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MEROMORPHIC BERGMAN SPACES

МЕРОМОРФНІ ПРОСТОРИ БЕРГМАНА

We introduce new spaces of holomorphic functions on the pointed unit disc in $\mathbb{C}$ that generalize classical Bergman spaces. We prove some fundamental properties of these spaces and their dual spaces. Finally, we extend the Hardy – Littlewood and Fejér – Riesz inequalities to these spaces with application of the Toeplitz operators.

Введено нові простори голоморфних функцій на загостреному одиничному диску в $\mathbb{C}$, які узагальнюють класичні простори Бергмана. Доведено деякі фундаментальні властивості цих просторів та дуальних до них. Насамкінець поширено нерівності Гарді – Літтльвуда та Феєра – Рісса на ці простори за допомогою операторів Тепліца.

1. Introduction and preliminary results. Since the seventeenth of the last century the notion of Bergman spaces has known an increasing use in mathematics and essentially in complex analysis and geometry. The fundamental concept of this notion is the Bergman kernel. This kernel was computed firstly for the unit disc $\mathbb{D}$ in $\mathbb{C}$ and then it was determined for any simply connected domain by the famous Riemann's theorem. However the determination of the Bergman kernels of domains in $\mathbb{C}^n$ is more delicate and it is determined for some type of domains and still unknown up to our day in general. In this paper we generalize most properties of Bergman spaces of the unit disk by introducing new spaces of holomorphic functions on the pointed unit disc $\mathbb{D}^*$ that are square integrable with respect to a probability measure $d\mu_{\alpha,\beta}$ for some $\alpha, \beta > -1$. In fact the classical Bergman space is reduced to the case $\beta = 0$ (see [3] for more details). We call these new spaces meromorphic Bergman spaces; indeed any element of such a space is a meromorphic function which has 0 as a pole of order controlled by the parameter β. The originality of our idea is that the Bergman kernels of these spaces may have zeros in the unit disk essentially when β is not an integer. This problem will be discussed in a separate paper as a continuity of the present paper. For this reason we will concentrate here on the topological properties of these spaces and prove some well-known inequalities.

Throughout this paper, $\mathbb{D}(a, r)$ will be the disc of $\mathbb{C}$ with center a and radius $r > 0$. In case $a = 0$, we use $\mathbb{D}(r)$ (resp., $\mathbb{D}$) in stead of $\mathbb{D}(0, r)$ (resp., $\mathbb{D}(0, 1)$). We set $\mathbb{S}(r) := \partial\mathbb{D}(r)$ the circle and $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$. For every $-1 < \alpha,\ \beta < +\infty$, we consider the positive measure $\mu_{\alpha,\beta}$ on $\mathbb{D}$ defined by

$$d\mu_{\alpha,\beta}(z) := \frac{1}{\mathscr{B}(\alpha+1, \beta+1)} |z|^{2\beta}(1-|z|^2)^\alpha dA(z),$$

where $\mathscr{B}$ is the beta-function defined by

$$\mathscr{B}(s,t) = \int_0^1 x^{s-1}(1-x)^{t-1}dx = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \quad \forall s,t > 0$$

and

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$$dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} rdrd\theta, \qquad z = x + iy = re^{i\theta},$$

the normalized area measure on $\mathbb{D}$.

The general aim of this paper is to study the properties of the Bergman type space $\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$ defined for $0 < p < +\infty$ as the set of holomorphic functions on $\mathbb{D}^*$ that belongs to the space

$$L^p(\mathbb{D}, d\mu_{\alpha,\beta}) = \{f : \mathbb{D} \longrightarrow \mathbb{C}; \text{ measurable function such that } \|f\|_{\alpha,\beta,p} < +\infty\},$$

where

$$\|f\|^p_{\alpha,\beta,p} := \int\limits_{\mathbb{D}} \big|f(z)\big|^p d\mu_{\alpha,\beta}(z).$$

When $1 \le p < +\infty$, the space $\big(L^p(\mathbb{D}, d\mu_{\alpha,\beta}), \|.\|_{\alpha,\beta,p}\big)$ is a Banach space; however for $0 < p < 1$, the space $L^p(\mathbb{D}, d\mu_{\alpha,\beta})$ is a complete metric space where the metric is given by $d(f, g) = \|f - g\|^p_{\alpha,\beta,p}$. The following proposition will be useful in the hole of the paper.

