DOI: 10.37863/umzh.v74i8.6134

UDC 517.5

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DEFORMED HANKEL TRANSFORM OF DINI – LIPSCHITZ FUNCTIONS

ДЕФОРМОВАНЕ ПЕРЕТВОРЕННЯ ГАНКЕЛЯ ФУНКЦІЙ ДІНІ – ЛІПШИЦЯ

By using a generalized symmetric difference Δ_h^m of order m and step $h > 0$, we obtain an analog of the Titchmarsh theorems [Introduction to the theory of Fourier integrals, Oxford Univ. Press (1948)] (Theorems 84 and 85) for the deformed Hankel transform. We also provide a further extension of the theorem cited above for functions in L_k^p with an abstract deformed Hankel – Dini – Lipschitz condition.

Використовуючи узагальнену симетричну різницю Δ_h^m порядку m і кроку $h > 0$, отримано аналог теорем Тітчмарша [Introduction to the theory of Fourier integrals, Oxford Univ. Press (1948)] (теореми 84 і 85) для деформованого перетворення Ганкеля. Крім того, наведено додаткове розширення вказаної теореми для функцій у L_k^p з абстрактною деформованою умовою Ганкеля – Діні – Ліпшиця.

1. Introduction. In [8], Titchmarsh gave a Lipschitz condition on a function $f \in L^p(\mathbb{R})$ for which its Fourier transform belongs to $L^\beta(\mathbb{R})$ for some values of β. His result reads as follows.

Theorem 1.1 ([8], Theorem 84). *Let f belong to $L^p(\mathbb{R})$, $1 < p \leq 2$, and*

$$\int\limits_{\mathbb{R}} \big|f(x+h) - f(x)\big|^p dx = O\big(h^{\alpha p}\big), \qquad 0 < \alpha \leq 1, \quad \textit{as} \quad h \to 0.$$

Then $\widehat{f}$ belong to $L^\beta(\mathbb{R})$ for

$$\frac{p}{p + p\alpha - 1} < \beta \leq \frac{p}{p-1}.$$

Theorem 1.2 ([8], Theorem 85). *Let $\alpha \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following statement are equivalents*:

$$\|f(.+h) - f\|_{L^2(\mathbb{R})} = O(h^\alpha),$$

$$\int\limits_{|\lambda| \geq r} |\widehat{f}(f)(\lambda)|^2 d\lambda = O\big(r^{-2\alpha}\big) \quad \textit{as} \quad r \to \infty,$$

where $\widehat{f}$ stands for the Fourier transform of f.

There are many different analogs of Theorems 1.1 and 1.2: for the Fourier transform on Riemannian symmetric spaces of rank 1 and, in particular, for the Fourier transform on the Lobachevski plane, for the Fourier – Jacobi transform, for the q-Bessel transform, etc. (see, for example, [3 – 7]).

In the present paper, we obtain a analog of Theorem 1.1 and Theorem 1.2 for the deformed Hankel transform newly introduced by S. Ben Saïd [1, 2]. The importance of this transform lies in

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the fact that it generalizes many integral transforms. Furthermore, we define the deformed Hankel – Dini – Lipschitz class $H^{m,p}_{\alpha,\gamma}$ and we obtain an extension of the Theorem 1.1 in this occurence.

This paper is organized as follows. Section 2 is a summary of the main results in the harmonic analysis associated with the deformed Hankel transform and we prove some auxiliary results required for the proofs of the main results. In Section 3, first, we define the deformed Hankel – Lipschitz class $H^{m,p}_{\alpha}$. Next, we prove analogous of the Theorem 1.1. In Section 4, we consider the particular case, when $p = 2$, and we provide a characterization of the space $H^{m,2}_{\alpha}$ of deformed Hankel – Lipschitz class functions by means of asymptotic estimate growth of the norm of their deformed Hankel transform for $\alpha \in (0, 1]$. In Section 5, we extend the Theorem 1.1 to the deformed Hankel – Dini – Lipschitz class $H^{m,p}_{\alpha,\gamma}$.

2. Definitions and auxiliary results. In this section, first, we briefly collect the pertinent definitions and facts relevant for deformed Hankel transform. Secondly, we prove some auxiliary results. For more details we refer to [1, 2].

