{t_{i+1}} \left\langle \tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big), \varphi\right\rangle dW\right)^{p'} \le$$

$$\le \left[\bar{\mathbb{E}}\left(\int_{t_i}^{t_{i+1}} \sup_{\varphi\in V', \|\varphi\|_{V'}=1}\left\langle \tilde{G}_{t_i}^M\big(t_i, u^M(t_i)\big), \varphi\right\rangle dW\right)^2\right]^{\frac{p'}{2}}.$$

Now, thanks to Itô's isometry, Cauchy – Schwarz's inequality, estimates (4), (15) and condition (iv) on G, we deduce that

$$\bar{\mathbb{E}} \left\| \tilde{G}_{t_i}^M \big(t_i, u^M(t_i)\big) \Delta W_i \right\|_{V'}^{p'} \le C\tau^{\frac{p'}{2}} \left[E\left(1 + \left\|u^M(t_i)\right\|_H^2\right)\right]^{\frac{p'}{2}} \le C\tau^{\frac{p'}{2}}. \tag{21}$$

Combining (20), (21) with (17) and (15), we infer that

$$\bar{\mathbb{E}} \left\|u^M(t_{i+1}) - u^M(t_i)\right\|_{V'}^{p'} ds \le C\left(\tau^{p'-1} + \tau^{\frac{p'}{2}}\right), \quad i = 0, \ldots, M-1,$$

and, therefore,

$$\bar{\mathbb{E}} \sup_{\|\theta\| \le \tau} \int\limits_0^T \left\|u^M(t+\theta) - u^M(t)\right\|_{V'}^{p'} ds \le CT\left(\tau^{p'-1} + \tau^{\frac{p'}{2}}\right). \tag{22}$$

We summarize our findings in the following lemma.

Lemma 1. *Under the assumptions* (i) – (vi), *the sequence* $\left\{u^M(t)\right\}_{M\in\mathbb{N}}$ *defined by the relation* (3) *satisfies the estimates* (15), (16), (18), (19) *and* (22).

Armed with this lemma, we are able to establish crucial compactness results in the next subsection.

3.2. Probabilistic compactness results. We start this subsection by introducing some auxiliary spaces which will be needed for the compactness of probability measures generated by the pair $\big(\bar{W}, u^M\big)$.

Following [2], for any sequences $(\mu_n), (\nu_n)$ such that $\mu_n, \nu_n \ge 0$ and $\mu_n, \nu_n \to 0$ as $n \to \infty$, we define the set U_{μ_n,ν_n} of functions

$$\varphi \in L^p(0,T;V) \cap L^\infty(0,T;H)$$

such that

$$\sup_n \frac{1}{\nu_n} \sup_{|\theta| \le \mu_n} \left(\int\limits_0^T \left\|\varphi(t+\theta) - \varphi(t)\right\|_{V'}^{p'} dt \right)^{\frac{1}{p'}} < \infty.$$

We endow U_{μ_n,μ_n} with the norm

$$\|\varphi\|_{U_{\mu_n,\nu_n}} = \sup_{0\le t\le T} \left\|\varphi(t)\right\|_{L^2(D)} + \left(\int\limits_0^T \left\|\varphi(t)\right\|_V^p dt\right)^{\frac{1}{p}} +$$

$$+ \sup_n \frac{1}{\nu_n} \left(\sup_{|\theta|\le\mu_n} \int\limits_0^T \left\|\varphi(t+\theta) - \varphi(t)\right\|_{V'}^{p'} dt \right)^{\frac{1}{p'}}.$$

U_{μ_n,ν_n} is a Banach space.

Due to the compact embedding of V into H, we have the following compactness result from [1] which is interesting in its own right.

Lemma 2. *The set* U_{μ_n,ν_n} *defined above is a compact subset of* $L^2(0,T;H)$.

Let $2 \leq p < \infty$ and let $\mathcal{U}_{\mu_n,\nu_n}$ be the space consisting of random variables $\varphi(t)$ on $\big(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}\big)$ such that

$$\bar{\mathbb{E}} \sup_{0 \leq t \leq T} \big\|\varphi(t)\big\|_H^2 < \infty,$$

$$\bar{\mathbb{E}} \int_0^T \big\|\varphi(t)\big\|_V^p dt < \infty,$$

$$\bar{\mathbb{E}} \sup_n \frac{1}{\nu_n} \left(\sup_{|\theta| \leq \mu_n} \int_0^T \big\|\varphi(t+\theta) - \varphi(t)\big\|_{V'}^{p'} dt \right)^{\frac{1}{p'}} < \infty.$$