Proposition 1. *For every $0 < r < 1$ and $0 < \varepsilon < 1$ there exists $c_\varepsilon(r) = c_{\varepsilon,\alpha,\beta}(r) > 0$ such that, for any $0 < p < +\infty$ and $f \in \mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$, we have*

$$\big|f(z)\big|^p \le \frac{\mathscr{B}(\alpha + 1, \beta + 1)}{c_\varepsilon(r)} \|f\|^p_{\alpha,\beta,p} \quad \forall z \in \mathbb{S}(r).$$

One can choose $c_\varepsilon(r) = r_\varepsilon^2 \mathfrak{a}_\varepsilon(r)\mathfrak{b}_\varepsilon(r)$ with $r_\varepsilon = \varepsilon \min(r, 1 - r)$,

$$\mathfrak{a}_\varepsilon(r) := \begin{cases} \big[1 - (r + r_\varepsilon)^2\big]^\alpha & \textit{if} \quad \alpha \ge 0, \\ \big[1 - (r - r_\varepsilon)^2\big]^\alpha & \textit{if} \quad -1 < \alpha < 0, \end{cases}$$

and

$$\mathfrak{b}_\varepsilon(r) := \begin{cases} (r - r_\varepsilon)^{2\beta} & \textit{if} \quad \beta \ge 0, \\ (r + r_\varepsilon)^{2\beta} & \textit{if} \quad -1 < \beta < 0. \end{cases}$$

Proof. Let $0 < r < 1$, $0 < \varepsilon < 1$ and $0 < p < +\infty$ be fixed reals. Let $f \in \mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$ and $z \in \mathbb{S}(r)$. We set $r_\varepsilon = \varepsilon \min(r, 1 - r)$. It is easy to see that $\overline{\mathbb{D}}(z, r_\varepsilon) \subset \mathbb{D}^*$; so thanks to the subharmonicity of $|f|^p$, we obtain

$$\big|f(z)\big|^p \le \frac{1}{r_\varepsilon^2} \int\limits_{\mathbb{D}(z,r_\varepsilon)} |f(w)|^p dA(w) \le$$

$$\le \frac{\mathscr{B}(\alpha + 1, \beta + 1)}{r_\varepsilon^2} \int\limits_{\mathbb{D}(z,r_\varepsilon)} \frac{|f(w)|^p}{|w|^{2\beta}\big(1 - |w|^2\big)^\alpha} d\mu_{\alpha,\beta}(w). \tag{1.1}$$

If $w \in \mathbb{D}(z, r_\varepsilon)$ then $r - r_\varepsilon \le |w| \le r + r_\varepsilon$. Thus, we obtain

$$|w|^{2\beta} \ge \mathfrak{b}_\varepsilon(r) := \begin{cases} (r - r_\varepsilon)^{2\beta} & \text{if} \quad \beta \ge 0, \\ (r + r_\varepsilon)^{2\beta} & \text{if} \quad -1 < \beta < 0, \end{cases}$$

and

$$\big(1 - |w|^2\big)^\alpha \ge \mathfrak{a}_\varepsilon(r) := \begin{cases} \big[1 - (r + r_\varepsilon)^2\big]^\alpha & \text{if} \quad \alpha \ge 0, \\ \big[1 - (r - r_\varepsilon)^2\big]^\alpha & \text{if} \quad -1 < \alpha < 0. \end{cases}$$

It follows that inequality (1.1) gives

$$\left|f(z)\right|^p \leq \frac{\mathscr{B}(\alpha+1,\beta+1)}{r_\varepsilon^2\mathfrak{a}_\varepsilon(r)\mathfrak{b}_\varepsilon(r)}\int\limits_{\mathbb{D}(z,r_\varepsilon)}\left|f(w)\right|^p d\mu_{\alpha,\beta}(w) \leq$$

$$\leq \frac{\mathscr{B}(\alpha+1,\beta+1)}{r_\varepsilon^2\mathfrak{a}_\varepsilon(r)\mathfrak{b}_\varepsilon(r)}\|f\|^p_{\alpha,\beta,p}.$$

Proposition 1 is proved.