We denote by L^p_k the space of measurable functions f on $\mathbb{R}$ with the finite norm

$$\|f\|_{k,p} = \left(\int\limits_{\mathbb{R}} |f(x)|^p d\mu(x) \right)^{1/p},$$

where $d\mu(x) = 2^{-1}\Gamma(2k)^{-1}|x|^{2k-1}dx$.

In [1], the author introduced a new transform $\mathcal{F}_k(f)$ called deformed Hankel transform which is a deformation of the Hankel transform by a parameter $k > 0$. Namely, for $k > \dfrac{1}{4}$ and $f \in L^1_k$, the integral transform $\mathcal{F}_k(f)$ is defined by

$$\mathcal{F}_k(f)(\lambda) = \int\limits_{\mathbb{R}} \mathcal{B}_k(\lambda, x) f(x) d\mu_k(x), \quad \lambda \in \mathbb{R},$$

where the kernel $\mathcal{B}_k(\lambda, x)$, called deformed Hankel kernel, given by

$$\mathcal{B}_k(\lambda, x) = j_{2k-1}\Big(2\sqrt{|\lambda x|}\Big) - \frac{\lambda x}{2k(2k+1)} j_{2k+1}\Big(2\sqrt{|\lambda x|}\Big).$$

Here, j_α is the normalized Bessel function of the first kind and order α defined by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n \left(\dfrac{z}{2}\right)^{2n}}{n!\Gamma(n+\alpha+1)}, \quad z \in \mathbb{C},$$

where $\Gamma(x)$ is the gamma-function.

Theorem 2.1. *For all $k > \dfrac{1}{4}$, we have*

$$\big|\mathcal{B}_k(\lambda, x)\big| \le 1, \quad \lambda, x \in \mathbb{R},$$

$$\mathcal{B}_k(\lambda, x) = j_{2k-1}\Big(2\sqrt{\lambda x}\Big) - \operatorname{sgn}(\lambda x)\Big(j_{2k}\Big(2\sqrt{\lambda x}\Big) - j_{2k-1}\Big(2\sqrt{\lambda x}\Big)\Big),$$

$\mathcal{F}_k(f)$ is an involutional unitary operator on $L^1(R, d\mu_k(x))$,

Parseval's identity, if $f \in L^1_k \bigcap L^2_k$, *then* $\left\|\mathcal{F}_k(f)\right\|_{k,2} = \left\|\mathcal{F}_k(f)\right\|_{k,2}$,
there exists a unique isometry on L^2_k *that coincides with* $\mathcal{F}_k(f)$ *on* $f \in L^1_k \cap L^2_k$.

Let f be a suitable function on $\mathbb{R}$. The translation operator T^y_k is defined by

$$T^y_k(f)(x) = \int\limits_{\mathbb{R}} f(z)\mathcal{K}_k(x,y,z)d\mu_k(z),$$

where

$$\mathcal{K}_k(x,y,z) = 2\Gamma(2k)\mathcal{W}_{2k-1}\Big(\sqrt{|x|}, \sqrt{|y|}, \sqrt{|z|}\Big)\nabla_k(x,y,z),$$

and

$$\mathcal{W}_k(x,y,z) = \frac{\Gamma(k+1)}{2^{2k-1}\Gamma\left(k+\dfrac{1}{2}\right)} \frac{\left[\left((x+y)^2 - z^2\right)\left(z^2 - (x-y)^2\right)\right]^{k-\frac{1}{2}}}{(xyz)} \mathbf{1}_{]|x-y|,x+y[}(z).$$

Here, $\mathbf{1}_A$ is the characteristic function of the set A and

$$\nabla_k(x,y,z) = \frac{1}{4}\left[1 + \frac{\operatorname{sgn}(xy)}{4k-1}\Big(4k\Delta\big(|x|,|y|,|z|\big)^2 - 1\Big)\right] +$$

$$+\frac{1}{4}\left[1 + \frac{\operatorname{sgn}(xz)}{4k-1}\Big(4k\Delta\big(|z|,|x|,|y|\big)^2 - 1\Big)\right] +$$

$$+\frac{1}{4}\left[1 + \frac{\operatorname{sgn}(yz)}{4k-1}\Big(4k\Delta\big(|z|,|y|,|x|\big)^2 - 1\Big)\right],$$

and

$$\Delta(a,b,c) = \frac{1}{2\sqrt{ab}}(a+b-c), \quad a,b,c \in \mathbb{R}^*_+.$$

Proposition 2.1. *If* $f \in L^p_k$, $1 \le p \le 2$, *and* $x \in \mathbb{R}$, *then*

$$\mathcal{F}_k\Big(T^h_k(f)\Big)(\lambda) = \mathcal{B}_k(\lambda,h)\mathcal{F}_k(f)(\lambda). \tag{2.1}$$

Below, we will define and prove several auxiliary assertions to be used in the proofs of our main results.

