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q-DEFORMED CONFORMABLE FRACTIONAL NATURAL TRANSFORM

q-ДЕФОРМОВАНЕ КОНФОРМНЕ ДРОБОВЕ НАТУРАЛЬНЕ ПЕРЕТВОРЕННЯ

We develop a new deformation and generalization of the natural integral transform based on the conformable fractional q-derivative. We obtain transformation of some deformed functions and apply the transform to solve linear differential equation with given initial conditions.

Розроблено нову деформацію та узагальнення натурального інтегрального перетворення на основі конформної дробової q-похідної. Отримано перетворення деяких деформованих функцій. Це перетворення застосовано до розв'язування лінійного диференціального рівняння з заданими початковими умовами.

1. Introduction. Differential equations appear in many problems of physics, engineering, and other sciences. So we need powerful mathematical tools to handle them. The integral transforms are one of the widely used techniques applied for solving differential equations. Generalizations of integral transforms turn them into more flexible tools to deal with complicated problems. Some generalizations are based on an extension of transforms to multivariate cases. Other generalizations can be done by deforming a differential operator and, consequently, an integral. Two leading kinds of deformations are fractional calculus and q-calculus. Some applications of fractional calculus can be found in [23], and of q-calculus and fractional q-calculus in [1].

In this paper, we develop a new deformation of the natural integral transform. We define new extensions of some special functions and apply our deformed transform to them. Among other tools, a new extension of q,α-Taylor series is proposed. We start here by recalling three integral transforms, namely the Laplace, the Sumudu, and the natural transform. Further, we develop a new deformation of the natural transform and show its application to some deformed differential equations.

The Laplace transform is one of the most famous of the integral transforms. It is defined as

$$F(s)=\int\limits_0^\infty f(t)e^{-st}dt.$$

This transform is very useful in solving differential equations with given initial and boundary conditions. Moreover, it can be used for evaluating new identities for functions and integrals (see [18, 21]). One of its important features is the transformation from the time domain to the frequency domain.

In 1993, Watugala [20] proposed a new integral transform, named Sumudu transform, which is defined as

$$S\{f(t)\}=F_s(u)=\int\limits_0^\infty \frac{1}{u}e^{-\frac{t}{u}}f(t)dt.$$

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Among its properties, Watugala remarks an easier visualization. This transform was further widely studied by different researchers (see, for example, [4, 9, 14] and references therein). In his works Belgacem (see [4] and references therein) denotes the Sumudu transform as an ideal tool for solving many engineering problems. The Sumudu transform, unlike the Laplace transform, is not focusing on transformation into the frequency domain. One of its main features is preserving units and scale during transformation [4].

In 2008, Khan and Khan [12] defined a new transform, called N-transform and renamed later to natural transform, as following:

$$R(u,s) = N(f(t)) = \int\limits_0^\infty f(ut)e^{-st}dt.$$

It is easy to see that in the case $u = 1$ we obtain the Laplace transform, and in the case $s = 1$ we obtain the Sumudu transform. Due to its dual nature and close relationship with both, Laplace and Sumudu, transforms, the natural transform is more flexible and lets easily to choose during the problem solution, what way is preferable in each concrete case. In 2017, Kiliçman and Omran generalized this transform for the two-dimensional case [13].

For both Sumudu and Laplace transforms their q-analogues were obtained and studied. The q-analogues of the Sumudu transform based on Jackson q-derivative and q-integral were studied by D. Albayrak and others (see [3] and references therein). The q-analogues of the Laplace transform based some on the Jackson's and some on the Tsallis q-derivative and q-integral were studied in [6, 10, 15–17, 19]. Recently a q-analogue was proposed also for the natural transform [2].

In this paper, we define and study a deformation of the natural transform based on the conformable fractional q-derivative defined by Chung [7]. This deformation is actually a generalization of the q-deformation based on the Jackson q-derivative. In case when certain transform's parameters equal 1, it proposes another definition for the q-Sumudu transform, different from [3]. Moreover, our transform generalizes some results for q analogue of the natural transform defined in [13]. Finally, we demonstrate some applications of the q-deformed conformable fractional natural transform.

2. Definitions and some properties of the conformable q-derivative. We start from a definition of a conformable fractional q-derivative $D_x^{q,\alpha}$ given by Chung in [7], namely,

$$D_x^{q,\alpha} f(x) = \frac{[\alpha](f(x) - f(qx)}{x^\alpha(1-q^\alpha)} = x^{1-\alpha} D_x^q f(x), \tag{2.1}$$

where $[\alpha] = \dfrac{1-q^\alpha}{1-q}$ is the q-number of α, and D_x^q is the Jackson q-derivative with respect to the variable x. It is easy to check that the operator $D_x^{q,\alpha}$ defined by (2.1) is a linear operator. One can see that in case $\alpha = 1$, this differential operator coincides with Jackson q-derivative. The following notation is widely used in q-calculus $(a+b)_q^n = \prod_{j=0}^{n-1}(a+q^j b)$. Accordingly, with this definition, we have

$$(a+b)_{q^\alpha}^n = \begin{cases} \prod_{j=0}^{n-1}(a+q^{\alpha j}b) & \text{for integer} \quad n>0, \\ 1 & \text{for} \quad n=0. \end{cases} \tag{2.2}$$

Another notation closely related to q-calculus is a q-Pochhammer symbol

$$(a;q)_n = \begin{cases} \prod_{j=0}^{n-1}(1-aq^j) & \text{for integer} \quad n>0, \\ 1 & \text{for} \quad n=0 \end{cases}$$

and

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

The conformable fractional q-integral is an inverse operation of the conformable fractional q-derivative

$$I_x^{q,\alpha} f(x) = \frac{1}{[\alpha]}(1-q^\alpha)x^\alpha \sum_{j\geq 0} q^{\alpha j} f(q^j x) = I_x^q(x^{\alpha-1}f(x)),$$

where I_x^q is the Jackson q-integral. Then, for α-monomial $x^{\alpha n}$, we have

$$D_x^{q,\alpha} x^{\alpha n} = [n\alpha]x^{\alpha(n-1)}, \qquad I_x^{q,\alpha} x^{\alpha n} = \frac{x^{\alpha(n+1)}}{[(n+1)\alpha]}. \tag{2.3}$$

It can be shown that the Leibniz rule for the conformable fractional q-derivative has the following form:

$$D_x^{q,\alpha}(f(x)g(x)) = f(qx)D_x^{q,\alpha}g(x) + \left(D_x^{q,\alpha}f(x)\right)g(x). \tag{2.4}$$

Therefore, by integrating both sides of (2.4), we obtain a rule for integrating by parts

$$\int \left(D_x^{q,\alpha}f(x)\right)g(x)d_{q,\alpha}x = f(x)g(x) - \int f(qx)D_x^{q,\alpha}g(x)d_{q,\alpha}x. \tag{2.5}$$

Chung defined also a conformable fractional q-exponential function

$$e_{q,\alpha}(x) = \sum_{j\geq 0}\frac{x^{\alpha j}}{[j\alpha]!} = \left((1-q)x^\alpha; q^\alpha\right)_\infty, \tag{2.6}$$

where $[n\alpha]! = [\alpha][2\alpha]\ldots[n\alpha]$, with the property

$$D_x^{q,\alpha}e_{q,\alpha}(ax) = a^\alpha e_{q,\alpha}(ax). \tag{2.7}$$

Note that, as usual, $[0]! = 1$.