Using the previous proof, one can improve the previous proposition as follows.

***Remark* 1.** For any $n\in\mathbb{N}$ and $0<r<1$, there exists $c=c(n,r,\alpha,\beta)>0$ such that, for every $f\in\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$, we have

$$|f^{(n)}(z)|^p \leq c\|f\|^p_{\alpha,\beta,p} \quad \forall z\in\mathbb{S}(r).$$

As a first consequence of Proposition 1, we have the following corollary.

***Corollary* 1.** *For every $-1<\alpha,\beta<+\infty$ and $0<p<+\infty$, the space $\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$ is closed in $L^p(\mathbb{D},\mu_{\alpha,\beta})$ and, for any $z\in\mathbb{D}^*$, the linear form $\delta_z:\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)\longrightarrow\mathbb{C}$ defined by $\delta_z(f)=f(z)$ is bounded on $\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$.*

Proof. As $L^p(\mathbb{D},d\mu_{\alpha,\beta})$ is complete, it suffices to consider a sequence $(f_n)_n\subset\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$ that converges to $f\in L^p(\mathbb{D},d\mu_{\alpha,\beta})$ and to prove that $f\in\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$. Thanks to Proposition 1, the sequence $(f_n)_n$ converges uniformly to f on every compact subset of $\mathbb{D}^*$. Hence, the function f is holomorphic on $\mathbb{D}^*$ and we conclude that $f\in\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$.

For the second statement, one can see that δ_z is a linear functional well defined on $\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$. For the continuity of δ_z, thanks to Proposition 1, for every $z\in\mathbb{D}^*$, there exists $c>0$ such that, for every $f\in\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$, we have $|\delta_z(f)|=\left|f(z)\right|\leq c\|f\|_{\alpha,\beta,p}$. Thus the linear functional δ_z is continuous on $\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$.

The corollary is proved.

In the following we give some immediate properties:

If $f\in\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$ then 0 can't be an essential singularity for f, hence either 0 is removable for f (so f is holomorphic on $\mathbb{D}$) or 0 is a pole for f with order $\nu_f=\nu_f(0)$ that satisfies

$$\nu_f \leq m_{p,\beta} = \begin{cases} \left\lfloor \dfrac{2(\beta+1)}{p} \right\rfloor & \text{if} \quad \dfrac{2(\beta+1)}{p}\notin\mathbb{N}, \\ \dfrac{2(\beta+1)}{p}-1 & \text{if} \quad \dfrac{2(\beta+1)}{p}\in\mathbb{N}, \end{cases} \tag{1.2}$$

where $\lfloor . \rfloor$ is the integer part.

If we set $\widetilde{f}(z)=z^{\nu_f}f(z)$ then $\widetilde{f}$ is a holomorphic function on $\mathbb{D}$ and $f\in\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$ if and only if $\widetilde{f}\in\mathcal{A}^p_{\alpha,\beta-\frac{p\nu_f}{2}}(\mathbb{D}^*)$ and

$$\|f\|_{\alpha,\beta,p} = \left(\frac{\mathscr{B}(\alpha+1,\beta-\frac{p\nu_f}{2}+1)}{\mathscr{B}(\alpha+1,\beta+1)}\right)^{\frac{1}{p}}\|\widetilde{f}\|_{\alpha,\beta-\frac{p\nu_f}{2},p}.$$

Using the two previous properties, if we replace f by $z^{m_{p,\beta}}f$ in the proof of Proposition 1, we can obtain a more sharp estimate in Proposition 1.

