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ON THE SOLVABILITY OF NONLINEAR ORDINARY DIFFERENTIAL EQUATION IN GRAND LEBESGUE SPACES

ПРО РОЗВ'ЯЗНІСТЬ НЕЛІНІЙНИХ ЗВИЧАЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ У ВЕЛИКИХ ПРОСТОРАХ ЛЕБЕГА

We study the relationship between the second-order nonlinear ordinary differential equations and the Hardy inequality in grand Lebesgue spaces. In particular, we give a characterization of the Hardy inequality by using nonlinear ordinary differential equations in grand Lebesgue spaces.

Вивчається зв'язок між нелінійними звичайними диференціальними рівняннями другого порядку та нерівністю Гарді у великих просторах Лебега. Зокрема, дано характеристику нерівності Гарді нелінійними звичайними диференціальними рівняннями у великих просторах Лебега.

1. Introduction. It is well-known that in 1925 G. H. Hardy [15] proved the integral inequality using the calculus of variations, which states that if $f \in L_p$ is a nonnegative function on $(0,\infty)$, then

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx\right)^{\frac{1}{p}} \le \frac{p}{p-1}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}, \quad p>1. \tag{1.1}$$

The constant $\dfrac{p}{p-1}$ in (1.1) is the best possible (see also [16]). Also, inequality (1.1) holds in any finite interval $[a,b],\ 0\le a<b<\infty$. The prehistory of the classical Hardy inequality has been described in [23]. Some important steps in the further development of what today is called Hardy type inequalities are described in [24]. A systematic investigation of the generalized Hardy inequality with weights that started in [3]. Namely, in [3] two-weight Hardy inequality in its equivalent differential form

$$\left(\int_0^\infty f^p(x)\omega(x)\,dx\right)^{\frac{1}{p}} \le C\left(\int_0^\infty \big(f'(x)\big)^p v(x)dx\right)^{\frac{1}{p}}, \quad f(0)=f(+0)=0, \tag{1.2}$$

was connected with the Euler – Lagrange differential equation. It should be mentioned that in [4] Hardy inequality was studied not only with the case $p>1$, but also with $p<0$ and even with $0<p<1$. Beesack's approach was extended to a class of inequalities containing the Hardy inequality (1.2) as a special case (see, e.g., [4] or [30]). In particular, a necessary and sufficient condition on weight functions for validity (1.2) was obtained in [31] and [32]. The study of the case with different parameters p and q was started in [5] and developed in [22, 24, 25]. In the case $p\ne q$

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the other type criterion on weight functions for validity (1.1) was obtained in [14] and [27]. Namely, in [14] and [27] the inequality (1.2) was connected with nonlinear ordinary differential equation in weighted Lebesgue spaces. Similar problems for two-dimensional Hardy operator in weighted Lebesgue spaces with mixed norm is studied in [1]. Moreover, the Hardy inequality has numerous applications in the spectral theory of operators, in the theory of integral equations, in the theory of function spaces etc. (see, e.g., [6, 7, 24 – 26]).

In this paper we study similar problem in grand Lebesgue spaces. Namely, we give a new characterization of Hardy inequality by nonlinear ordinary differential equation in grand Lebesgue spaces. The main contribution in this paper is the characterization of best possible constant in Hardy inequality by specially introduced quantity.

Grand Lebesgue spaces proved to be useful in application to partial differential equations (see, e.g., [11, 13, 18, 19, 28, 29]). In particular, in the theory of PDE's, it turned out that these are the right spaces in which some nonlinear equations have to be considered (see, e.g., [8, 10, 12, 34]).

We note that the boundedness of classical Hardy operator in grand and small Lebesgue spaces was first proved in [9]. Later, the characterization of boundedness of the Hardy type operators between weighted grand Lebesgue spaces was studied in [20] (see, e.g., [21]). Similar results for one-dimensional and multidimensional Hardy operators in grand Lebesgue spaces on unbounded domains were proved in [33]. Recently, the boundedness of Hausdorff operator in grand Lebesgue spaces was obtained in [2].