Two new deformations of trigonometric functions were proposed in [7]

$$e_{q,\alpha}(i^{\frac{1}{\alpha}}x) = c_{q,\alpha}(x) + is_{q,\alpha}(x), \tag{2.8}$$

where

$$c_{q,\alpha}(x) = \sum_{n\geq 0}\frac{(-1)^n}{[2n\alpha]!}x^{2\alpha n}, \qquad s_{q,\alpha}(x) = \sum_{n\geq 0}\frac{(-1)^n}{[(2n+1)\alpha]!}x^{\alpha(2n+1)}.$$

From (2.8) one can obtain

$$c_{q,\alpha}(x) = \frac{1}{2}\Big(e_{q,\alpha}\big(i^{\frac{1}{\alpha}}x\big) + e_{q,\alpha}((-i)^{\frac{1}{\alpha}}x)\Big), \tag{2.9}$$

$$s_{q,\alpha}(x) = \frac{1}{2i}\Big(e_{q,\alpha}\big(i^{\frac{1}{\alpha}}x\big) - e_{q,\alpha}((-i)^{\frac{1}{\alpha}}x)\Big). \tag{2.10}$$

By applying (2.7), it is easy to show that

$$D_x^{q,\alpha}c_{q,\alpha}(x) = -s_{q,\alpha}(x), \qquad D_x^{q,\alpha}s_{q,\alpha}(x) = c_{q,\alpha}(x).$$

By using the definition of the deformed conformable fractional derivative (2.1), we evaluate conformable derivative of a function $\dfrac{1}{e_{q,\alpha}(ax)}$:

$$D_x^{q,\alpha}\frac{1}{e_{q,\alpha}(ax)} = x^{1-\alpha}D_x^q\frac{1}{e_{q,\alpha}(ax)} = x^{1-\alpha}\frac{\dfrac{1}{e_{q,\alpha}(ax)} - \dfrac{1}{e_{q,\alpha}(qax)}}{x-qx} =$$

$$= x^{1-\alpha}\frac{e_{q,\alpha}(qax) - e_{q,\alpha}(ax)}{e_{q,\alpha}(qax)e_{q,\alpha}(ax)(x-qx)} = -\frac{D_x^{q,\alpha}e_{q,\alpha}(ax)}{e_{q,\alpha}(qax)e_{q,\alpha}(ax)},$$

and, by applying (2.7), we obtain

$$D_x^{q,\alpha}\frac{1}{e_{q,\alpha}(ax)} = -\frac{a^\alpha e_{q,\alpha}(ax)}{e_{q,\alpha}(qax)e_{q,\alpha}(ax)} = -\frac{a^\alpha}{e_{q,\alpha}(qax)}. \tag{2.11}$$

It is easy to see that $D_x^{q,\alpha}C = 0$, where C is a constant (C does not depend on x). Indeed, from (2.1), we have $D_x^{q,\alpha}C = x^{1-\alpha}D_x^qC = 0$.

We can see that $D_x^{q,\alpha}x^\alpha = [\alpha]$. We would like to build a sequence of polynomials $P_0(x), P_1(x), \ldots$ $\ldots, P_n(x)$ of degrees $0, \alpha, \ldots, n\alpha$, respectively, so that

$$D_x^{q,\alpha}P_n(x) = P_{n-1}(x),$$

$$P_n\big(a^{\frac{1}{\alpha}}\big) = 0$$

with initial condition $P_0(x) = 1$. Therefore, the polynomial $P_1(x)$ has the form $P_1(x) = (x^\alpha -$ $- a)/[\alpha]$. Obviously, $P_1\big(a^{\frac{1}{\alpha}}\big) = \Big(\big(a^{\frac{1}{\alpha}}\big)^\alpha - a\Big)/[\alpha] = 0$ and $D_x^{q,\alpha}P_1(x) = 1 = P_0(x)$.

Proposition 2.1. *For all* $n \in \mathbb{N}$, *we have*

$$D_x^{q,\alpha}(x^\alpha - a)_{q^\alpha}^n = [n\alpha](x^\alpha - a)_{q^\alpha}^{n-1}.$$

Proof. We proceed the proof by induction on n. It is easy to see that, for $n=1$, we have

$$D_x^{q,\alpha}(x^\alpha - a)_{q^\alpha}^1 = D_x^{q,\alpha}(x^\alpha - a) = [\alpha],$$

and statement holds. Let us assume that statement holds for some integer k. We will prove it for $k+1$. By (2.2), we have $(x^\alpha - a)_{q^\alpha}^{k+1} = (x^\alpha - a)_{q^\alpha}^k(x^\alpha - q^{\alpha k}a)$. By applying (2.4), we obtain

$$D_x^{q,\alpha}(x^\alpha - a)_{q^\alpha}^{k+1} = D_x^{q,\alpha}\Big((x^\alpha - a)_{q^\alpha}^k(x^\alpha - q^{\alpha k}a)\Big) =$$

$$= (q^\alpha x^\alpha - q^{\alpha k} a)[k\alpha](x^\alpha - a)_{q^\alpha}^{k-1} + [\alpha](x^\alpha - a)_{q^\alpha}^{k} =$$

$$= q^\alpha (x^\alpha - q^{\alpha(k-1)} a)[k\alpha](x^\alpha - a)_{q^\alpha}^{k-1} + [\alpha](x^\alpha - a)_{q^\alpha}^{k} =$$

$$= q^\alpha [k\alpha](x^\alpha - a)_{q^\alpha}^{k} + [\alpha](x^\alpha - a)_{q^\alpha}^{k} =$$

$$= (x^\alpha - a)_{q^\alpha}^{k} \left(q^\alpha \frac{1 - q^{k\alpha}}{1 - q} + \frac{1 - q^\alpha}{1 - q} \right) =$$

$$= (x^\alpha - a)_{q^\alpha}^{k} \frac{q^\alpha - q^{\alpha(k+1)} + 1 - q^\alpha}{1 - q} =$$

$$= (x^\alpha - a)_{q^\alpha}^{k} \frac{1 - q^{\alpha(k+1)}}{1 - q} = [(k+1)\alpha](x^\alpha - a)_{q^\alpha}^{k},$$

which completes the induction.

Proposition 2.2. *For all* $n \in \mathbb{N}$, *we have*

$$D_x^{q,\alpha}(a - x^\alpha)_{q^\alpha}^{n} = -[n\alpha](a - q^\alpha x^\alpha)_{q^\alpha}^{n-1}.$$

Proof. By (2.2), we have

$$(a - x^\alpha)_{q^\alpha}^{n} = (a - x^\alpha)(a - q^\alpha x^\alpha)(a - q^{2\alpha} x^\alpha) \ldots (a - q^{(n-1)\alpha} x^\alpha) =$$

$$= (a - x^\alpha) q^\alpha (q^{-\alpha} a - x^\alpha) q^{2\alpha} (q^{-2\alpha} a - x^\alpha) \ldots q^{(n-1)\alpha} (q^{-(n-1)\alpha} a - x^\alpha) =$$

$$= (-1)^n q^{\frac{\alpha n(n-1)}{2}} (x^\alpha - q^{-(n-1)\alpha} a) \ldots (x^\alpha - q^{-2\alpha} a)(x^\alpha - q^{-\alpha} a)(x^\alpha - a) =$$

$$= (-1)^n q^{\frac{\alpha n(n-1)}{2}} (x^\alpha - q^{-(n-1)\alpha} a)_{q^\alpha}^{n}.$$

Now, by using Proposition 2.1, we obtain

$$D_x^{q,\alpha}(a - x^\alpha)_{q^\alpha}^{n} = D_x^{q,\alpha}\left((-1)^n q^{\frac{\alpha n(n-1)}{2}} (x^\alpha - q^{-(n-1)\alpha} a)_{q^\alpha}^{n} \right) =$$

$$= [n\alpha](-1)^n q^{\frac{\alpha n(n-1)}{2}} (x^\alpha - q^{-(n-1)\alpha} a)_{q^\alpha}^{n-1} =$$

$$= (-1)^n [n\alpha] q^\alpha q^{2\alpha} \ldots q^{(n-1)\alpha} (x^\alpha - q^{-(n-1)\alpha} a) \ldots (x^\alpha - q^{-2\alpha} a)(x^\alpha - q^{-\alpha} a) =$$

$$= -[n\alpha] q^\alpha q^{2\alpha} \ldots q^{(n-1)\alpha} (q^{-(n-1)\alpha} a - x^\alpha) \ldots (q^{-2\alpha} a - x^\alpha)(q^{-\alpha} a - x^\alpha) =$$

$$= -[n\alpha](a - q^{(n-1)\alpha} x^\alpha) \ldots (a - q^{2\alpha} x^\alpha)(a - q^\alpha x^\alpha) =$$

$$= -[n\alpha](a - q^\alpha x^\alpha)_{q^\alpha}^{n-1},$$

and the proof is complete.