The first and the higher order finite differences of $f(x)$ with step h are defined as follows:

$$\triangle^1_h f(x) = T^h_k f(x) - f(x)$$

and

$$\triangle^k_h f(x) = \triangle^1_h\Big(\triangle^{k-1}_h f(x)\Big) =$$

$$= \sum_{i=1}^{k} (-1)^{k-i}\binom{k}{i}\Big(T^h_k\Big)^i f(x) \quad \text{for} \quad k \in \mathbb{N}^*,$$

where $\Big(T^h_k\Big)^0 f(x) = f(x)$ and $\Big(T^h_k\Big)^i = T^h_k\Big(\Big(T^h_k\Big)^{i-1} f(x)\Big)$ for $i \in \mathbb{N}^*$.

Lemma 2.1. *There exists* $C_1 > 0,\ C_2 > 0$ *and* $\eta > 0$ *such that*

$$C_1|\lambda x| \leq 1 - \mathcal{B}_k(\lambda, x) \leq C_2|\lambda x| \quad \textit{for all} \quad |\lambda x| < \eta. \tag{2.2}$$

Proof. By the second statement of Theorem 2.1, we have

$$\mathcal{B}_k(\lambda, x) = j_{2k-1}\left(2\sqrt{|\lambda x|}\right) - \operatorname{sgn}(\lambda x)\left(j_{2k}\left(2\sqrt{|\lambda x|}\right) - j_{2k-1}\left(2\sqrt{|\lambda x|}\right)\right).$$

Let

$$\varphi_k(x) = j_{2k-1}\left(2\sqrt{|x|}\right) - \operatorname{sgn}(\lambda x)\left(j_{2k}\left(2\sqrt{|x|}\right) - j_{2k-1}\left(2\sqrt{|x|}\right)\right).$$

If $x < 0$, we have

$$\varphi_k(x) = j_{2k}\left(2\sqrt{-x}\right) =$$

$$= \Gamma(2k+1)\sum_{m=0}^{+\infty}\frac{x^m}{\Gamma(2k+m+1)} =$$

$$= 1 + \frac{x}{2k+1} + \Gamma(2k+1)\sum_{m=2}^{+\infty}\frac{x^m}{\Gamma(2k+m+1)}.$$

Then

$$\lim_{x\to 0^-}\frac{\varphi_k(x)-1}{x} = \frac{1}{2k+1} \neq 0. \tag{2.3}$$

If $x \geq 0$, we have

$$\varphi_k(x) = 2j_{2k-1}\left(2\sqrt{x}\right) - j_{2k}\left(2\sqrt{x}\right) =$$

$$= 2\Gamma(2k)\sum_{m=0}^{+\infty}\frac{(-x)^m}{\Gamma(2k+m)} - \Gamma(2k+1)\sum_{m=0}^{+\infty}\frac{(-x)^m}{\Gamma(2k+m+1)} =$$

$$= 1 - \frac{x}{k} + \frac{x}{2k+1} + 2\Gamma(2k)\sum_{m=2}^{+\infty}\frac{(-x)^m}{\Gamma(2k+m+1)} - \Gamma(2k+1)\sum_{m=2}^{+\infty}\frac{(-x)^m}{\Gamma(2k+m+1)}.$$

Then

$$\lim_{x\to 0^+}\frac{\varphi_k(x)-1}{x} = \frac{1}{2k+1} - \frac{1}{k} \neq 0. \tag{2.4}$$

Hence, by (2.3) and (2.4), there exist $C_1 > 0,\ C_2 > 0$ and $\eta > 0$ such that

$$C_1|x| \leq | - \varphi_k(x)| \leq C_2|x| \quad \text{for all} \quad |x| < \eta, \tag{2.5}$$

and (2.2) follows from (2.5).