This paper is organized as follows. Section 2 contains some preliminaries along with the standard ingredients used in the proofs. The main results are stated and proved in Section 3. Namely, in Section 3, we establish necessary condition and sufficient condition on the best possible constant in Hardy inequality on grand Lebesgue spaces.

2. Preliminaries. Let $1 < p < \infty$ and $p' = \dfrac{p}{p-1}$. In 1992 T. Iwaniec and C. Sbordone [17], in their studies related with the integrability properties of the Jacobian in a bounded open set $\Omega \subset \mathbb{R}^n$, introduced a new type of function spaces $L^{p)}(\Omega)$ called grand Lebesgue spaces. Namely, the grand Lebesgue space is defined as the space of the Lebesgue measurable functions f on Ω such that

$$\|f\|_{L_{p)}(\Omega)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|\Omega|} \int_{\Omega} |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} < \infty,$$

where $|\Omega|$ is the Lebesgue measure of Ω. Throughout this paper we assume that all functions are Lebesgue measurable. Let $n = 1$ and let $\Omega = (0,1)$. Then the norm in grand Lebesgue space has the form

$$\|f\|_{L_{p)}(0,1)} = \|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} =$$

$$= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|f\|_{p-\varepsilon} < \infty.$$

We denote by $C^1(0,1)$ the space of continuously differentiable functions on $(0,1)$. The set of all absolutely continuous functions on $(0,1)$ is denoted by $AC(0,1)$.

Let $1 < a < p$ be a fixed number and $0 < \varepsilon < p - a$. Suppose that λ is a positive measurable function defined on $(0, p - a)$ such that $\operatorname{ess\,inf}_{0<\varepsilon<p-a} \lambda(\varepsilon) > 0$. Let us consider the nonlinear differential equation

$$\lambda(\varepsilon)\,\frac{d}{dt}\big([y'(t)]^{p-\varepsilon-1}\big) + t^{\varepsilon-p}\left[y(t)\right]^{p-\varepsilon-1} = 0, \tag{2.1}$$

where

$$y(t) > 0, \qquad y'(t) > 0, \quad 0 < t < 1, \qquad y' \in AC(0,1). \tag{2.2}$$

We say that y is a solution of the problem (2.1), (2.2), if y satisfies the differential equation (2.1) almost everywhere on $(0,1)$ and the condition (2.2). We denote $y(0) = \lim_{t\to+0} y(t)$ and let $A(p,\lambda) = \operatorname{ess\,sup}_{0<\varepsilon<p-a} \big(\lambda(\varepsilon)\big)^{\frac{1}{p-\varepsilon}}$.

We denote by $L_{p),a}(0,1)$ the grand Lebesgue space the set of all measurable functions with the finite norm

$$\|u\|_{L_{p),a}(0,1)} = \|u\|_{p),a} = \sup_{0<\varepsilon<p-a} \varepsilon^{\frac{1}{p-\varepsilon}} \|u\|_{p-\varepsilon}.$$

It is obvious that $L_{p),1}(0,1) = L_{p)}(0,1)$ and $L_{p)}(0,1) \hookrightarrow L_{p),a}(0,1)$.

First we prove the following theorem.

Theorem 2.1. *Let $a < p < \infty$ and λ be a positive measurable function defined by* (2.1) *and $A(p,\lambda) < \infty$. Suppose that u is an absolutely continuous function on $(0,1)$ satisfies condition $u(0) = u(+0) = 0$. If the problem* (2.1), (2.2) *has a solution u, then*

$$\left\|\frac{u}{x}\right\|_{p),a} \le A(p,\lambda)\big\|u'\big\|_{p),a}.$$

Proof. It is well-known that for any absolutely continuous function the representation

$$u(x) = u(0) + \int_0^x u'(t)dt$$

holds. Since $u(0) = 0$, it follows that

$$u(x) = \int_0^x u'(t)dt.$$

Let a function y be a solution of problem (2.1), (2.2). Suppose that $0 < \varepsilon < p - a$ be any number. Then, using Hölder inequality with exponents $p - \varepsilon$ and $(p - \varepsilon)'$, we have