Now, we can state that $P_n(x) = \dfrac{(x^\alpha - a)_{q^\alpha}^{n}}{[n\alpha]!}$. Indeed, $P_n\big(a^{\frac{1}{\alpha}}\big) = 0$ and

$$D_x^{q,\alpha} P_n(x) = D_x^{q,\alpha} \frac{(x^\alpha - a)_{q^\alpha}^{n}}{[n\alpha]!} = \frac{[n\alpha](x^\alpha - a)_{q^\alpha}^{n-1}}{[n\alpha]!} = \frac{(x^\alpha - a)_{q^\alpha}^{n-1}}{[(n-1)\alpha]!} = P_{n-1}(x).$$

Therefore, by using the [11] (Theorems 2.1 and 8.1), we can state the following result.

Theorem 2.1. *Any polynomial or formal power series function $f(x)$ can be expressed via the generalized conformable fractional q-Taylor expansion about $x = a^{\frac{1}{\alpha}}$ as*

$$f(x) = \sum_{n\geq 0} \left(D_x^{q,\alpha}\right)^n f\left(a^{\frac{1}{\alpha}}\right) \frac{(x^\alpha - a)_{q^\alpha}^n}{[n\alpha]!}.$$

Let us define one more q-deformed conformable fractional exponential function

$$E_{q,\alpha}(x) = \sum_{j\geq 0} q^{\alpha j(j-1)/2} \frac{x^{\alpha j}}{[j\alpha]!} = (1 + (1-q)x^\alpha)_{q^\alpha}^\infty. \tag{2.12}$$

It is easy to see that $E_{q,\alpha}(0) = 1$. By using (2.3), we obtain

$$D_x^{q,\alpha} E_{q,\alpha}(ax) = \sum_{j\geq 1} q^{\alpha j(j-1)/2} \frac{a^{\alpha j} x^{\alpha(j-1)}}{[(j-1)\alpha]!} = \sum_{j\geq 0} q^{\alpha(j+1)j/2} \frac{a^{\alpha(j+1)} x^{\alpha j}}{[j\alpha]!} =$$

$$= a^\alpha \sum_{j\geq 0} q^{\alpha j(j-1)/2} \frac{q^{\alpha j} a^{\alpha j} x^{\alpha j}}{[j\alpha]!} = a^\alpha E_{q,\alpha}(qax).$$

Let us define the q-deformed conformable fractional Gamma function for some $n \geq 1$:

$$\Gamma_{q,\alpha}(n+1) = \int_0^\infty x^{\alpha n} \frac{1}{e_{q,\alpha}(qx)} d_{q,\alpha}x. \tag{2.13}$$

Proposition 2.3. *For all $n > 0$, the function $\Gamma_{q,\alpha}(n+1)$ defined by* (2.13) *satisfies the recurrence relation*

$$\Gamma_{q,\alpha}(n+1) = [n\alpha]\Gamma_{q,\alpha}(n)$$

with the initial condition $\Gamma_{q,\alpha}(1) = 1$.

Proof. We proceed the proof by induction on n. For $n = 0$, we have

$$\Gamma_{q,\alpha}(1) = \int_0^\infty \frac{1}{e_{q,\alpha}(qx)} d_{q,\alpha}x = -\int_0^\infty D_x^{q,\alpha} \frac{1}{e_{q,\alpha}(x)} = -\frac{1}{e_{q,\alpha}(x)}\bigg|_0^\infty = 1.$$

Let us assume that the claim holds for $k-1$ and let us prove it for k. Let us consider now the function $\Gamma_{q,\alpha}(k+1)$ for some k. By (2.13), we have

$$\Gamma_{q,\alpha}(k+1) = \int_0^\infty x^{\alpha k} \frac{1}{e_{q,\alpha}(qx)} d_{q,\alpha}x,$$

from where, by rearranging and using (2.4), we get

$$\Gamma_{q,\alpha}(k+1) = -\int_0^\infty x^{\alpha k} \left(D_x^{q,\alpha} \frac{1}{e_{q,\alpha}(x)}\right) d_{q,\alpha}x =$$

$$= -x^{\alpha k}\frac{1}{e_{q,\alpha}(x)}\Big|_0^{\infty} + \int\limits_0^{\infty}\left(D_x^{q,\alpha}x^{\alpha k}\right)\frac{1}{e_{q,\alpha}(qx)}\,d_{q,\alpha}x =$$

$$= [k\alpha]\int\limits_0^{\infty} x^{\alpha(k-1)}\frac{1}{e_{q,\alpha}(qx)}\,d_{q,\alpha}x = [k\alpha]\Gamma_{q,\alpha}(k),$$

which completes the proof.

One can immediately obtain from the last proposition the following result.

***Corollary* 2.1.** *For all* $n \in \mathbb{N}$, *it holds that*

$$\Gamma_{q,\alpha}(n+1) = [n\alpha]!.$$

The function $\Gamma_{q,\alpha}(n)$ defined as (2.13) is a q-deformed conformable fractional extension of the Γ-function. It is well-known that Γ-function is closely related to the B-function, that is, for the B-function, defined as

$$B(m,n) = \int\limits_0^1 x^{m-1}(1-x)^{n-1}dx,$$

holds that

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{2.14}$$

Let us define the function

$$B_{q,\alpha}(m,n) = \int\limits_0^1 x^{\alpha(m-1)}(1-q^{\alpha}x^{\alpha})_{q^{\alpha}}^{n-1}d_{q,\alpha}x.$$

Proposition 2.4. *For all natural* m, n, *it holds that*

$$B_{q,\alpha}(m,n) = \frac{\Gamma_{q,\alpha}(m)\Gamma_{q,\alpha}(n)}{\Gamma_{q,\alpha}(m+n)}.$$

Proof. With the notations $f(qx) = (1-q^{\alpha}x^{\alpha})_{q^{\alpha}}^{n-1}$ and $D_x^{q,\alpha}g(x) = x^{\alpha(m-1)}d_{q,\alpha}x$, we obtain $f(x) = (1-x^{\alpha})_{q^{\alpha}}^{n-1}$ and $g(x) = \dfrac{x^{\alpha m}}{[\alpha m]}$. Therefore, by using Proposition 2.2, we get $D_x^{q,\alpha}f(x) =$ $= -[(n-1)\alpha](1-q^{\alpha}x^{\alpha})_{q^{\alpha}}^{n-2}$. Applying (2.5) with our notations yields

$$B_{q,\alpha}(m,n) = (1-x^{\alpha})_{q^{\alpha}}^{n-1}\frac{x^{\alpha m}}{[\alpha m]}\Big|_0^1 + \int\limits_0^1[(n-1)\alpha](1-q^{\alpha}x^{\alpha})_{q^{\alpha}}^{n-2}\frac{x^{\alpha m}}{[\alpha m]}d_{q,\alpha}x =$$

$$= \frac{[(n-1)\alpha]}{[\alpha m]}\int\limits_0^1 x^{\alpha m}(1-q^{\alpha}x^{\alpha})_{q^{\alpha}}^{n-2}d_{q,\alpha}x.$$