Lemma 2.2. *For a fixed* $h > 0$, *we have*

$$\mathcal{F}_k(\triangle_h^m f)(x) = (\mathcal{B}_k(x,h) - 1)^m \mathcal{F}_k(f)(x). \tag{2.6}$$

Proof. Let $h > 0$. On the basis of (2.1), we obtain

$$\mathcal{F}_k\Big(T_k^h(f)\Big)(\lambda) = \mathcal{B}_k(\lambda, h)\mathcal{F}_k(f)(\lambda).$$

Then, by recurrence on i, we get

$$\mathcal{F}_k\left(\left(T_k^h\right)^i(f)\right)(\lambda) = (\mathcal{B}_k(\lambda, h))^i\mathcal{F}_k(f)(\lambda).$$

Hence,

$$\mathcal{F}_k\big(\triangle_h^m f\big)(x) = \sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}\big(\mathcal{B}_k(\lambda, h)\big)^i\mathcal{F}_k(f)(x) =$$

$$= \left(\sum_{i=1}^{k}(-1)^{k-i}\binom{k}{i}\big(\mathcal{B}_k(\lambda, h)\big)^i\right)\mathcal{F}_k(f)(x).$$

Using Newton's formula, we obtain (2.6).

3. Lipschitz conditions in deformed Hankel setting. In this section, we state and prove an analogous of Titchmarsh's theorem [8] (Theorem 84) for the deformed Hankel transform. Before, we need to define the deformed Hankel – Lipschitz class $H_\alpha^{m,p}$.

Definition 3.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is said belong to $H_\alpha^{m,p}$ for $\alpha > 0$ if*

$$\big\|\triangle_h^m f(x)\big\|_{k,p} = O(h^\alpha) \quad as \quad h \to 0.$$

***Remark* 3.1.** The spaces $H_\alpha^{1,p}$ for $\alpha > 0$ are called the Lipshitz classes $\operatorname{lip}_p(\alpha)$. The spaces $H_\alpha^{2,p}$ for $\alpha > 0$ are called the Zygmund classes $\operatorname{zyg}_p(\alpha)$.

Theorem 3.1. *Let f belong to L_k^p, $1 < p \le 2$. If*

$$\int\limits_{\mathbb{R}} \big|\triangle_h^m f(x)\big|^p d\mu_k(x) = O(h^{\alpha p}) \quad as \quad h \to 0, \tag{3.1}$$

then $\mathcal{F}_k(f)$ belong to L_k^β for

$$\frac{2kp}{2kp + \alpha p - 2k} < \beta \le \frac{p}{p-1}.$$

***Remark* 3.2.** The statement (3.1) is equivalent to $f \in H_\alpha^{m,p}$.

Proof. By virtue of the Lemma 2.2 and Hausdorff – Young inequality, we have

$$\int\limits_{\mathbb{R}} \big|(1 - \mathcal{B}_k(x, h))^m\mathcal{F}_k(f)(x)\big|^{p'} d\mu_k(x) \le \left(\int\limits_{\mathbb{R}} \big|\triangle_h^m f(x)\big|^p d\mu_k(x)\right)^{\frac{p'}{p}}.$$

Thus, using (3.1), we obtain

$$\int\limits_{\mathbb{R}} \big|(1 - \mathcal{B}_k(x, h))^m\mathcal{F}_k(f)(x)\big|^{p'} d\mu_k(x) = O\big(h^{\alpha p'}\big).$$

By the Lemma 2.1, there exist $C1 > 0$ and $\eta > 0$ such that $1 - \mathcal{B}_k(x,h) > C_1 xh$ for $|x| < \dfrac{\eta}{h}$. Then

$$C_1 \int\limits_0^{\eta/h} \left|(xh)^m \mathcal{F}_k(f)(x)\right|^{p'} d\mu_k(x) \leq \int\limits_{\mathbb{R}^2} \left|(1-\mathcal{B}_k(x,h))^m \mathcal{F}_k(f)(x)\right|^{p'} d\mu_k(x) = O(h^{\alpha p'}).$$

Hence,

$$\int\limits_0^{\eta/h} x^{mp'} \left|\mathcal{F}_k(f)(x)\right|^{p'} d\mu_k(x) = O\big(h^{(\alpha-m)p'}\big).$$