$\mathcal{U}_{\mu_n,\nu_n}$ is a Banach space under the norm

$$\|\varphi\|_{\mathcal{U}_{\mu_n,\nu_n}} = \left(\bar{\mathbb{E}} \sup_{0 \leq t \leq T} \big\|\varphi(t)\big\|_H^2 \right)^{\frac{1}{2}} + \left(\bar{\mathbb{E}} \int_0^T \|\varphi(t)\|_V^p dt \right)^{\frac{1}{p}} +$$

$$+ \bar{\mathbb{E}} \sup_n \frac{1}{\nu_n} \left(\sup_{|\theta| \leq \mu_n} \int_0^T \big\|\varphi(t+\theta) - \varphi(t)\big\|_{V'}^{p'} dt \right)^{\frac{1}{p'}}.$$

The a priori estimates established in the previous lemmas allow us to assert that for any $p \geq 2$, and for $\mu_n,\ \nu_n$ such that the series $\sum_{n=1}^{\infty} \dfrac{(\mu_n)^{1/[p'(p-1)]}}{\nu_n}$ converges, the sequence $\big\{u^M(t), M \in \mathbb{N}\big\}$ remains in a bounded subset of $\mathcal{U}_{\mu_n,\nu_n}$.

Next, let $S = C(0,T;\mathbb{R}^d) \times L^2(0,T;H)$ and $\mathcal{B}(S)$ the σ-algebra of the Borel sets of S. For each M, we construct the probability measure Π^M on $\big(S, \mathcal{B}(S)\big)$ as follows.
Consider the mapping

$$\varphi \colon \omega \mapsto \big(\bar{W}(.,\omega), u^M(.,\omega)\big)$$

defined on $\big(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}\big)$ and taking values in $\big(S, \mathcal{B}(S)\big)$. Then

$$\Pi^M(A) = \bar{\mathbb{P}}(\varphi^{-1}(A)) \quad \text{for all} \quad A \in \mathcal{B}(S).$$

We now formulate the following key tightness result.

Lemma 3. *The family of probability measures $\{\Pi^M\}_{M=1}^{\infty}$ is tight on $\big(S, \mathcal{B}(S)\big)$. That is, for any $\varepsilon > 0$, there exist some compact subsets $\Sigma_\varepsilon \subset C\big(0,T;\mathbb{R}^d\big)$ and $Z_\varepsilon \subset L^2(0,T;H)$ such that*

$$\Pi^M\left(\Sigma_\varepsilon \times Z_\varepsilon\right) \geq 1 - \varepsilon \quad \forall M \in \mathbb{N}.$$

Proof. For the proof, we refer for instance to [2, 32, 33].

The above tightness of the family of probability measures $\big(\Pi^M\big)$ and Prokhorov's theorem imply that $\{\Pi^M\}_{M=1}^{\infty}$ is relatively compact. Therefore, we can extract a subsequence $\big\{\Pi^{M_j}\big\}_{j=1}^{\infty}$ which weakly converges to a probability measure Π. Hence by Skorokhod's theorem, there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (the expectation of which we denote by $\mathbb{E}$) and pairs of random variables (W_{M_j}, u^{M_j}) and (W, u) on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in S such that

$$\text{the probability law of } (W_{M_j}, u^{M_j}) \quad \text{is} \quad \Pi^{M_j}, \quad \text{the probability law of} \quad (W, u) \quad \text{is} \quad \Pi, \tag{23}$$

$$W_{M_j}(.,\omega) \longrightarrow W(.,\omega) \quad \text{in} \quad C(0,T;\mathbb{R}^d) \quad \text{as} \quad j \longrightarrow \infty, \quad \mathbb{P}\text{-a.s.}, \tag{24}$$

$$u^{M_j}(.,\omega) \longrightarrow u(.,\omega) \quad \text{in} \quad L^2(0,T;H) \quad \text{as} \quad j \longrightarrow \infty \quad \mathbb{P}\text{-a.s.} \tag{25}$$

Next, we choose the filtration $(\mathcal{F}_t)$ by setting

$$\mathcal{F}_t = \sigma\big\{(W(s), u(s)) : 0 \leq s \leq t\big\}.$$

It turns out, according to similar reasoning used in [2, 32, 33], that W is a d-dimensional $\mathcal{F}_t$-standard Wiener process.

It follows also that

$$u^{M_j}(t_{i+1}) = u^{M_j}(t_i) - \tau A_{t_i}^{M_j}\big(u^{M_j}(t_{i+1})\big) + \tilde{G}_{t_i}^{M_j}\big(t_i, u^{M_j}(t_i)\big)\Delta W_{M_j,i}, \quad \mathbb{P}\text{-a.s.}, \tag{26}$$

where

$$\Delta W_{M_j,i} = W_{M_j}(t_{i+1}) - W_{M_j}(t_i).$$

Then

$$u^{M_j}(t) = u^{M_j}(t_i) \quad \text{if} \quad t \in [t_i, t_{i+1}), \quad \mathbb{P}\text{-a.s.},$$

and, therefore, gluing the relations (26) by means of a summation over $i = 1, \ldots, M_j - 1$, we have

$$u^{M_j}(t) + \int_0^t A_s\big(u^{M_j}(s)\big)ds = \int_0^t G\big(s, u^{M_j}(s)\big)dW_{M_j}(s), \quad \mathbb{P}\text{-a.s.} \tag{27}$$

as an equality between random variables with values in V' for any $t \in [0,T]$.