$$\big|u(x)\big| \le \int_0^x \big|u'(t)\big|\,dt = \int_0^x \big|u'(t)\big|\left[y'(t)\right]^{-\frac{1}{(p-\varepsilon)'}}\left[y'(t)\right]^{\frac{1}{(p-\varepsilon)'}} dt \le$$

$$\le \left(\int_0^x y'(t)dt\right)^{\frac{1}{(p-\varepsilon)'}} \left(\int_0^x \big|u'(t)\big|^{p-\varepsilon}\left[y'(t)\right]^{-\frac{p-\varepsilon}{(p-\varepsilon)'}} dt\right)^{\frac{1}{p-\varepsilon}} =$$

$$= \big(y(x) - y(0)\big)^{\frac{1}{(p-\varepsilon)'}} \left(\int_0^x \big|u'(t)\big|^{p-\varepsilon}\left[y'(t)\right]^{\varepsilon-p+1} dt\right)^{\frac{1}{p-\varepsilon}} \le$$

$$\le [y(x)]^{\frac{1}{(p-\varepsilon)'}} \left(\int\limits_0^x \left|u'(t)\right|^{p-\varepsilon} \left[y'(t)\right]^{\varepsilon-p+1} dt \right)^{\frac{1}{p-\varepsilon}} =$$

$$= \left[-x^{p-\varepsilon}\, \lambda(\varepsilon)\, \frac{d}{dx}\Big(\left[y'(x)\right]^{p-\varepsilon-1} \Big) \right]^{\frac{1}{p-\varepsilon}} \left(\int\limits_0^x \left|u'(t)\right|^{p-\varepsilon} \left[y'(t)\right]^{\varepsilon-p+1} dt \right)^{\frac{1}{p-\varepsilon}} =$$

$$= x(\lambda(\varepsilon))^{\frac{1}{p-\varepsilon}} \left(\int\limits_0^x \left|u'(t)\right|^{p-\varepsilon} \left[y'(t)\right]^{\varepsilon-p+1} \left(-\frac{d}{dx}\Big(\left[y'(x)\right]^{p-\varepsilon-1} \Big) \right) dt \right)^{\frac{1}{p-\varepsilon}}.$$

Hence, we have

$$\frac{|u(x)|}{x} \le (\lambda(\varepsilon))^{\frac{1}{p-\varepsilon}} \left(\int\limits_0^x \left|u'(t)\right|^{p-\varepsilon} \left[y'(t)\right]^{\varepsilon-p+1} \left(-\frac{d}{dx}\big([y'(x)]^{p-\varepsilon-1} \big) \right) dt \right)^{\frac{1}{p-\varepsilon}}.$$

Thus, one has

$$\left(\int\limits_0^1 \left(\frac{|u(x)|}{x} \right)^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} \le$$

$$\le A(p,\lambda) \left(\int\limits_0^1 \left(\int\limits_0^x \left|u'(t)\right|^{p-\varepsilon} \left[y'(t)\right]^{\varepsilon-p+1} \left(-\frac{d}{dx}\Big(\left[y'(x)\right]^{p-\varepsilon-1} \Big) \right) dt \right) dx \right)^{\frac{1}{p-\varepsilon}} =$$

$$= A(p,\lambda) \left(\int\limits_0^1 \int\limits_0^1 \left|u'(t)\right|^{p-\varepsilon} \left[y'(t)\right]^{\varepsilon-p+1} \left(-\frac{d}{dx}\Big(\left[y'(x)\right]^{p-\varepsilon-1} \Big) \right) \chi_{(0,x)}(t)\, dt\, dx \right)^{\frac{1}{p-\varepsilon}} =$$

$$= A(p,\lambda) \left(\int\limits_0^1 \left|u'(t)\right|^{p-\varepsilon} \left[y'(t)\right]^{\varepsilon-p+1} \left(\int\limits_t^1 \left(-\frac{d}{dx}\Big(\left[y'(x)\right]^{p-\varepsilon-1} \Big) \right) dx \right) dt \right)^{\frac{1}{p-\varepsilon}} =$$

$$= A(p,\lambda) \left(\int\limits_0^1 \left|u'(t)\right|^{p-\varepsilon} \left[y'(t)\right]^{\varepsilon-p+1} \Big(\left[y'(t)\right]^{p-\varepsilon-1} - \left[y'(1)\right]^{p-\varepsilon-1} \Big) dt \right)^{\frac{1}{p-\varepsilon}}. \qquad (2.3)$$