If $-1 < \beta < \beta'$ and $-1 < \alpha < \alpha'$ then $\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*) \subseteq \mathcal{A}^p_{\alpha',\beta'}(\mathbb{D}^*)$ and the canonical injection is continuous. This is a consequence of the fact that we have

$$\mathscr{B}(\alpha'+1,\beta'+1)\|f\|_{\alpha',\beta',p} \leq \mathscr{B}(\alpha+1,\beta+1)\|f\|_{\alpha,\beta,p}$$

for every $f \in \mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$.

Claim that if we set $\mathbb{D}_\zeta := \mathbb{D} \setminus \{\zeta\}$ for any $\zeta \in \mathbb{D}$, then all results on $\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$ can be extended to the space $\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}_\zeta)$ of holomorphic functions on $\mathbb{D}_\zeta$ that are p-integrable with respect to the positive measure $|z-\zeta|^{2\beta}\big(1-|z|^2\big)^\alpha dA(z)$. Indeed $h \in \mathcal{A}^p_{\alpha,\beta}(\mathbb{D}_\zeta)$ if and only if $h \circ \varphi_\zeta \in \mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$, where $\varphi_\zeta(z) = \dfrac{\zeta - z}{1-\overline{\zeta} z}$.

2. Meromorphic Bergman kernels. In the case $p = 2$ we have $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*)$ is a Hilbert space and $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*) = \mathcal{A}^2_{\alpha,m}(\mathbb{D}^*)$ for every $\beta \in]m-1,m]$ with $m \in \mathbb{N}$. If we set

$$e_n(z) = \sqrt{\frac{\mathscr{B}(\alpha+1,\beta+1)}{\mathscr{B}(\alpha+1,n+\beta+1)}}\, z^n \tag{2.1}$$

for every $n \geq -m$, then the sequence $(e_n)_{n\geq -m}$ is a Hilbert basis of $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*)$. Furthermore, if $f, g \in \mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*)$ with

$$f(z) = \sum_{n=-m}^{+\infty} a_n z^n, \qquad g(z) = \sum_{n=-m}^{+\infty} b_n z^n,$$

then

$$\langle f, g\rangle_{\alpha,\beta} = \sum_{n=-m}^{+\infty} a_n \overline{b}_n \frac{\mathscr{B}(\alpha+1,n+\beta+1)}{\mathscr{B}(\alpha+1,\beta+1)},$$

where $\langle .,.\rangle_{\alpha,\beta}$ is the inner product in $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*)$ inherited from $L^2(\mathbb{D}, d\mu_{\alpha,\beta})$.

Lemma 1. *Let $-1 < \alpha < +\infty$ and $m \in \mathbb{N}$. Then the reproducing* (*Bergman*) *kernel $\mathbb{K}_{\alpha,m}$ of $\mathcal{A}^2_{\alpha,m}(\mathbb{D}^*)$ is given by*

$$\mathbb{K}_{\alpha,m}(w,z) = \frac{(\alpha+1)\mathscr{B}(\alpha+1,m+1)}{(w\overline{z})^m(1-w\overline{z})^{2+\alpha}}.$$

Proof. The sequence $(e_n)_{n\geq -m}$ given by (2.1) is a Hilbert basis of $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*)$, hence the reproducing kernel of $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*)$ is given by

$$\mathbb{K}_{\alpha,\beta}(w,z) = \sum_{n=-m}^{+\infty} e_n(w)\overline{e_n(z)} = \sum_{n=-m}^{+\infty} \frac{\mathscr{B}(\alpha+1,\beta+1)}{\mathscr{B}(\alpha+1,n+\beta+1)} w^n \overline{z}^n =$$

$$= \frac{1}{(w\overline{z})^m} \sum_{n=0}^{+\infty} \frac{\mathscr{B}(\alpha+1,\beta+1)}{\mathscr{B}(\alpha+1,n+\beta-m+1)} (w\overline{z})^n.$$

The computation of this kernel in the general case is more complicated. However, in our case for $\beta = m$, we obtain

$$\mathbb{K}_{\alpha,m}(w,z) = \frac{1}{(w\overline{z})^m} \sum_{n=0}^{+\infty} \frac{\mathscr{B}(\alpha+1,m+1)}{\mathscr{B}(\alpha+1,n+1)} (w\overline{z})^n =$$

$$= \frac{(\alpha+1)\mathscr{B}(\alpha+1,m+1)}{(w\overline{z})^m(1-w\overline{z})^{2+\alpha}}.$$

Lemma 1 is proved.