We are now in the position to prove Theorem 1. This will be carried through in the next section.

4. Proof of Theorem 1. The proof of Theorem 1 proceeds in several steps.

***Step* 1.** This step is devoted to some weak convergence results. Owing to relation (27), we see that Lemma 1 holds for the sequence $u^{M_j}(t)$; that is, for any $p \in [2,\infty)$,

$$\sup_M \mathbb{E}\int_0^T \big\|u^{M_j}(s)\big\|_V^p\, ds \leq C, \tag{28}$$

$$\sup_M \sup_{t\in[0,T]} \mathbb{E}\big\|u^{M_j}(t)\big\|_H^4 \leq C, \tag{29}$$

$$\mathbb{E}\sup_{\|\theta\|\leq\tau}\int_0^T \big\|u^{M_j}(t+\theta) - u^{M_j}(t)\big\|_{V'}^{p'} ds \leq CT\Big(\tau^{p'-1} + \tau^{\frac{p'}{2}}\Big), \tag{30}$$

$$\mathbb{E}\int_0^T \big\|A_s\big(u^{M_j}(s)\big)\big\|_{V'}^{p'} ds < \infty, \tag{31}$$

and similarly

$$\mathbb{E}\int_0^T \big\|G_s\big(u^{M_j}(s)\big)\big\|_H^4 ds < \infty. \tag{32}$$

Thus, there exists a new subsequence of $u^{M_j}(t)$, which we still denote by the same symbol, such that

$$u^{M_j} \rightharpoonup u \quad \text{weakly in} \quad L^p\big(\Omega, L^p(0,T;V)\big), \tag{33}$$

$$u^{M_j} \rightharpoonup u \quad \text{weakly in} \quad L^4\big(\Omega, L^r(0,T;H)\big) \quad \forall r \in [2,\infty), \tag{34}$$

$$u^{M_j}(\omega) \rightharpoonup u(\omega) \quad \text{weakly star in} \quad L^\infty(0,T;H) \quad \text{for almost all} \quad \omega \in \Omega.$$

Furthermore, u satisfies

$$\mathbb{E}\int_0^T \|u(t)\|_V^p dt \le C,$$

$$\mathbb{E}\int_0^T \|u(t)\|_H^r dt \le C \quad \forall r \in [1,\infty),$$

$$\big\|u(\omega)\big\|_{L^\infty(0,T;H)} < \infty, \quad \mathbb{P}\text{-a.s.}$$

It follows from (31) that there exists a random function $\chi \in L^{p'}\big(\Omega, L^{p'}(0,T;V')\big)$ such that up to extraction of a subsequence

$$A_t\big(u^{M_j}(\cdot)\big) \rightharpoonup \chi\,(\cdot) \quad \text{weakly in} \quad L^{p'}\Big(\Omega, L^{p'}(0,T;V')\Big). \tag{35}$$

Thanks to (29) and Vitali's theorem, we obtain

$$u^{M_j} \longrightarrow u \quad \text{strongly in} \quad L^2(\Omega, L^2(0,T;H)) \quad \text{and almost everywhere.} \tag{36}$$

Thus, there exists a new subsequence still denoted as u^{M_j}, such that for almost every (t,ω), we have

$$u^{M_j} \longrightarrow u \quad \text{strongly in}\, H \text{ (with respect to the measure}\, d\mathbb{P}\times dt). \tag{37}$$

Owing to the condition (v) on G, the estimates (32), the a. e. convergence of u^{M_j} to u on $\Omega\times[0,T]$, we see that $G(s,u^{M_j})$ is uniformly integrable in $L^2\big(\Omega, L^2(0,T;H)\big)$ and $G(s,u^{M_j}(s))$ converges to $G(s,u)$ a. e. on $\Omega\times[0,T]$. Therefore, Vitali's theorem implies that

$$G(s,u^{M_j}) \to G(s,u) \quad \text{strongly in} \quad L^2\left(\Omega, L^2(0,T;H)\right). \tag{38}$$

***Step* 2.** We prove in this step the convergence of the stochastic integral

$$\int_0^T G(t,u^{M_j}(t))dW_{M_j}(t).$$

We intend to use integration by parts. But since the integrand is not smooth with respect to t, we introduce a suitable regularization in order to overcome that obstacle. For that purpose, letting ϱ be a standard mollifier, we define, for $v\in L^2(D)$, the function

$$G^\varepsilon(t,v)=\frac{1}{\varepsilon}\int\limits_0^T \varrho\left(\frac{s-t}{\varepsilon}\right)G(s,v)ds;$$

G^ε is smooth in t and continuous in v, and we have the uniform estimate

$$\mathbb{E}\int\limits_0^T \|G^\varepsilon(t,v)\|^2_{H^d}dt\le \mathbb{E}\int\limits_0^T \|G(t,v)\|^2_{H^d}dt \tag{39}$$

and

$$G^\varepsilon(.,u)\longrightarrow G(.,u)\quad\text{in}\quad L^2\left(\Omega,L^2\left(0,T;H^d\right)\right) \tag{40}$$

as $\varepsilon\to 0$.