From equation (2.1), it follows that $y'' < 0$. Therefore y' is a decreasing function on $(0,1)$. Thus, (2.3) implies that

$$\left(\int\limits_0^1 \left(\frac{|u(x)|}{x} \right)^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} \le A(p,\lambda) \left(\int\limits_0^1 \left|u'(x)\right|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}. \qquad (2.4)$$

Multiply both side of (2.4) by $\varepsilon^{\frac{1}{p-\varepsilon}}$ and passing to supremum over all $\varepsilon \in (0, p-a)$, we complete the proof of Theorem 2.1.

Let us set

$$M(\varepsilon) = \frac{1}{p-\varepsilon-1} \inf_{g} \sup_{0<x<1} \frac{1}{g(x)-x} \int_0^x (g(t))^{p-\varepsilon}\, t^{\varepsilon-p}\, dt, \quad 0<\varepsilon<p-a, \tag{2.5}$$

where the infimum is taken over the class of all measurable functions g such that $g(x) > x$ for $0<x<1$.

***Remark* 2.1.** Let $0<x<1$ and $g(x)=2x$ or $g(x)=x(1+e^x)$. Then $\sup_{0<\varepsilon<p-a} M(\varepsilon) < \infty$.

The following lemma establishes a connection between problem (2.1), (2.2) and $M(\varepsilon)$.

Lemma 2.1. *Let λ and M be two functions defined on $(0, p-a)$. Then the following statements are equivalent*:

(i) *if* $\operatorname{ess\,inf}_{0<\varepsilon<p-a} \lambda(\varepsilon) > 0$ *and the problem* (2.1), (2.2) *has a solution with an absolutely continuous first derivative, then $\lambda(\varepsilon) \geq M(\varepsilon)$ for all $0<\varepsilon<p-1$*;

(ii) *if* $\sup_{0<\varepsilon<p-a} M(\varepsilon) < \infty$, *then the problem* (2.1), (2.2) *has a solution for every $\lambda(\varepsilon) > M(\varepsilon)$*.

Proof. Suppose that (i) holds. Let y be a solution of (2.1) – (2.3). Let us take $\omega = \dfrac{y}{y'}$. Then ω is positive solution of the nonlinear differential equation

$$\omega' = \frac{1}{(p-\varepsilon-1)\,\lambda(\varepsilon)}\, t^{\varepsilon-p}\, \omega^{p-\varepsilon} + 1. \tag{2.6}$$

By (2.6), we have

$$\omega(x) \geq \int_0^x \omega'(t)dt = \frac{1}{(p-\varepsilon-1)\lambda(\varepsilon)} \int_0^x t^{\varepsilon-p} \big(\omega(t)\big)^{p-\varepsilon} dt + x. \tag{2.7}$$

This implies that $\omega(x) \geq x$. By (2.7), one has

$$\lambda(\varepsilon) \geq \frac{1}{p-\varepsilon-1} \inf_{g} \sup_{0<x<1} \frac{1}{g(x)-x} \int_0^x t^{\varepsilon-p} \big(g(t)\big)^{p-\varepsilon} dt. \tag{2.8}$$

Therefore, by (2.8) and (2.5), we conclude that $\lambda(\varepsilon) \geq M(\varepsilon)$ for all $0<\varepsilon<p-1$. This completes the proof of (i).