Thus, by assuming m and n are natural numbers, we obtain

$$B_{q,\alpha}(m,n) = \int\limits_0^1 x^{\alpha(m-1)}(1-q^\alpha x^\alpha)_{q^\alpha}^{n-1} d_{q,\alpha}x =$$

$$= \frac{[(n-1)\alpha]}{[\alpha m]} \int\limits_0^1 x^{\alpha m}(1-q^\alpha x^\alpha)_{q^\alpha}^{n-2} d_{q,\alpha}x =$$

$$= \frac{[(n-1)\alpha]}{[\alpha m]} \frac{[(n-2)\alpha]}{[\alpha(m+1)]} \int\limits_0^1 x^{\alpha(m+1)}(1-q^\alpha x^\alpha)_{q^\alpha}^{n-3} d_{q,\alpha}x = \ldots$$

$$\ldots = \frac{[(n-1)\alpha]}{[m\alpha]} \frac{[(n-2)\alpha]}{[(m+1)\alpha]} \cdots \frac{[2\alpha]}{[(m+n-3)\alpha]} \int\limits_0^1 x^{\alpha(m+n-3)}(1-q^\alpha x^\alpha)_{q^\alpha}^{1} d_{q,\alpha}x =$$

$$= \frac{[(n-1)\alpha]}{[m\alpha]} \frac{[(n-2)\alpha]}{[(m+1)\alpha]} \cdots \frac{[2\alpha]}{[(m+n-3)\alpha]} \times$$

$$\times \left(\frac{x^{\alpha(m+n-2)}}{[(m+n-2)\alpha]}(1-x^\alpha)\Big|_0^1 + \int\limits_0^1 [\alpha] \frac{x^{\alpha(m+n-2)}}{[(m+n-2)\alpha]} d_{q,\alpha}x \right) =$$

$$= \frac{[(n-1)\alpha][(n-2)\alpha]\ldots[2\alpha][\alpha]}{[m\alpha][(m+1)\alpha]\ldots[(m+n-3)\alpha][(m+n-2)\alpha]} \frac{x^{\alpha(m+n-1)}}{[\alpha(m+n-1)]}\Bigg|_0^1 =$$

$$= \frac{[(n-1)\alpha]!\,[(m-1)\alpha]!}{[(m+n-1)\alpha]!} = \frac{\Gamma_{q,\alpha}(n)\Gamma_{q,\alpha}(m)}{\Gamma_{q,\alpha}(m+n)}. \tag{2.15}$$

Proposition 2.4 is proved.

***Remark* 2.1.** It is easy to see that for $\alpha = q = 1$, (2.15) turns into (2.14). Thus, functions $\Gamma_{q,\alpha}$ and $B_{q,\alpha}$ are q-deformed conformable fractional extensions of the well-known Γ- and B-functions, respectively. This proposition may be extended for all positive m, n.

3. q-Deformed conformable fractional natural transform. We define now a q-deformed conformable fractional natural transform as

$$N_{q,\alpha}(f(t)) = \int\limits_0^\infty f(ut)\frac{1}{e_{q,\alpha}(qst)}\, d_{q,\alpha}t, \quad s>0. \tag{3.1}$$

Then we have

$$N_{q,\alpha}(1) = \int\limits_0^\infty \frac{1}{e_{q,\alpha}(qst)}\, d_{q,\alpha}t =$$

$$= -\frac{1}{s^\alpha}\int\limits_0^\infty D_t^{q,\alpha}\frac{1}{e_{q,\alpha}(st)}\, d_{q,\alpha}t = -\frac{1}{s^\alpha}\frac{1}{e_{q,\alpha}(st)}\Bigg|_0^\infty = \frac{1}{s^\alpha}.$$

Let us now to obtain a transform of α-monomial.

Proposition 3.1. *For all integer $N \geq 0$,*

$$N_{q,\alpha}(t^{\alpha N}) = \frac{u^{\alpha N}}{s^{\alpha(N+1)}} \Gamma_{q,\alpha}(N+1).$$

Proof. By definition (3.1), we have

$$N_{q,\alpha}(t^{\alpha N}) = \int\limits_0^\infty u^{\alpha N} t^{\alpha N} \frac{1}{e_{q,\alpha}(qst)}\, d_{q,\alpha} t =$$

$$= -\frac{1}{s^\alpha} u^{\alpha N} \int\limits_0^\infty \left(D_t^{q,\alpha} \frac{1}{e_{q,\alpha}(st)} \right) t^{\alpha N} d_{q,\alpha} t.$$

Integrating by parts of the last equation leads to

$$N_{q,\alpha}(t^{\alpha N}) = -\frac{u^{\alpha N}}{s^\alpha} \left\{ \frac{1}{e_{q,\alpha}(st)}\, t^{\alpha N} \Big|_0^\infty - \int\limits_0^\infty \frac{1}{e_{q,\alpha}(qst)} \big(D_t^{q,\alpha} t^{\alpha N}\big) d_{q,\alpha} t \right\} =$$

$$= \frac{u^{\alpha N}}{s^\alpha} \int\limits_0^\infty [N\alpha]\, t^{\alpha(N-1)} \frac{1}{e_{q,\alpha}(qst)}\, d_{q,\alpha} t =$$

$$= \frac{u^\alpha}{s^\alpha} [N\alpha] \int\limits_0^\infty u^{\alpha(N-1)} t^{\alpha(N-1)} \frac{1}{e_{q,\alpha}(qst)}\, d_{q,\alpha} t =$$

$$= \frac{u^\alpha}{s^\alpha} [N\alpha] N_{q,\alpha}(t^{\alpha(N-1)}). \tag{3.2}$$

Thus, by (3.2), we obtain

$$N_{q,\alpha}(t^{\alpha N}) = \frac{u^\alpha}{s^\alpha}[N\alpha] \frac{u^\alpha}{s^\alpha}[(N-1)\alpha] \ldots \frac{u^\alpha}{s^\alpha}[\alpha] \frac{1}{s^\alpha} = \frac{u^{\alpha N}}{s^{\alpha(N+1)}}[N\alpha]!. \tag{3.3}$$

Applying Corollary 2.1 completes the proof.

Now let us consider the transform of two deformed exponential functions.

Proposition 3.2. *The q-deformed conformable natural transforms of the q-deformed conformable exponential functions are given by*

$$N_{q,\alpha}(e_{q,\alpha}(at)) = \frac{1}{s^\alpha - a^\alpha u^\alpha},$$

$$N_{q,\alpha}(E_{q,\alpha}(at)) = \sum_{n=0}^\infty q^{\frac{\alpha n(n-1)}{2}} \frac{(ua)^{\alpha n}}{s^{\alpha(n+1)}}.$$

Proof. By applying transform (3.1) to the deformed exponential funciton (2.6), we obtain

$$N_{q,\alpha}(e_{q,\alpha}(at)) = \int\limits_0^\infty e_{q,\alpha}(aut) \frac{1}{e_{q,\alpha}(qst)}\, d_{q,\alpha} t =$$

$$= \int_0^\infty \sum_{n=0}^\infty \frac{a^{\alpha n} u^{\alpha n} t^{\alpha n}}{[n\alpha]!} \frac{1}{e_{q,\alpha}(qst)}\, d_{q,\alpha} t =$$

$$= \sum_{n=0}^\infty \frac{a^{\alpha n}}{[n\alpha]!} \int_0^\infty u^{\alpha n} t^{\alpha n} \frac{1}{e_{q,\alpha}(qst)}\, d_{q,\alpha} t =$$

$$= \sum_{n=0}^\infty \frac{a^{\alpha n}}{[n\alpha]!} N_{q,\alpha}(t^{\alpha n}) = \sum_{n=0}^\infty \frac{a^{\alpha n}}{[n\alpha]!} \frac{u^{\alpha n}}{s^{\alpha(n+1)}} [n\alpha]! =$$

$$= \frac{1}{s^\alpha} \sum_{n=0}^\infty \frac{(au)^{\alpha n}}{s^{\alpha n}} = \frac{1}{s^\alpha - a^\alpha u^\alpha}.$$

By applying transform (3.1) to the deformed exponential funciton (2.12), we obtain

$$N_{q,\alpha}(E_{q,\alpha}(at)) = \int_0^\infty E_{q,\alpha}(aut) \frac{1}{e_{q,\alpha}(qst)}\, d_{q,\alpha} t =$$

$$= \int_0^\infty \sum_{n=0}^\infty q^{\frac{\alpha n(n-1)}{2}} \frac{a^{\alpha n} u^{\alpha n} t^{\alpha n}}{[n\alpha]!} \frac{1}{e_{q,\alpha}(qst)}\, d_{q,\alpha} t =$$

$$= \sum_{n=0}^\infty q^{\frac{\alpha n(n-1)}{2}} \frac{a^{\alpha n}}{[n\alpha]!} \int_0^\infty u^{\alpha n} t^{\alpha n} \frac{1}{e_{q,\alpha}(qst)}\, d_{q,\alpha} t =$$

$$= \sum_{n=0}^\infty q^{\frac{\alpha n(n-1)}{2}} \frac{a^{\alpha n}}{[n\alpha]!} \frac{u^{\alpha n}}{s^{\alpha(n+1)}} [n\alpha]! =$$

$$= \sum_{n=0}^\infty q^{\frac{\alpha n(n-1)}{2}} \frac{(ua)^{\alpha n}}{s^{\alpha(n+1)}},$$

and the proof is complete.