Integrating by parts, we get

$$\int\limits_0^t G^\varepsilon\big(s,u^{M_j}(s)\big)dW_{M_j}(s)=G^\varepsilon(t,u^{M_j})W_{M_j}(t)-\int\limits_0^t G^{\varepsilon\prime}\big(s,u^{M_j}(s)\big)W_{M_j}(s)ds. \tag{41}$$

By Fubini's theorem, Burkholder – Davis – Gundy's inequality and (39), we obtain

$$\mathbb{E}\left\|\int\limits_0^t G^\varepsilon\big(s,u^{M_j}(s)\big)dW_{M_j}(s)\right\|^2_H\le \mathbb{E}\int\limits_0^t \left\|G^\varepsilon\big(s,u^{M_j}(s)\big)\right\|^2_{H^d}ds\le C. \tag{42}$$

Similarly,

$$\int\limits_0^t G^\varepsilon(s,u)dW(s)=G^\varepsilon(t,u)W(t)-\int\limits_0^t G^{\varepsilon\prime}(s,u)W(s)ds. \tag{43}$$

Owing to (38), we have that

$$G^\varepsilon(t,u^{M_j})\longrightarrow G^\varepsilon(t,u)\quad\text{a. e. in}\quad \Omega\times(0,T). \tag{44}$$

It then follows from the definition of $G^\varepsilon (G^{\varepsilon\prime}(t,\cdot)$ is still continuous in $\cdot)$, (41) and (24) that

$$\lim_{j\longrightarrow\infty}\int\limits_0^t G^\varepsilon(s,u^{M_j})dW_{M_j}(s)=G^\varepsilon(s,u)W(t)-\int\limits_0^t G^{\varepsilon\prime}(s,u)W(s)ds \tag{45}$$

for almost all ω. Hence, by (43) and (45), we get

$$\lim_{j\longrightarrow\infty}\int\limits_0^t G^\varepsilon(s,u^{M_j})dW_{M_j}(s)\to\int\limits_0^t G^\varepsilon(s,u)dW(s) \tag{46}$$

for almost all ω.

By (42), the sequence of stochastic integrals $\left(\int_0^t G^\varepsilon(s,u^{M_j})dW_{M_j}(s)\right)_{i\in\mathbb{N}}$ is uniformly bounded in $L^2(\Omega,H)$ for any $t\in[0,T]$, then it is uniformly integrable in the space $L^r(\Omega,H)$ for any

$1 \leq r < 2$. Combining this with (46), we are able to use Vitali's theorem in order to obtain that

$$\int_0^t \int_0^t G^\varepsilon(s, u^{M_j}) dW_{M_j}(s) \to \int_0^t G^\varepsilon(s, u) dW(s) \quad \text{strongly in} \quad L^r(\Omega, H). \tag{47}$$

On the other hand, we also have that

$$\int_0^t G^\varepsilon(s, u^{M_j}) dW_{M_j}(s) \rightharpoonup \psi(t) \quad \text{weakly in} \quad L^2(\Omega, H),$$

for some random function ψ. Therefore,

$$\int_0^t G^\varepsilon(s, u^{M_j}) dW_{M_j}(s) \rightharpoonup \psi(t) \quad \text{weakly in} \quad L^r(\Omega, H) \quad \text{for} \quad 1 \leq r < 2.$$

Since the convergence (47) holds also weakly in $L^r(\Omega, H)$, we get

$$\psi(t) = \int_0^t G^\varepsilon(s, u) dW(s),$$

by uniqueness of weak limits. Thus,

$$\int_0^t G^\varepsilon(s, u^{M_j}) dW_{M_j}(s) \rightharpoonup \int_0^t G^\varepsilon(s, u) dW(s) \quad \text{weakly in} \quad L^2(\Omega, H).$$

This can be expressed as: for fixed ε let j tends to ∞ to have, for any $\kappa \in L^2(\Omega, H)$,

$$\mathbb{E}\left(\kappa, \int_0^t G^\varepsilon(s, u^{M_j}) dW_{M_j}(s)\right) \to \mathbb{E}\left(\kappa, \int_0^t G^\varepsilon(s, u) dW(s)\right). \tag{48}$$

We obviously have that the sequence $\left(\int_0^t G^\varepsilon(s, u^{M_j}) dW_{M_j}(s)\right)_{j \in \mathbb{N}}$ is uniformly bounded in $L^2(\Omega, H)$. Thus, there exists $\eta \in L^2(\Omega, H)$ such that for any $\kappa \in L^2(\Omega, H)$

$$\mathbb{E}\left(\kappa, \int_0^t G^\varepsilon(s, u^{M_j}) dW_{M_j}(s)\right) \to \mathbb{E}(\kappa, \eta) \quad \text{as} \quad j \to \infty.$$