Let

$$\phi(y) = \int\limits_1^{y} \left|x^m \mathcal{F}_k(f)(x)\right|^{\beta} d\mu_k(x). \tag{3.2}$$

Then, if $\beta < p'$, we get

$$\phi(y) \leq \left(\int\limits_1^{y} |x^m \mathcal{F}_k(f)(x)|^{p'} d\mu_k(x)\right)^{\beta/p'} \left(\int\limits_1^{y} d\mu_k(x)\right)^{1-\beta/p'} =$$

$$= O\Big(y^{(m-\alpha)\beta + 2k(1-\beta/p')}\Big).$$

On the basis of (3.2), we obtain

$$\int\limits_1^{y} \left|\mathcal{F}_k(f)(x)\right|^{\beta} d\mu_k(x) = \int\limits_1^{y} x^{-m\beta} \phi'(x) dx.$$

An integrate by parts, we get

$$\int\limits_1^{y} \Big|\mathcal{F}_k(f)(x)\Big|^{\beta} d\mu_k(x) = y^{-m\beta}\phi(y) + m\beta \int\limits_1^{y} x^{-m\beta-1}\phi(x) dx =$$

$$= O\left(y^{-m\beta + (m-\alpha)\beta + 2k\left(1-\beta/p'\right)}\right) =$$

$$= O\left(y^{-\alpha\beta + 2k\left(1-\beta/p'\right)}\right),$$

the quantity is bonded as $y \to +\infty$ if $-\alpha\beta + 2k\left(1 - \dfrac{\beta}{p'}\right) < 0$, i.e.,

$$\beta > \frac{2kp}{2kp + \alpha p - 2k}.$$

Similarly for $\displaystyle\int_{-\infty}^{-1} \left|\mathcal{F}_k(f)(x)\right|^{\beta} d\mu_k(x)$.

The case $\beta = p'$ is true by Hausdorff – Young inequality.

4. An equivalence theorem for deformed Hankel – Lipschitz class functions. This section deals with a particular case, when $p = 2$ and $\alpha \in (0, 1]$. We can put the Theorem 3.1 into a form in which it is reversible. Hence, we give a characterization of the space $H_\alpha^{m,2}$ of deformed Hankel – Lipschitz class functions by means of asymptotic estimate growth of the norm of their deformed Hankel transform.

Theorem 4.1. *Let $\alpha \in (0, 1]$ and $f \in L_k^2$. Then the following statements are equivalent*:

$$\int_{\mathbb{R}} |\triangle_h^m f(x)|^2 d\mu_k(x) = O\big(h^{2\alpha}\big), \tag{4.1}$$

$$\int_{\lambda \geq r} \mathcal{F}_k(f)(\lambda) d\mu_k(\lambda) = O\big(r^{-2\alpha}\big) \quad as \quad r \to \infty. \tag{4.2}$$

Proof. By Parseval's identity, we get

$$\int_{\mathbb{R}} \big|(1 - \mathcal{B}_k(\lambda, x))^m \mathcal{F}_k(f)(x)\big|^2 d\mu_k(x) = \int_{\mathbb{R}} \big|\triangle_h^m f(x)\big|^2 d\mu_k(x). \tag{4.3}$$

Suppose that (4.1) holds. Then, by virtue of (4.3) yields

$$\int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} |\mathcal{F}_k(f)(x)|^2 d\mu_k(x) \leq K \int_{\mathbb{R}} \big|(1 - \mathcal{B}_k(\lambda, x))^m \mathcal{F}_k(f)(x)\big|^2 d\mu_k(x) =$$

$$= O(h^{2\alpha}). \tag{4.4}$$

Let $r > 0$ and $h = \dfrac{\eta}{2^{i+1} r}$ for $i \in \mathbb{N}$. Then, from (4.4), we find

$$\int_{2^i r}^{2^{i+1} r} |\mathcal{F}_k(f)(x)|^2 d\mu_k(x) = \int_{\frac{\eta}{2h}}^{\frac{\eta}{h}} |\mathcal{F}_k(f)(x)|^2 d\mu_k(x) = O(h^{2\alpha}) = O\bigg(\Big(\frac{\eta}{2^{i+1} r}\Big)^{2\alpha}\bigg) =$$

$$= O((2^i r)^{-2\alpha}).$$

Hence,

$$\int_{r}^{+\infty} |\mathcal{F}_k(f)(x)|^2 d\mu_k(x) = \sum_{i=0}^{+\infty} \int_{2^i r}^{2^{i+1} r} |\mathcal{F}_k(f)(x)|^2 d\mu_k(x) =$$

$$= \sum_{i=0}^{+\infty} O\Big(\big(2^i r\big)^{-2\alpha}\Big) = \sum_{i=0}^{+\infty} O\Big(\big(2^i r\big)^{-2\alpha}\Big) = O\big(r^{-2\alpha}\big).$$