Let us consider now the transform of the deformed trigonometric functions (2.9), (2.10).

Proposition 3.3. *The deformed conformable fractional natural transform of deformed trigonometric functions*, *defined by* (2.9) *and* (2.10), *is given by*

$$N_{q,\alpha}(c_{q,\alpha}(t)) = \frac{s^\alpha}{s^{2\alpha} + u^{2\alpha}},$$

$$N_{q,\alpha}(s_{q,\alpha}(t)) = \frac{u^\alpha}{s^{2\alpha} + u^{2\alpha}}.$$

Proof. It follows from the definition (2.9) and the linearity of the transform $N_{q,\alpha}$ that

$$N_{q,\alpha}(c_{q,\alpha}(at)) = \frac{1}{2}\left(N_{q,\alpha}\left(e_{q,\alpha}\big(i^{\frac{1}{\alpha}} at\big) + N_{q,\alpha}\big(e_{q,\alpha}\big((-i)^{\frac{1}{\alpha}} at\big)\big)\right)\right) =$$

$$= \frac{1}{2}\left(\frac{1}{s^\alpha - (i^{\frac{1}{\alpha}}a)^\alpha u^\alpha} + \frac{1}{s^\alpha - ((-i)^{\frac{1}{\alpha}}a)^\alpha u^\alpha}\right) =$$

$$= \frac{1}{2}\left(\frac{1}{s^\alpha - i(au)^\alpha} + \frac{1}{s^\alpha + i(au)^\alpha}\right) =$$

$$= \frac{1}{2}\frac{s^\alpha + i(au)^\alpha + s^\alpha - i(au)^\alpha}{(s^\alpha - i(au)^\alpha)(s^\alpha + i(au)^\alpha)} = \frac{s^\alpha}{s^{2\alpha} + (au)^{2\alpha}}.$$

In the same way, from the definition (2.10) and the linearity of the transform $N_{q,\alpha}$, we obtain

$$N_{q,\alpha}(s_{q,\alpha}(at)) = \frac{1}{2i}\left(N_{q,\alpha}\left(e_{q,\alpha}(i^{\frac{1}{\alpha}}at) - N_{q,\alpha}\big(e_{q,\alpha}((-i)^{\frac{1}{\alpha}}at)\big)\right)\right) =$$

$$= \frac{1}{2i}\left(\frac{1}{s^\alpha - (i^{\frac{1}{\alpha}}a)^\alpha u^\alpha} - \frac{1}{(s^\alpha - ((-i)^{\frac{1}{\alpha}}a)^\alpha u^\alpha}\right) =$$

$$= \frac{1}{2i}\left(\frac{1}{s^\alpha - i(au)^\alpha} - \frac{1}{s^\alpha + i(au)^\alpha}\right) =$$

$$= \frac{1}{2i}\frac{s^\alpha + i(au)^\alpha - s^\alpha + i(au)^\alpha}{(s^\alpha - i(au)^\alpha)(s^\alpha + i(au)^\alpha)} =$$

$$= \frac{(au)^\alpha}{s^{2\alpha} + (au)^{2\alpha}},$$

and the proof is complete.

Suppose that function $f(t)$ has a polynomial or formal power series expansion in α-monomials $t^{\alpha n}$. Let us denote such function $f(t)$ by $f_\alpha(t)$. We consider now the transform of a derivative $D_t^{q,\alpha} f_\alpha(t)$:

$$N_{q,\alpha}\big(D_t^{q,\alpha} f_\alpha(t)\big) = \int\limits_0^\infty \left(D_t^{q,\alpha} f_\alpha\right)(ut)\frac{1}{e_{q,\alpha}(qst)}\, d_{q,\alpha}t =$$

$$= \frac{1}{u^\alpha}\int\limits_0^\infty \left(D_t^{q,\alpha} f_\alpha\right)(y)\frac{1}{e_{q,\alpha}\left(qs\dfrac{y}{u}\right)}\, d_{q,\alpha}y =$$

$$= \frac{1}{u^\alpha} f_\alpha(y)\, \frac{1}{e_{q,\alpha}\left(qs\dfrac{y}{u}\right)}\Bigg|_0^\infty + \frac{1}{u^\alpha}\frac{s^\alpha}{u^\alpha}\int\limits_0^\infty f_\alpha(y)\frac{1}{e_{q,\alpha}\left(qs\dfrac{y}{u}\right)}\, d_{q,\alpha}y =$$

$$= -\frac{1}{u^\alpha} f_\alpha(0) + \frac{s^\alpha}{u^\alpha}\int\limits_0^\infty f_\alpha(ut)\frac{1}{e_{q,\alpha}(qst)}\, d_{q,\alpha}t =$$

$$= -\frac{1}{u^\alpha} f_\alpha(0) + \frac{s^\alpha}{u^\alpha} N_{q,\alpha}(f_\alpha(t)).$$

Let us rewrite it as

$$N_{q,\alpha}(D_t^{q,\alpha} f_\alpha(t)) = \frac{s^\alpha}{u^\alpha} N_{q,\alpha}(f_\alpha(t)) - \frac{1}{u^\alpha} f_\alpha(0). \tag{3.4}$$

Therefore,

$$N_{q,\alpha}\big((D_t^{q,\alpha})^2 f_\alpha(t)\big) = \frac{s^\alpha}{u^\alpha} N_{q,\alpha}\big(D_t^{q,\alpha} f_\alpha(t)\big) - \frac{1}{u^\alpha}\big(D_t^{q,\alpha} f_\alpha\big)(0) =$$

$$= \frac{s^\alpha}{u^\alpha}\left(\frac{s^\alpha}{u^\alpha} N_{q,\alpha}(f_\alpha(t)) - \frac{1}{u^\alpha} f_\alpha(0)\right) - \frac{1}{u^\alpha}\big(D_t^{q,\alpha} f_\alpha\big)(0) =$$

$$= \left(\frac{s^\alpha}{u^\alpha}\right)^2 N_{q,\alpha}(f_\alpha(t)) - \frac{1}{u^\alpha}\frac{s^\alpha}{u^\alpha} f_\alpha(0) - \frac{1}{u^\alpha}\big(D_t^{q,\alpha} f_\alpha\big)(0).$$

Thus, we can state the following result.