Lastly we need to prove that $\int_0^t G(s, u) dW(s) = \eta$. For that purpose, we rewrite (48) as follows:

$$\mathbb{E}\left(\kappa, \int_0^t G(s, u^{M_j}) dW_{M_j}(s) - \int_0^t G(s, u) dW(s)\right) = I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon, \tag{49}$$

where κ is an arbitrary element of $L^2(\Omega, H)$ and

$$I_1^\varepsilon = \mathbb{E}\left(\kappa, \int_0^t \left[G(s, u^{M_j}) - G^\varepsilon(s, u^{M_j})\right] dW_{M_j}(s)\right),$$

$$I_2^\varepsilon = \mathbb{E}\left(\kappa, \int_0^t \left[G^\varepsilon(s, u) - G(s, u)\right] dW(s)\right),$$

$$I_2^\varepsilon = \mathbb{E}\left(\kappa, \int_0^t G^\varepsilon(s, u^{M_j}) dW_{M_j}(s) - \int_0^t G^\varepsilon(s, u) dW(s)\right).$$

By Burkholder – Davis – Gundy's inequality

$$I_1^\varepsilon \leq \mathbb{E}\|\kappa\|_{L^2(D)} \left\| \int_0^t \left[G(s, u^{M_j}) - G^\varepsilon(s, u^{M_j})\right] dW_{M_j}(s) \right\|_H \leq$$

$$\leq C\mathbb{E}\left[\int_0^t \left\|G(s, u^{M_j}) - G^\varepsilon(s, u^{M_j})\right\|_{H^d}^2 ds\right]^{\frac{1}{2}}$$

and

$$I_2^\varepsilon \leq C\mathbb{E}\left[\int_0^t \left\|G(s, u) - G^\varepsilon(s, u)\right\|_{H^d}^2 ds\right]^{\frac{1}{2}}.$$

Passing to the limit as $\varepsilon \longrightarrow 0$ in the above inequalities and using (40), we get that $\lim_{\varepsilon \longrightarrow 0} \left(I_1^\varepsilon + + I_2^\varepsilon\right) = 0$. By (48), we have

$$I_3^\varepsilon = \mathbb{E}\left(\kappa, \int_0^t G^\varepsilon(s, u^{M_j}) dW_{M_j}(s) - \int_0^t G^\varepsilon(s, u) dW(s)\right) \to 0.$$

Thus, it follows from (49) that

$$\int_0^t G(s, u^{M_j}) dW_{M_j}(s) \rightharpoonup \int_0^t G(s, u) dW(s) \quad \text{weakly in} \quad L^2(\Omega, H). \tag{50}$$

***Step* 3.** In this step, we prove that $\chi = A_t(u)$. This will close the arguments leading to the complete proof of Theorem 1, since passing to the limit in equation (27) and using the convergences (35), (37), (50), we have that

$$u(t) + \int_0^t \chi(s)\, ds = \int_0^t G(s, u(s))\, dW(s), \quad \mathbb{P}\text{-a.s. in} \quad V'. \tag{51}$$

Then the needed relation will follow from the definition of pseudomonotone operators as given in the introduction.

Let us prove (51). We recall the equation (27)

$$u^{M_j}(t) + \int_0^t A_s\big(u^{M_j}(s)\big)ds = \int_0^t G\big(s, u^{M_j}(s)\big)dW_{M_j}(s), \quad \mathbb{P}\text{-a.s. in} \quad V'. \tag{52}$$

Testing the equation (52) with $u^{M_j} - u$ with respect to the inner product of H, we get

$$\int_0^t \big(u^{M_j}(s) - u(s), du^{M_j}\big) + \int_0^t \big\langle A_s\big(u^{M_j}(s)\big), u^{M_j}(s) - u(s)\big\rangle\, ds =$$

$$= \int_0^t \big(G\big(s, u^{M_j}(s)\big), u^{M_j}(s) - u(s)\big)\, dW_{M_j}(s).$$

By Burkholder – Davis – Gundy's inequality and (36), it is clear that

$$\lim_{j\to\infty} E \int_0^t \big(G(s, u^{M_j}(s)), u^{M_j}(s) - u(s)\big)\, dW_{M_j}(s) = 0.$$

Therefore, (51) will follow, if we can show the following lemma.