Here we give some fundamental properties of the Bergman kernel as consequences of Lemma 1.

***Corollary* 2.** *Let $-1 < \alpha < +\infty$ and $m \in \mathbb{N}$. Let $\mathbb{P}_{\alpha,m}$ be the orthogonal projection from $L^2(\mathbb{D}, d\mu_{\alpha,m})$ onto $\mathcal{A}^2_{\alpha,m}(\mathbb{D}^*)$. Then, for every $f \in L^2(\mathbb{D}, d\mu_{\alpha,m})$, we have*

$$\mathbb{P}_{\alpha,m} f(z) = (\alpha+1)\mathscr{B}(\alpha+1,m+1)\int\limits_{\mathbb{D}} \frac{f(w)}{(z\overline{w})^m(1-z\overline{w})^{2+\alpha}} d\mu_{\alpha,m}(w).$$

Proof. This is a simple consequence of Lemma 1 and the fact that for every $f \in L^2(\mathbb{D}, d\mu_{\alpha,m})$ we have $\mathbb{P}_{\alpha,m} f(z) = \langle f, \mathbb{K}_{\alpha,m}(.,z)\rangle_{\alpha,m}$.

Using the density of $\mathcal{A}^2_{\alpha,m}(\mathbb{D}^*)$ in $\mathcal{A}^1_{\alpha,m}(\mathbb{D}^*)$, one can prove the following corollary.

***Corollary* 3.** *Let $-1 < \alpha < +\infty$ and $m \in \mathbb{N}$. Then, for every $f \in \mathcal{A}^1_{\alpha,m}(\mathbb{D}^*)$, we have*

$$f(z) = (\alpha+1)\mathscr{B}(\alpha+1,m+1)\int\limits_{\mathbb{D}} \frac{f(w)}{(z\overline{w})^m(1-z\overline{w})^{2+\alpha}} d\mu_{\alpha,m}(w).$$

The following result is well-known in general, its proof is based essentially on the fact that $\mathbb{K}_{\alpha,\beta}(z,z) \neq 0$ for every $z \in \mathbb{D}^*$.

Proposition 2. *Let $-1 < \alpha,\ \beta < +\infty$ and $\mathbb{K}_{\alpha,\beta}$ be the reproducing (Bergman) kernel of $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*)$. Then, for every $z \in \mathbb{D}^*$, we have $\mathbb{K}_{\alpha,\beta}(z,z) > 0$ and satisfies*

$$\mathbb{K}_{\alpha,\beta}(z,z) = \sup\Big\{\big|f(z)\big|^2; f \in \mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*), \|f\|_{\alpha,\beta,2} \le 1\Big\} =$$

$$= \sup\left\{\frac{1}{\|f\|_{\alpha,\beta,2}}; f \in \mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*), f(z) = 1\right\}. \tag{2.2}$$

In particular, the norm of the Dirac form δ_z on $\mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*)$ is given by

$$\|\delta_z\| = \|\mathbb{K}_{\alpha,\beta}(.,z)\|_{\alpha,\beta,2} = \sqrt{\mathbb{K}_{\alpha,\beta}(z,z)}.$$

One can find the proof of the first equality in Krantz book [5], however the second one is due to Kim [4]. For the completeness of our paper we give the proof.