Theorem 3.1. *Suppose that function $f_\alpha(t)$ has polynomials or formal power series expansion in α-monomials t^α. Then, for all integer $n > 0$, it holds that*

$$N_{q,\alpha}\left((D_t^{q,\alpha})^n f_\alpha(t)\right) = \left(\frac{s^\alpha}{u^\alpha}\right)^n N_{q,\alpha}(f_\alpha(t)) - \frac{1}{u^\alpha}\sum_{j=0}^{n-1}\left(\frac{s^\alpha}{u^\alpha}\right)^{n-1-j} (D_t^{q,\alpha})^j f_\alpha(0).$$

Proof. The proof is by induction on n. The theorem statement holds for $n = 1$ as it shown in (3.4). Let us assume the formula holds for k, and let us prove it for $k+1$. By using (3.4), we have

$$N_{q,\alpha}\left((D_t^{q,\alpha})^{k+1} f_\alpha(t)\right) = \frac{s^\alpha}{u^\alpha} N_{q,\alpha}\left((D_t^{q,\alpha})^k f_\alpha(t)\right) - \frac{1}{u^\alpha}\big((D_t^{q,\alpha})^k f_\alpha\big)(0),$$

and, by induction's assumption, we get

$$N_{q,\alpha}\left((D_t^{q,\alpha})^{k+1} f_\alpha(t)\right) = -\frac{1}{u^\alpha}\big((D_t^{q,\alpha})^k f_\alpha\big)(0) +$$

$$+\frac{s^\alpha}{u^\alpha}\left(\left(\frac{s^\alpha}{u^\alpha}\right)^k N_{q,\alpha}(f_\alpha(t)) - \frac{1}{u^\alpha}\sum_{j=0}^{k-1}\left(\frac{s^\alpha}{u^\alpha}\right)^{k-1-j} (D_t^{q,\alpha})^j f_\alpha(t)\right) =$$

$$= \left(\frac{s^\alpha}{u^\alpha}\right)^{k+1} N_{q,\alpha}(f_\alpha(t)) - \frac{1}{u^\alpha}\sum_{j=0}^{k-1}\left(\frac{s^\alpha}{u^\alpha}\right)^{k-j} (D_t^{q,\alpha})^j f_\alpha(t) - \frac{1}{u^\alpha}\big((D_t^{q,\alpha})^k f_\alpha\big)(0) =$$

$$= \left(\frac{s^\alpha}{u^\alpha}\right)^{k+1} N_{q,\alpha}(f_\alpha(t)) - \frac{1}{u^\alpha}\sum_{j=0}^{k}\left(\frac{s^\alpha}{u^\alpha}\right)^{k-j} (D_t^{q,\alpha})^j f_\alpha(t),$$

which completes the proof.

Let us consider now two examples of applying the conformable fractional q-deformed natural transform for solving differential equations.

***Example* 3.1.** This example is an extension of the Example 4.2.4 in [8]. We have a differential equation

$$\left((D_t^{q,\alpha})^3 + (D_t^{q,\alpha})^2 - 6D_t^{q,\alpha}\right) f(t) = 0,$$

with the initial condition

$$f(0)=1, \qquad D_t^{q,\alpha}f(0)=0, \qquad \left(D_t^{q,\alpha}\right)^2 f(0)=5.$$

We apply our conformable fractional natural q-transform to the differential equation, and, by using the Theorem 3.1, obtain

$$\left(\frac{s^\alpha}{u^\alpha}\right)^3 \bar{f} - \frac{1}{u^\alpha}\sum_{j=0}^{2}\left(\frac{s^\alpha}{u^\alpha}\right)^{2-j}\left(D_t^{q,\alpha}\right)^j f(0) + \left(\frac{s^\alpha}{u^\alpha}\right)^2 \bar{f} -$$

$$-\frac{1}{u^\alpha}\sum_{j=0}^{1}\left(\frac{s^\alpha}{u^\alpha}\right)^{1-j}\left(D_t^{q,\alpha}\right)^j f(0) - 6\cdot\frac{s^\alpha}{u^\alpha}\bar{f} + \frac{6}{u^\alpha}f(0)=0,$$

where $\bar{f}=N_{q,\alpha}(f(t))$. Let $w=\dfrac{s^\alpha}{u^\alpha}$. Then we get

$$w^3\bar{f} - \frac{w^2}{u^\alpha}f(0) - \frac{w}{u^\alpha}D_t^{q,\alpha}f(0) - \frac{1}{u^\alpha}\left(D_t^{q,\alpha}\right)^2 f(0) + w^2\bar{f} -$$

$$-\frac{w}{u^\alpha}f(0) - \frac{1}{u^\alpha}D_t^{q,\alpha}f(0) - 6w\bar{f} + \frac{6}{u^\alpha}f(0)=0,$$

and, by applying the initial conditions, we obtain

$$\bar{f} = \frac{1}{u^\alpha}\frac{w^2+w-1}{w(w^2+w-6)} =$$

$$= \frac{1}{u^\alpha}\left(\frac{1}{6w} + \frac{1}{3(w+3)} + \frac{1}{2(w-2)}\right) =$$

$$= \frac{1}{6}\frac{1}{s^\alpha} + \frac{1}{3}\frac{1}{s^\alpha+3u^\alpha} + \frac{1}{2}\frac{1}{s^\alpha-2u^\alpha} =$$

$$= \frac{1}{6}\frac{1}{s^\alpha} + \frac{1}{3}\frac{1}{s^\alpha - \left((-3)^{\frac{1}{\alpha}}\right)^\alpha u^\alpha} + \frac{1}{2}\frac{1}{s^\alpha - \left(2^{\frac{1}{\alpha}}\right)^\alpha u^\alpha}.$$

Now, by using the results of the Proposition 3.2, we can find the original function $f(t)$ as following:

$$f(t) = \frac{1}{6} + \frac{1}{3}e_{q,\alpha}\left((-3)^{\frac{1}{\alpha}}t\right) + \frac{1}{2}e_{q,\alpha}\left(2^{\frac{1}{\alpha}}t\right).$$

One can easily see that this solution for $q=1,\ \alpha=1$ becomes $f(t)=\dfrac{1}{6}+\dfrac{1}{3}e^{-3t}+\dfrac{1}{2}e^{2t}$, which coincides with the solution of [8].

***Example* 3.2.** Let us consider now an extension of the differential equation appearing in Example 4 of [22]:

$$D_t^{q,\alpha}f(t) + 3f(t) = 13s_{q,\alpha}\left(2^{\frac{1}{\alpha}}t\right),$$

with the initial condition $f(0)=6$. Again, let $\bar{f}=N_{q,\alpha}(f(t))$. By applying the integral transform to this differential equation, we obtain the equation

$$\frac{s^\alpha}{u^\alpha}\bar{f} - \frac{1}{u^\alpha}f(0) + 3\bar{f} = 13 \cdot \frac{2u^\alpha}{s^{2\alpha} + 4u^{2\alpha}},$$

which, by applying the initial condition, can be rewritten as

$$\frac{s^\alpha + 3u^\alpha}{u^\alpha}\bar{f} = \frac{26u^\alpha}{s^{2\alpha} + 4u^{2\alpha}} + \frac{6}{u^\alpha}.$$

Now we can express the transformation $\bar{f}$ as

$$\bar{f} = \frac{1}{s^\alpha + 3u^\alpha}\frac{5 - u^{2\alpha} + 6s^{2\alpha}}{s^{2\alpha} + 4u^{2\alpha}} =$$

$$= \frac{A}{s^\alpha + 3u^\alpha} + \frac{Bs^\alpha + Cu^\alpha}{s^{2\alpha} + 4u^{2\alpha}}. \tag{3.5}$$

The unknown constants $A,\ B,\ C$ can be found by comparing two expressions for $\bar{f}$. One can easily check that $A = 8,\ B = -2$, and $C = 6$. Therefore (3.5) can be rewritten as

$$\bar{f} = \frac{8}{s^\alpha + 3u^\alpha} - 2\frac{s^\alpha}{s^{2\alpha} + 4u^{2\alpha}} + 3\frac{2u^\alpha}{s^{2\alpha} + 4u^{2\alpha}},$$

where, by Propositions 3.2 and 3.3, we can obtain the original function $f(t)$ as following:

$$f(t) = 8e_{q,\alpha}\big((-3)^{\frac{1}{\alpha}}t\big) - 2c_{q,\alpha}\big(2^{\frac{1}{\alpha}}t\big) + 3s_{q,\alpha}\big(2^{\frac{1}{\alpha}}t\big).$$

Note that for $q = 1,\ \alpha = 1$ we obtain the solution of [22].

We have considered transforms of functions and their derivatives. Let us consider now derivatives of the transform. We would like to emphasize that a function $f(t)$ is, actually, polynomial or formal power series in t^α-monomials. Let us denote by $R_{q,\alpha}(u,s)$ the conformable fractional q-deformed natural transform (3.1). With the notation $e^{-1}_{q,\alpha}(t) = \dfrac{1}{e_{q,\alpha}(t)}$ we can state the following lemma.