Similarly for $\displaystyle\int_{-\infty}^{-r} |\mathcal{F}_k(f)(x)|^2 d\mu_k(x)$.

On the other hand, if (4.2), holds, we can write

$$\phi(r) = \int\limits_r^{+\infty} |\mathcal{F}_k(f)(x)|^2 d\mu_k(x).$$

Then, by (4.2), we have

$$\int\limits_0^r \left|x^m \mathcal{F}_k(f)(x)\right|^2 d\mu_k(x) = \int\limits_0^r -x^{2m}\phi'(x)dx =$$

$$= -r^{2m}\phi(r) + 2m\int\limits_0^r x^{2m-1}\phi(x)dx.$$

Hence,

$$\int\limits_0^r |x^m \mathcal{F}_k(f)(x)|^2 d\mu_k(x) = O\big(r^{2m-2\alpha}\big). \tag{4.5}$$

By Lemma 2.1, there exist $C > 0$ and $\eta > 0$ such that

$$1 - \mathcal{B}_k(\lambda, h) \leq C\lambda h \quad \text{for} \quad |\lambda h| < \eta. \tag{4.6}$$

Then, (4.5) and (4.6) gives

$$\int\limits_{\mathbb{R}} \big(1 - \mathcal{B}_k(x, h)\big)^m |\mathcal{F}_k(f)(x)|^2 d\mu_k(x) \leq C_2 h^m \int\limits_{-\eta/h}^{\eta/h} |x^m \mathcal{F}_k(f)(x)|^2 d\mu_k(x) +$$

$$+ \int\limits_{-\infty}^{-\eta/h} |\mathcal{F}_k(f)(x)|^2 d\mu_k(x) + \int\limits_{\eta/h}^{+\infty} \left|\mathcal{F}_k(f)(x)\right|^2 d\mu_k(x) = O(h^{2\alpha}).$$

Hence, (4.1) follows from (4.3).

5. Deformed Hankel – Dini – Lipschitz conditions. In this section, we consider a new larger class of functions $H^{m,p}_{\alpha,\gamma}$ defined by

Definition 5.1. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is said belong to $H^{m,p}_{\alpha,\gamma}$ for $\alpha, \gamma > 0$ if*

$$\left\|\triangle^m_h f(x)\right\|_{k,p} = O\left(h^\alpha \log\left(\frac{1}{h}\right)^\gamma\right).$$

***Remark* 5.1.** Let $\alpha_2 > \alpha_1 > 0$, we have

$$H^{m,p}_{\alpha_2} \subset H^{m,p}_{\alpha_1}$$

and

$$H^{m,p}_{\alpha_2} \subset H^{m,p}_{\alpha_2,\gamma} \subset H^{m,p}_{\alpha_1}.$$

Now, we are able to establish an extension of Theorem 3.1 to $H^{m,p}_{\alpha,\gamma}$.

Theorem 5.1. *Let f belong to L_k^p, $1 < p \leq 2$. If*

$$\int\limits_{\mathbb{R}} |\triangle_h^m f(x)|^p d\mu_k(x) = O\left(h^{\alpha p} \log\left(\frac{1}{h}\right)^{\gamma p}\right) \quad as \quad h \to 0, \tag{5.1}$$

then $\mathcal{F}_k(f)$ belong to L_k^β for

$$\frac{2kp}{2kp + \alpha p - 2k} < \beta \leq \frac{p}{p-1}.$$

***Remark* 5.2.** The statement (5.1) is equivalent to $f \in H_{\alpha,\gamma}^{m,p}$.

Proof. By virtue of the Lemma 2.2 and Hausdorff–Young inequality, we have

$$\int\limits_{\mathbb{R}} \left|(1 - \mathcal{B}_k(x,h))^m \mathcal{F}_k(f)(x)\right|^{p'} d\mu_k(x) \leq \left(\int\limits_{\mathbb{R}} |\triangle_h^m f(x)|^p d\mu_k(x)\right)^{\frac{p'}{p}}.$$