Lemma 4. *Under our conditions*

$$\lim_{j\to\infty} \inf \mathbb{E} \int_0^t \big(u^{M_j}(s) - u(s), du^{M_j}\big) \geq 0.$$

This result is crucial for our work. It is the stochastic version of a result by Landes (see [21]). The proof is partly based on the following integration by parts of stochastic integrals:

$$\big(u^{M_j}(t), u(t)\big) = \int_0^t \big(du^{M_j}(s), u(s)\big) +$$

$$+ \int_0^t \big(du(s), u^{M_j}(s)\big) + \big\langle\big\langle u^{M_j}, u\big\rangle\big\rangle_t^H, \tag{53}$$

where the last term denotes the quadratic covariation of u^{M_j} and u, namely, if $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of H,

$$\big\langle\big\langle u^{M_j}, u\big\rangle\big\rangle_t^H =$$

$$= \sum_{i=1}^{\infty} \left(\int_0^t G\big(s, u^{M_j}(s)\big)dW_{M_j}(s), e_i\right)\left(\int_0^t G(s, u(s))dW(s), e_i\right) =$$

$$= \left(\int_0^t G(s, u^{M_j}(s))dW_{M_j}(s), \int_0^t G(s, u(s))dW(s)\right).$$

In view of (50),

$$\lim_{j\to\infty}\mathbb{E}\left\langle\left\langle u^{M_j},u\right\rangle\right\rangle_t^H=\mathbb{E}\left\|\int\limits_0^t G(s,u(s))dW(s)\right\|_H^2=$$

$$=\mathbb{E}\int\limits_0^t\left\|G(s,u(s))\right\|_H^2 ds, \tag{54}$$

thanks to Fubini's theorem and Itô's isometry.

Next, we show that

$$\lim_{j\to\infty}\mathbb{E}\int\limits_0^t\big(du^{M_j}(s),u(s)\big)=\lim_{j\to\infty}\mathbb{E}\int\limits_0^t\big(du(s),u^{M_j}(s)\big). \tag{55}$$

Testing (52) with u, and using the convergences (35) and (50), it readily follows that

$$\lim_{j\to\infty}\mathbb{E}\int\limits_0^t\big(du^{M_j}(s),u(s)\big)=-\mathbb{E}\int\limits_0^t\langle\chi(s),u\rangle+\mathbb{E}\int\limits_0^t\big(G(s,u(s)),\,u\big)dW(s).$$

Similarly

$$\lim_{j\to\infty}\mathbb{E}\int\limits_0^t\big(du(s),u^{M_j}(s)\big)=-\mathbb{E}\int\limits_0^t\langle\chi(s),u\rangle+\mathbb{E}\int\limits_0^t\big(G(s,u(s)),\,u\big)dW(s).$$

Hence, (55) holds.

Since $u^{M_j}(t)$ weakly converges to $u(t)$, by (28) for almost every t, we deduce from (53)–(55) that

$$\lim_{j\to\infty}\mathbb{E}\int\limits_0^t\big(du^{M_j}(s),u(s)\big)=\frac{1}{2}\mathbb{E}\left(\left\|u(t)\right\|_H^2-\int\limits_0^t\left\|G(s,u(s))\right\|_H^2 ds\right). \tag{56}$$

By Itô's formula,

$$\mathbb{E}\int\limits_0^t\big(du^{M_j}(s),u^{M_j}(s)\big)=$$

$$=\frac{1}{2}\mathbb{E}\left(\left\|u^{M_j}(t)\right\|_H^2-\int\limits_0^t\left\|G(s,u^{M_j}(s))\right\|_H^2 ds\right). \tag{57}$$

We then deduce from (57), (38), Fatou's lemma and the weak semicontinuity of the norm $\|\cdot\|_H$ that

$$\liminf_{j\to\infty}\mathbb{E}\int\limits_0^t\big(du^{M_j}(s),u^{M_j}(s)\big)=$$

$$= \frac{1}{2} \liminf_{j\to\infty} \mathbb{E}\left(\left\| u^{M_j}(t) \right\|_H^2 - \int_0^t \left\| G(s, u^{M_j}(s)) \right\|_H^2 ds \right) \geq$$

$$\geq \frac{1}{2}\mathbb{E}\left(\left\| u(t) \right\|_H^2 - \int_0^t \left\| G(s, u(s)) \right\|_H^2 ds \right). \qquad (58)$$

The Lemma 4 now follows from (56) and (58), using the subadditivity property of $\liminf$. This subsequently proves (51), and we can deduce from the definition of pseudomonotone operators that $\chi(t) = A_t(u)$. We therefore conclude that u is a weak solution of problem (P).

Theorem 1 is proved.

***Remark* 1.** The arguments used in the proof of our main result readily apply to the non vanishing initial value $(u(0) \neq 0)$ case and to the case when the process $W(t)$ is a cylindrical Hilbert space-valued Wiener process. We omitted these generalities in order to focus on the key ideas leading to the settling of the main problematic. Since pseudomonotone operators arise naturally in variational inequalities, the approach developed here is a decisive stepping stone for the generalization of our result to stochastic variational inequalities featuring pseudomontone operators.

In the next section, we provide an example of application of our main result which includes several important particular cases of stochastic partial differential equations arising in applied sciences.