Lemma 3.1. *For all integer $n > 0$,*

$$\big(D^{q,\alpha}_s\big)^n e^{-1}_{q,\alpha}\big(q^{-(n-1)}st\big) = (-1)^n t^{\alpha n} q^{-\binom{n}{2}\alpha} e^{-1}_{q,\alpha}(qst).$$

Proof. By applying the operator $D^{q,\alpha}$ with respect to s consequently $n-1$ times to the function $e^{-1}_{q,\alpha}\big(q^{-(n-1)}st\big)$ and using (2.11), we obtain

$$\big(D^{q,\alpha}_s\big)^n e^{-1}_{q,\alpha}\big(q^{-(n-1)}st\big) = \big(-(q^{-(n-1)}t)^\alpha\big)\ldots\big(-(q^{-1}t)^\alpha\big)D^{q,\alpha}_s e^{-1}_{q,\alpha}(st).$$

Applying (2.11) one more time, we have

$$\big(D^{q,\alpha}_s\big)^n e^{-1}_{q,\alpha}\big(q^{-(n-1)}st\big) = \big(-(q^{-(n-1)}t)^\alpha\big)\ldots\big(-(q^{-1}t)^\alpha\big)(-t^\alpha)e^{-1}_{q,\alpha}(qst) =$$

$$= (-1)^n t^{\alpha n} q^{-\binom{n}{2}\alpha} e^{-1}_{q,\alpha}(qst),$$

where from the lemma's statement follows.

Proposition 3.4. *Suppose that function $f_\alpha(t)$ has polynomials or formal power series expansion in α-monomials t^α. Then, for all integer $n > 0$, it holds that*

$$N_{q,\alpha}\big(t^{\alpha n}f_\alpha(t)\big) = (-1)^n q^{\binom{n}{2}\alpha}u^{\alpha n}\big(D^{q,\alpha}_s\big)^n R_{q,\alpha}(u, q^{-n}s),$$

where $R_{q,\alpha}(u,s) = N_{q,\alpha}(f_\alpha(t))$.

Proof. We have

$$\big(D_s^{q,\alpha}\big)^n R_{q,\alpha}(u, q^{-n}s) = \big(D_s^{q,\alpha}\big)^n \int\limits_0^\infty f_\alpha(ut) e_{q,\alpha}^{-1}\big(q^{-(n-1)}st\big) d_{q,\alpha}t =$$

$$= \int\limits_0^\infty f_\alpha(ut)\big(D_s^{q,\alpha}\big)^n e_{q,\alpha}^{-1}\big(q^{-(n-1)}st\big) d_{q,\alpha}t.$$

By using Lemma 3.1, we get

$$\big(D_s^{q,\alpha}\big)^n R_{q,\alpha}(u, q^{-n}s) = \int\limits_0^\infty f_\alpha(ut)(-1)^n t^{\alpha n} q^{-\binom{n}{2}\alpha} e_{q,\alpha}^{-1}(qst) d_{q,\alpha}t =$$

$$= (-1)^n \frac{q^{-\binom{n}{2}\alpha}}{u^{\alpha n}} \int\limits_0^\infty (ut)^{\alpha n} f_\alpha(ut) e_{q,\alpha}^{-1}\big(q^{-(n-1)}st\big) d_{q,\alpha}t =$$

$$= (-1)^n \frac{q^{-\binom{n}{2}\alpha}}{u^{\alpha n}} N_{q,\alpha}\big(t^{\alpha n} f_\alpha(t)\big).$$

The rearrangement of the last equation completes the proof.

The natural transform is a function of two variables, namely u and s. The previous proposition establishes a connection between the transform of product of $f_\alpha(t)$ with a positive power of α-monomials t^α and q,α-deformed derivative with respect to one of the variables, namely s, of q,α-transform. Let us consider now a derivative of deformed transform with respect to its another variable u.

Proposition 3.5. *Suppose that function $f_\alpha(t)$ has the following expansion:*

$$f_\alpha(t) = \sum_{m=0}^\infty a_m t^{\alpha m}.$$

Then, for all integer $n > 0$, it holds that

$$N_{q,\alpha}\big(t^{\alpha n} f_\alpha(t)\big) = \frac{u^{\alpha n}}{s^{\alpha n}} (D_u^{q,\alpha})^n u^{\alpha n} R_{q,\alpha}(u, s).$$

Proof. If $f_\alpha(t) = \sum_{m=0}^\infty a_m t^{\alpha m}$, then

$$N_{q,\alpha}\big(t^{\alpha n} f_\alpha(t)\big) = N_{q,\alpha}\left(\sum_{m=0}^\infty a_m t^{\alpha(m+n)}\right).$$

By using the linearity of the transform and applying (3.3), we get

$$N_{q,\alpha}\big(t^{\alpha n} f_\alpha(t)\big) = \sum_{m=0}^\infty \frac{u^{\alpha(n+m)}}{s^{\alpha(n+m+1)}} [(n+m)\alpha]! a_m =$$

$$= \frac{u^{\alpha n}}{s^{\alpha n}} \sum_{m=0}^{\infty} \frac{u^{\alpha m}}{s^{\alpha(m+1)}} [(n+m)\alpha]! a_m =$$

$$= \frac{u^{\alpha n}}{s^{\alpha n}} \sum_{m=0}^{\infty} (D_u^{q,\alpha})^n \frac{[m\alpha]! a_m u^{\alpha(n+m)}}{s^{\alpha(m+1)}} =$$

$$= \frac{u^{\alpha n}}{s^{\alpha n}} \sum_{m=0}^{\infty} (D_u^{q,\alpha})^n \frac{u^{\alpha n} \cdot [m\alpha]! a_m u^{\alpha m}}{s^{\alpha(m+1)}} =$$

$$= \frac{u^{\alpha n}}{s^{\alpha n}} (D_u^{q,\alpha})^n u^{\alpha n} N_{q,\alpha}(f_\alpha(t)).$$

The replacement $N_{q,\alpha}(f_\alpha(t))$ by $R_{q,\alpha}(u,s)$ in the last equation completes the proof.

These results are in complete agreement with those obtained for non-deformed Sumudu and natural transform investigated by Belgacem and others (see [5] and references therein). Now we will give another representation of the q,α-deformed natural transform of the product of $f_\alpha(t)$ with positive degree of α-monomial t^α. This proposition extends the [5] (Theorem 4.2).

Proposition 3.6. *Suppose that function $f_\alpha(t)$ has polynomial or formal power series expansion in α-monomials t^α. Then, for all integer $n > 0$, we have*

$$N_{q,\alpha}\big(t^{\alpha n} f_\alpha(t)\big) = \frac{u^{\alpha n}}{s^{\alpha n}} \sum_{k=0}^{n} b_{n,k} u^{\alpha k} (D_u^{q,\alpha})^k R_{q,\alpha}(u,s),$$

where the coefficients $b_{n,k}$ satisfy the recurrence relationship

$$b_{n,k} = \begin{cases} [n\alpha] b_{n-1,0}, & k = 0, \\ [(n+k)\alpha] b_{n-1,k} + q^{\alpha(n-1+k)} b_{n-1,k-1}, & 0 < k < n, \\ q^{\alpha(2n-1)} b_{n-1,n-1}, & k = n, \end{cases}$$

with initial condition $b_{0,0} = 1$.