Let us assume that (ii) holds. Let us fix $\lambda(\varepsilon) > M(\varepsilon)$. By the definition of $M(\varepsilon)$ there exists a measurable functions g such that

$$g(x) \geq \frac{1}{(p-\varepsilon-1)\lambda(\varepsilon)} \int_0^x t^{\varepsilon-p} (g(t))^{p-\varepsilon}\, dt + x.$$

We define a sequence of functions $\omega_n(x)$ by setting

$$\omega_0(x) = g(x), \qquad \omega_{n+1}(x) = \frac{1}{(p-\varepsilon-1)\,\lambda(\varepsilon)} \int_0^x t^{\varepsilon-p} (\omega_n(t))^{p-\varepsilon}\, dt + x, \quad n = 0, 1, 2, \ldots .$$

It is obvious that $\omega_0(x) \geq \omega_1(x)$. Let $\omega_{n-1}(x) \geq \omega_n(x)$. So, one has

$$\omega_n(x) - \omega_{n+1}(x) = \frac{1}{(p-\varepsilon-1)\,\lambda(\varepsilon)} \int\limits_0^x t^{\varepsilon-p}\big[(\omega_{n-1}(t))^{p-\varepsilon} - (\omega_n(t))^{p-\varepsilon}\big]dt \geq 0.$$

This implies that $\{\omega_n(x)\}_{n=0}^{\infty}$ is a nonincreasing by n on $x \in (0,1)$. Since $\omega_n(x) \geq 0$, this implies that a sequence $\{\omega_n(x)\}_{n=0}^{\infty}$ is converges. Let $\omega(x) = \lim_{n\to\infty} \omega_n(x)$ for a.e. $x \in (0,1)$. By the Levi monotone convergence theorem, it follows that ω is a nonnegative solution of the integral equation

$$\omega(x) = \frac{1}{(p-\varepsilon-1)\,\lambda(\varepsilon)} \int\limits_0^x t^{\varepsilon-p}\big(\omega(t)\big)^{p-\varepsilon}\,dt + x.$$

So, ω is an absolutely continuous function and satisfies the differential equation

$$\omega'(x) = \frac{1}{(p-\varepsilon-1)\,\lambda(\varepsilon)}\, x^{\varepsilon-p}(\omega(x))^{p-\varepsilon} + 1.$$

Therefore, for any fixed number $a \in (0,1)$ the function

$$y(x) = Ce^{\int_a^x \frac{dt}{\omega(t)}\,dt}, \quad C = y(a),$$

satisfies problem (2.1), (2.2).

The lemma is proved.

3. Main results. In this section, we proved the solvability of problem (2.1), (2.2) in grand Lebesgue space $L_{p),\,a}(0,1)$.

We need the following theorem.

Theorem 3.1. *Let* $1 < a \leq p < \infty$, M *be a positive function defined on* $(0, p-a)$ *by* (2.5) *and* $A(p,M) < \infty$. *Suppose that* u *is an absolutely continuous function on* $(0,1)$ *satisfies condition* $u(0) = 0$. *Let* $C > 0$ *be the best constant such that*

$$\left\|\frac{u}{x}\right\|_{p),a} \leq C\|u'\|_{p),a}. \tag{3.1}$$

Then

$$1 \leq C \leq A(p,M) \leq \frac{a}{a-1}. \tag{3.2}$$

Proof. Let us suppose that (3.1) holds and we choose the test function as $u(x) = x$, $0 < x < 1$. Then $u'(x) = 1$ and

$$\|1\|_{p),a} = \sup_{0<\varepsilon<p-a} \varepsilon^{\frac{1}{p-\varepsilon}} = (p-a)^{\frac{1}{a}}.$$

On the other hand, one has

$$\left\|\frac{u}{x}\right\|_{p),a} = \|1\|_{p),a}.$$

Hence $C \geq 1$.