5. Example of application of Theorem 1. From now on, we set $V = W_0^{m,p}(D),\ H = L^2(D)$ and $V' = W^{-m,p'}(D)$ with p', the conjugate of p and $p \geq 2$. By Rellich – Kondrachov embedding theorem, V is compactly embedded in H which in its turn is continuously embedded in V'. As applications of the theory developed in the paper, we consider the higher-order stochastic quasilinear parabolic problem

$$(P) \begin{cases} du + \left[A_t(u) + g(t,x,u)\right]dt = f(t,x)dt + G(t,u)dW(t) & \text{in} \quad Q_T, \\ D^\alpha u = 0 = 0 & \text{on}\, (0,T)\times\partial D \text{ for } |\alpha| \leq m-1, \\ u(x,0) = 0 & \text{in} \quad D, \end{cases}$$

where $T > 0$ is fixed real number, D is a bounded domain in $\mathbb{R}^n$, Q_T is the cylinder $(0,T)\times D$, the stochastic process $u = u(t,x)$ and the standard d-dimensional Wiener process W together with the probability space on which they are defined are the unknowns, the functions f, G and g are given, A_t is an elliptic operator of order $2m$ in the generalized divergence form, that is,

$$A_t(u) = \sum_{|\beta|\leq m} (-1)^{|\beta|} D^\beta A_\beta(t,x,u,Du,\ldots,D^m u),$$

with the functions A_β satisfying the Carathéodory conditions, that is each $A_\beta(t,x,\xi)$ is measurable in (t,x) and continuous in ξ. Here ξ is an element of the vector space $\mathbb{R}^N$ of m-jets on $\mathbb{R}^n$ which assumes the representation $\xi = \{\xi_\alpha : |\alpha| \leq m\}$. To each ξ, there corresponds a couple (η,ζ), with $\eta = \{\eta_\alpha : |\alpha| \leq m-1\}$ and $\zeta = \{\zeta_\alpha : |\alpha| = m\}$. Let $Q = [0,T]\times D$ and $2 \leq p < \infty$. We now formulate the conditions on A_β following Browder [11].

(i) For each multiindex β with $|\beta| \leq m$, $A_\beta(t,x,\xi)$ is Carathéodory; that is, it is measurable in (t,x) on $Q = [0,T]\times D$ for each fixed m-jet $\xi = \left\{\xi_\alpha : |\alpha| \leq m\right\}$ and continuous in ξ for almost all (t,x). In addition, there exist a constant $c_0 > 0$ and nonnegative function $h_0 \in L^{p'}\left(0,T; L^{p'}(Q)\right)$

such that

$$\big|A_\beta(t,x,\xi)\big| \le c_0\big\{|\xi|^{p-1} + h_0(t,x)\big\}$$

for all $(t,x) \in [0,T] \times D$ and all m-jets ξ.

(ii) If we divide up the m-jet ξ into its pure mth order part ζ and the lower-order jets η, then

$$\sum_{|\beta|=m} \Big[A_\beta(t,x,\eta,\zeta) - A_\beta(t,x,\eta,\tilde{\zeta})\Big]\,(\zeta_\beta - \tilde{\zeta}_\beta) > 0$$

for $\zeta_\beta \neq \tilde{\zeta}_\beta$ and for all $(t,x) \in Q$.

(iii) There exist a constant $c_1 > 0$ and a positive function $h_1 \in L^1(Q)$ such that

$$\sum_{|\beta|\le m} A_\beta(t,x,\xi)\xi_\beta \ge c_1|\xi|^p - h_1(t,x)$$

for all $(t,x) \in Q$ and all ξ.

(iv) $g(t,x,u)$ is Carathéodory. It satisfies the sign condition $rg(t,x,r) \ge 0,\ g(t,x,0) = 0$ and $g \in L^\infty(Q)$.

(v) The intensity of the noise $G(t,u)\colon [0,T] \times L^2(D) \longrightarrow (L^2(D))^d$ is continuous in (t,u), and there exists a positive constant C such that

$$\big\|G(t,u)\big\|_{(L^2(D))^d} \le C\big(1 + \|u(t)\|_{L^2(D)}\big).$$

(vi) We assume $f(t,x)$ is measurable in Q and there exists a positive constant C such that

$$\int_0^T \|f(t)\|^{p'}_{L^{p'}(D)}dt \le C.$$

We consider the operator family $\mathcal{A}_t\colon W_0^{m,p}(D) \longrightarrow W^{-m,p'}(D)$, defined by

$$\langle \mathcal{A}_t(u), v\rangle = \sum_{|\beta|\le m} \int_D A_\beta(t,x,u,Du,\ldots,D^m u)D^\beta v dx$$

for any $u, v \in W_0^{m,p}(D)$ and for any $t \in [0,T]$.

We note that under the conditions (i)–(iii), the operator $\mathcal{A}_t$ is pseudomonotone, as proved by Browder in [10] (see also [6, 8]). Alongside $\mathcal{A}_t$, we consider the operator $S\colon V \to V'$ such that

$$\langle S(u),\, v\rangle = \int_D g(t,x,u(x))v(x)dx.$$

Based on the above conditions, as in [41], we have the following crucial result.