Proof. We proceed the proof by induction on n. For $n = 0$, we have $N_{q,\alpha}(f_\alpha(t)) = R_{q,\alpha}(u,s)$, so that $b_{0,0} = 1$. By previous proposition, for $n = 1$ we get

$$N_{q,\alpha}(t^\alpha f_\alpha(t)) = \frac{u^\alpha}{s^\alpha} D_u^{q,\alpha} u^\alpha R_{q,\alpha}(u,s),$$

which, by applying deformed Leibniz rule (2.4), can be rewritten as

$$N_{q,\alpha}(t^\alpha f_\alpha(t)) = \frac{u^\alpha}{s^\alpha} \big([\alpha] R_{q,\alpha}(u,s) + q^\alpha u^\alpha D_u^{q,\alpha} R_{q,\alpha}(u,s)\big).$$

Thus, $b_{1,0} = [\alpha] = [\alpha] b_{0,0}$, $b_{1,1} = q^\alpha = q^\alpha b_{0,0}$, and the claim holds. Assuming that the claim holds for $m \le n$, we will prove it, for $m = n+1$,

$$N_{q,\alpha}\big(t^{\alpha(n+1)} f_\alpha(t)\big) = N_{q,\alpha}\left(t^\alpha \big(t^{\alpha n} f_\alpha(t)\big)\right) =$$

$$= \frac{u^\alpha}{s^\alpha} D_u^{q,\alpha} u^\alpha \frac{u^{\alpha n}}{s^{\alpha n}} \sum_{k=0}^{n} b_{n,k} u^{\alpha k} (D_u^{q,\alpha})^k R_{q,\alpha}(u,s) =$$

$$= \frac{u^\alpha}{s^\alpha} D_u^{q,\alpha} \sum_{k=0}^{n} b_{n,k} \frac{u^{\alpha(1+n+k)}}{s^{\alpha n}} (D_u^{q,\alpha})^k R_{q,\alpha}(u,s) =$$

$$= \frac{u^\alpha}{s^\alpha} \sum_{k=0}^{n} b_{n,k} \frac{[(n+k+1)\alpha] u^{\alpha(n+k)}}{s^{\alpha n}} (D_u^{q,\alpha})^k R_{q,\alpha}(u,s) +$$

$$+ \frac{u^\alpha}{s^\alpha} \sum_{k=0}^{n} b_{n,k} \frac{q^{\alpha(1+n+k)} u^{\alpha(1+n+k)}}{s^{\alpha n}} (D_u^{q,\alpha})^{k+1} R_{q,\alpha}(u,s) =$$

$$= \frac{u^{\alpha(n+1)}}{s^{\alpha(n+1)}} \sum_{k=0}^{n} b_{n,k} [(n+k+1)\alpha] u^{\alpha k} (D_u^{q,\alpha})^k R_{q,\alpha}(u,s) +$$

$$+ \frac{u^{\alpha(n+1)}}{s^{\alpha(n+1)}} \sum_{k=1}^{n-1} b_{n,k-1} q^{\alpha(n+k)} u^{\alpha k} (D_u^{q,\alpha})^k R_{q,\alpha}(u,s) =$$

$$= \frac{u^{\alpha(n+1)}}{s^{\alpha(n+1)}} \sum_{k=0}^{n+1} b_{n+1,k} u^{\alpha k} (D_u^{q,\alpha})^k R_{q,\alpha}(u,s),$$

where

$$b_{n+1,0} = b_{n,0}[(n+1)\alpha],$$

$$b_{n+1,k} = b_{n,k}[(n+k+1)\alpha] + b_{n,k-1} q^{\alpha(n+k)}, \quad 1 \le k \le n,$$

$$b_{n+1,n+1} = b_{n,n} q^{\alpha(2n+1)},$$

which completes the proof.

We end this paper by the following conclusion. Our new generalization of the natural transform proposes also new generalizations of other widely used integral transforms. By applying the techniques described here, one can solve a k-order linear q-differential equation with constant coefficients. There is no need to find separately homogeneous solution and a particular solution. In order to solve a differential equation by applying the integral transform one need to know the integral transform of the right-hand side function of the differential equation

$$\sum_{0 \le j \le k} a_j \left(D_x^{q,\alpha}\right)^{k-j} f(x) = b(x)$$

and the initial conditions $\left(D_x^{q,\alpha}\right)^j y(0) = y_j$ for $j = 0, \ldots, k-1$.

References

1. B. Ahmad, S. K. Ntouyas, J. Tariboon, *Quantum calculus. New concepts, impulsive IVPs and BVPs*, inequalities, Trends in Abstract and Applied Analysis, **4**, World Sci. Publ. Co. Pte. Ltd., Hackensack, NJ (2016).
2. S. K. Q. Al-Omari, *On the quantum theory of the natural transform and some applications*, J. Difference Equat. and Appl., **25**, № 1, 21 – 37 (2019).
3. D. Albayrak, S. D. Purohit, F. Uçar, *On q-Sumudu transforms of certain q-polynomials*, Filomat, **27**, № 2, 411 – 427 (2013).

4. F. B. M. Belgacem, A. A. Karaballi, *Sumudu transform fundamental properties investigations and applications*, J. Appl. Math. Stoch. Anal., **2006**, № 4, Article ID 91083 (2006).
5. F. B.M. Belgacem, R. Silambarasan, *Advances in the natural transform*, AIP Conf. Proc., **1493**, 106 – 110 (2012).
6. M. Bohner, G. S. Guseinov, *The h-Laplace and q-Laplace transforms*, J. Math. Anal. and Appl., **365**, 75 – 92 (2010).
7. W. S. Chung, *On the q-deformed conformable fractional calculus and the q-deformed generalized conformable fractional calculus*, preprint (2016).
8. L. Debnath, D. Bhatta, *Integral transforms and their applications*, 2nd. ed., Chapman & Hall/CRC (2007).
9. H. Eltayeb, A. Kiliçman, B. Fisher, *A new integral transform and associated distributions*, Integral Transforms and Spec. Funct., **21**, № 5-6, 367 – 379 (2010).
10. W. Hahn, *Beiträge zur Theorie der Heineschen Reihen. Die 24 Integrale der hypergeometrischen q-Differenzengleichung. Das q-Analogon der Laplace-Transformation*, Math. Nachr., **2**, 340 – 379 (1949).
11. V. Kac, P. Cheung, *Quantum calculus*, Springer (2002).
12. Z. H. Khan, W. A. Khan, *N-transform — properties and applications*, NUST J. Eng. Sci., **1**, № 1, 127 – 133 (2008).
13. A. Kiliçman, M. Omran, *On double natural transform and its applications*, J. Nonlinear Sci. and Appl., **10**, № 4, 1744 – 1754 (2017).
14. A. Kiliçman, R. Silambarasan, *Computing new solutions of algebro-geometric equation using the discrete inverse Sumudu transform*, Adv. Difference Equat., Paper 323 (2018).
15. E. K. Lenzi, E. P. Borges, R. S. Mendes, *A q-generalization of Laplace transforms*, J. Phys. A, **32**, № 48, 8551 – 8562 (1999).
16. A. Plastino, M. C. Rocca, *The Tsallis – Laplace transform*, J. Phys. A, **392**, 5581 – 5591 (2013).
17. S. D. Purohit, S. L. Kalla, *On q-Laplace transforms of the q-Bessel functions*, Fract. Calc. and Appl. Anal., **10**, № 2, 189 – 196 (2007).
18. Y. Simsek, *Functional equations from generating functions: a novel approach to deriving identities for the Bernstein basis functions*, Fixed Point Theory and Appl., **2013**, Article 80 (2013).
19. F. Uçar, D. Albayrak, *On q-Laplace type integral operators and their applications*, J. Difference Equat. and Appl., **18**, № 6, 1001 – 1014 (2012).
20. G. K. Watugala, *Sumudu transform: a new integral transform to solve differential equations and control engineering problems*, Internat. J. Math. Ed. Sci. and Tech., **24**, № 1, 35 – 43 (1993).
21. O. Yürekli, *Identities on fractional integrals and various integral transforms*, Appl. Math. and Comput., **187**, № 1, 559 – 566 (2007).
22. D. G. Zill, W. S. Wright, *Advanced engineering mathematics*, 5th ed., Jones & Bartlett Learning (2014).
23. H. G. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Q. Chen, *A new collection of real world applications of fractional calculus in science and engineering*, Commun. Nonlinear Sci. and Numer. Simul., **64**, 213 – 231 (2018).

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