Thus, using (5.1), we obtain

$$\int\limits_{\mathbb{R}} |(1 - \mathcal{B}_k(x,h))^m \mathcal{F}_k(f)(x)|^{p'} d\mu_k(x) = O\left(h^{\alpha p'} \log\left(\frac{1}{h}\right)^{\gamma p'}\right).$$

By the Lemma 2.1, there exist $C1 > 0$ and $\eta > 0$ such that $1 - \mathcal{B}_k(x,h) > C_1 xh$ for $|x| < \dfrac{\eta}{h}$. Then

$$C_1 \int\limits_0^{\eta/h} \left|(xh)^m \mathcal{F}_k(f)(x)\right|^{p'} d\mu_k(x) \leq \int\limits_{\mathbb{R}^2} |(1 - \mathcal{B}_k(x,h))^m \mathcal{F}_k(f)(x)|^{p'} d\mu_k(x) =$$

$$= O\left(h^{\alpha p'} \log\left(\frac{1}{h}\right)^{\gamma p'}\right).$$

Hence,

$$\int\limits_0^{\eta/h} x^{mp'} \left|\mathcal{F}_k(f)(x)\right|^{p'} d\mu_k(x) = O\left(h^{(\alpha - m)p'} \log\left(\frac{1}{h}\right)^{\gamma p'}\right).$$

Let

$$\phi(y) = \int\limits_1^y \left|x^m \mathcal{F}_k(f)(x)\right|^\beta d\mu_k(x). \tag{5.2}$$

Then, if $\beta < p'$, we get

$$\phi(y) \leq \left(\int\limits_1^y \left|x^m \mathcal{F}_k(f)(x)\right|^{p'} d\mu_k(x)\right)^{\beta/p'} \left(\int\limits_1^y d\mu_k(x)\right)^{1-\beta/p'} =$$

$$= O\left(y^{(m-\alpha)\beta + 2k\left(1 - \beta/p'\right)} \log\left(\frac{1}{h}\right)^{\gamma\beta}\right).$$

On the basis of (5.2), we obtain

$$\int\limits_2^y |\mathcal{F}_k(f)(x)|^\beta d\mu_k(x) = \int\limits_2^y x^{-m\beta}\phi'(x)dx.$$

An integrate by parts, we get

$$\int\limits_2^y \left|\mathcal{F}_k(f)(x)\right|^\beta d\mu_k(x) = y^{-m\beta}\phi(y) + m\beta\int\limits_2^y x^{-m\beta-1}\phi(x)dx =$$

$$= O\left(y^{-\alpha\beta+2k\left(1-\beta/p'\right)}\right) + O\left(\int\limits_2^y y^{-\alpha\beta+2k\left(1-\beta/p'\right)-1}\log\left(\frac{1}{h}\right)^{\gamma\beta}\right).$$

The quantity $y^{-\alpha\beta+2k\left(1-\beta/p'\right)}$ is bonded as $y\to+\infty$ if $-\alpha\beta+2k\left(1-\dfrac{\beta}{p'}\right)<0$, i.e.,

$$\beta > \frac{2kp}{2kp+\alpha p-2k}.$$

Since $-\gamma\beta<1$, then on the basis of Bertrand rule, the quantity $\int_2^y y^{-\alpha\beta+2k\left(1-\beta/p'\right)-1}\log\left(\dfrac{1}{h}\right)^{\gamma\beta}$ is bonded as $y\to+\infty$ if $\alpha\beta-2k\left(1-\dfrac{\beta}{p'}\right)+1>1$, i.e.,

$$\beta > \frac{2kp}{2kp+\alpha p-2k}.$$

Hence, $\int_2^y \left|\mathcal{F}_k(f)(x)\right|^\beta d\mu_k(x)$ is bonded.

Similarly for $\int_{-\infty}^{-1} \left|\mathcal{F}_k(f)(x)\right|^\beta d\mu_k(x)$.

The case $\beta = p'$ is true by Hausdorff – Young inequality. Then, the proof is completed.

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Received 24.05.20