Lemma 5. *The operators $\mathcal{A}_t$ and S induce the operator*

$$\mathcal{A}_t + S\colon L^p(0,T;V) \to L^{p'}(0,T;V'),$$

which is pseudomonotone.

Now we can invoke Theorem 1, to infer that problem (P) has at least a martingale solution

$$u \in L^2\Big(\Omega, L^\infty\big(0,T; L^2(D)\big)\Big) \cap L^p\Big(\Omega, L^p\big(0,T; W_0^{m,p}(D)\big)\Big),$$

in the sense of Definition 1.

The result of this example is to the best of our knowledge new for higher-order quasilinear stochastic parabolic equations of pseudomonotone type. The merit of the general approach undertaken in the work, is that the arguments are independent of the order of the equations. In the second-order case, arguments relying on truncation functions may lead to particular versions of our result, but it is well-know that such arguments break in the higher order case, due to lack of appropriate corresponding notion of truncation.

6. Closing remarks. We are deeply grateful to one of the reviewers for her/his insighful comments and for suggesting that we compare our work to those of Liu and Röckner on local monotonicity which she/he brought to our attention. This section is devoted to that task.

The main difference between our work and those of Liu and Röckner [22 – 24] is that while we are dealing in the paper with genuine pseudomonotone operators, as defined by Brezis and Browder [5, 9, 10], for the class of stochastic evolution equations studied, they consider local monotonicity, generalizing the results of Pardoux [27], Krylov and Rozovskii [20].

Namely, they assume the following condition appearing in these papers as

(H_2) (local monotonicity)

$$2\langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle_{V^*\times V} + \big\|B(t,v_1) - B(t,v_2)\big\|_2^2 \leq$$
$$\leq \big(K + \rho(v_2)\big)|v_1 - v_2|_H^2,$$

where $\rho\colon V \to [0,+\infty)$ is a measurable function and locally bounded in V.

In [24] under (H_2) and some additional conditions such as coercivity and growth conditions, they establish unique strong probabilistic solution for evolution equations of the type

$$dX_t = A(t,X_t)dt + B(t,X_t)dW_t.$$

Condition (H_2) is further weakened in [23] to

(H_2'') (local monotonicity)

$$2\langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle_{V^*\times V} + \big\|B(t,v_1) - B(t,v_2)\big\|_2^2 \leq$$
$$\leq (f + \eta(v_1) + \rho(v_2))|v_1 - v_2|_H^2,$$

where $f \in L^1\big([0,T],\mathbb{R}\big)$, $\eta,\rho\colon V \to [0,+\infty)$ are measurable functions and locally bounded in V.

In [23], Liu shows that (H_2'') together with some additional conditions imply pseudomonotonicity of $A(t,\cdot)$ in the same sense as ours. So pseudomonotonicity as considered by us is weaker than the above (H_2'') version of local monotonicity. But the results of [23] are essentially deterministic since they are established for evolution equations of the form

$$u'(t) = A(t,u(t)) + b(t). \tag{59}$$

For the applications of (H_2'') to SDEs, Liu announced in [23] the investigation stochastic evolution equations of the form

$$dX_t = A(t, X_t)dt + BdN_t,$$

with Levy type additive noise. This project was fulfilled in [22] jointly with Röckner, for the equation

$$dX_t = A(t, X_t)dt + B(t)dW_t \tag{60}$$

driven by Wiener noise and $B(t)$ is a functional depending on t only, by reducing (60) to the deterministic like evolution equation (59), thanks to a transformation made possible by the special form of the random forcing $B(t)dW_t$. Hence the results for equation (60) are derived from those of this deterministic equation established in [23].

The results of [24] were extended to the case of SPDEs driven by Levy noises in [14].

These facts show that the existence result for the genuine pseudomonotone operator that we prove was never established, to the best of our knowledge, in a previous paper and neither in Liu and Röckner's works.

The stochastic evolution equation that we consider in its current form can't be reduced to a deterministic like equation.

Due to lack of monotonicity and local monotonicity, Galerkin's method is not applicable to our stochastic evolution problem due to unavailability of a Carathéodory like existence result for the corresponding system of stochastic ordinary differential equations arising in the Galerkin approximation scheme. This compels us to use Gyöngy – Millet's numerical scheme.

Due to lack of local Lipschitzity on the intensity of the noise in our equation, the natural solution is martingale like (probabilistic weak in the sense of Skorokhod) as considered by us. The papers considering local monotonicity which implicitly implies local Lipschitzity of the intensity of the noise, are able to establish strong solutions.

A key point in our work is a sign condition introduced by Landes [21] for deterministic equations involving pseudomonotone operators that we successfully extend to the stochastic case.

Our work therefore generalize the results obtained for SPDEs under local monotonicity and methodologically, our approach is different as well.

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