Now we show that $C \leq A(p,M)$. Let $u'(x) = f(x)$. Since $u(0) = 0$, it follows that

$$u(x) = \int\limits_0^x f(t)dt.$$

We set $Hf(x) = \frac{1}{x}\int_0^x f(t)dt$. Obviously,

$$C = \sup_{\|f\|_{p),a}=1} \|Hf\|_{p),a}.$$

Assume the contrary. Let $C > A(p,M)$. Then there exists a number $\mu > 0$ such that $C > \mu >$ $> A(p,M)$. So $\mu > \big(M(\varepsilon)\big)^{\frac{1}{p-\varepsilon}}$ for all $0 < \varepsilon < p-1$. This implies that $\mu^{p-\varepsilon} > M(\varepsilon)$. We choose $\lambda(\varepsilon) = \mu^{p-\varepsilon}$. Since $A(p,M) < \infty$, by Lemma 2.1, problem (2.1), (2.2) has a solution for every $\lambda(\varepsilon) > M(\varepsilon)$. Therefore, by Theorem 2.1, we have

$$\Big\|\frac{u}{x}\Big\|_{p),a} \le \mu\|u'\|_{p),a}.$$

Hence C is not the best possible constant in (3.1). This contradiction completes the proof.

Finally we show that $A(p,M) \le \dfrac{a}{a-1}$. By the definition of $M(\varepsilon)$ for every function g satisfying condition $g(x) > x$, we have

$$M(\varepsilon) \le \frac{1}{p-\varepsilon-1} \sup_{0<x<1} \frac{1}{g(x)-x} \int\limits_0^x t^{\varepsilon-p}\big(g(t)\big)^{p-\varepsilon}dt.$$

We choose $g_\varepsilon(x) = (p-\varepsilon)'x$. It is obvious that $g_\varepsilon(x) > x$. So, one has

$$M(\varepsilon) \le \frac{1}{p-\varepsilon-1}\,\frac{\big((p-\varepsilon)'\big)^{p-\varepsilon}}{(p-\varepsilon)'-1} = \big((p-\varepsilon)'\big)^{p-\varepsilon}.$$

Hence $\big(M(\varepsilon)\big)^{\frac{1}{p-\varepsilon}} \le (p-\varepsilon)'$ and passing to supremum over all $\varepsilon \in (0, p-a)$, we get

$$A(p,M) \le \sup_{0<\varepsilon<p-a} \frac{p-\varepsilon}{p-\varepsilon-1} = \frac{a}{a-1}.$$

Theorem 3.1 is proved.

Now we proved our main theorem.

Theorem 3.2. *Let* $1 < a \le p < \infty$ *and* $\varepsilon \in (0, p-a)$. *Suppose that* u *is an absolutely continuous function on* $(0,1)$ *satisfies condition* $u(0) = 0$. *Then, for the solvability of problem* (2.1), (2.2), *it is necessary and sufficient that there exists a constant* $C_0 > 0$ *such that the inequality*

$$\Big\|\frac{u}{x}\Big\|_{p),a} \le C_0\,\|u'\|_{p),a} \tag{3.3}$$

holds.

Proof. The sufficiency part of the Theorem 3.2 follows from Theorem 2.1. On the other hand, the inequality (3.3) holds with constant $C_0 = A(p,\lambda)$. We shall prove only the necessity part. Let u be an absolutely continuous function satisfying condition $u(0) = 0$ and let the inequality (3.3) holds. Then $C \le C_0 < \infty$, where C is the constant in (3.1). By (3.2) for all $\varepsilon \in (0, p-a)$, we get that

$$M(\varepsilon) \leq \left(\frac{a}{a-1}\right)^{p-\varepsilon} \leq \left(\frac{a}{a-1}\right)^{p} \leq \left(\frac{a}{a-1}\right)^{p} C < \infty.$$

So $A(p, M) \leq \left(\dfrac{a}{a-1}\right)^{p} C < \infty$. Then, by Lemma 2.1, problem (2.1), (2.2) has a solution for any $\lambda(\varepsilon) > M(\varepsilon)$.

Theorem 3.2 is proved.

***Example* 3.1.** Let $1 < a \leq p < \infty$, $0 < \alpha < 1$, and $\lambda(\varepsilon) = \dfrac{\alpha^{\varepsilon - p + 1}}{(1-\alpha)\,(p-\varepsilon-1)}$. Then $y(t) = t^{\alpha}$ is the solution of problem (2.1), (2.2). It is easy to see that $A(p, \lambda) < \infty$. Thus, by Theorem 3.2, there exists a constant $C_0 > 0$ such that (3.3) holds.

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