Proof. Thanks to the proof of Lemma 1, we have $\mathbb{K}_{\alpha,\beta}(z,z) > 0$ for every $z \in \mathbb{D}^*$. To prove the first equality in (2.2), we fix $z \in \mathbb{D}^*$ and we consider

$$\mathscr{Q}(z) := \sup\big\{|f(z)|^2; f \in \mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*), \|f\|_{\alpha,\beta,2} \le 1\big\}.$$

Let $f \in \mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*)$ such that $\|f\|_{\alpha,\beta,2} \le 1$. Then, thanks to the Cauchy – Schwarz inequality,

$$|f(z)|^2 = \big|\langle f, \mathbb{K}_{\alpha,\beta}(.,z)\rangle_{\alpha,\beta}\big|^2 \le \|f\|_{\alpha,\beta,2}\big\|\mathbb{K}_{\alpha,\beta}(.,z)\big\|^2_{\alpha,\beta,2} \le \mathbb{K}_{\alpha,\beta}(z,z).$$

It follows that $\mathscr{Q}(z) \le \mathbb{K}_{\alpha,\beta}(z,z)$. Conversely, we set

$$g(\xi) = \frac{\mathbb{K}_{\alpha,\beta}(\xi,z)}{\sqrt{\mathbb{K}_{\alpha,\beta}(z,z)}}, \quad \xi \in \mathbb{D}^*.$$

Hence we have $g \in \mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*)$, $\|g\|_{\alpha,\beta,2} = 1$ and $|g(z)|^2 = \mathbb{K}_{\alpha,\beta}(z,z)$ and the converse inequality $\mathscr{Q}(z) \geq \mathbb{K}_{\alpha,\beta}(z,z)$ is proved.

Now to prove the second equality in (2.2), we let

$$\mathscr{M}(z) := \inf\big\{\|f\|_{\alpha,\beta,2};\ f \in \mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*),\ f(z) = 1\big\}.$$

If we set

$$h(\xi) = \frac{\mathbb{K}_{\alpha,\beta}(\xi,z)}{\mathbb{K}_{\alpha,\beta}(z,z)}, \quad \xi \in \mathbb{D}^*,$$

then $h(z) = 1$ and $h \in \mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*)$. Indeed,

$$\|h\|^2_{\alpha,\beta,2} = \int\limits_{\mathbb{D}} |h(\xi)|^2 d\mu_{\alpha,\beta}(\xi) = \int\limits_{\mathbb{D}} \frac{\mathbb{K}_{\alpha,\beta}(\xi,z)}{\mathbb{K}_{\alpha,\beta}(z,z)} \frac{\mathbb{K}_{\alpha,\beta}(z,\xi)}{\mathbb{K}_{\alpha,\beta}(z,z)} d\mu_{\alpha,\beta}(\xi) =$$

$$= \frac{1}{\mathbb{K}_{\alpha,\beta}(z,z)^2}\mathbb{K}_{\alpha,\beta}(z,z) = \frac{1}{\mathbb{K}_{\alpha,\beta}(z,z)}.$$

It follows that

$$\mathscr{M}(z) \leq \frac{1}{\mathbb{K}_{\alpha,\beta}(z,z)}.$$

Conversely, for every $f \in \mathcal{A}^2_{\alpha,\beta}(\mathbb{D}^*)$ such that $f(z) = 1$, we have

$$\big|f(\zeta)\big|^2 = \big|\langle f, \mathbb{K}_{\alpha,\beta}(.,\zeta)\rangle_{\alpha,\beta}\big|^2 \leq \|f\|^2_{\alpha,\beta,2}\mathbb{K}_{\alpha,\beta}(\zeta,\zeta).$$

Thus we obtain

$$\frac{|f(\zeta)|^2}{\mathbb{K}_{\alpha,\beta}(\zeta,\zeta)} \leq \|f\|^2_{\alpha,\beta,2} \quad \forall \zeta \in \mathbb{D}^*.$$

In particular, for $\zeta = z$,

$$\frac{1}{\mathbb{K}_{\alpha,\beta}(z,z)} \leq \|f\|^2_{\alpha,\beta,2}.$$

We conclude that

$$\frac{1}{\mathbb{K}_{\alpha,\beta}(z,z)} \leq \mathscr{M}(z).$$

Proposition 2 is proved.

3. Duality of meromorphic Bergman spaces. The aim of this part is to prove that the dual of $\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$ is related to $\mathcal{A}^q_{\alpha,\beta}(\mathbb{D}^*)$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. This will be a consequence of the main result (Theorem 1). But to prove the main result we need the following lemma.

Lemma 2. *For every $-1 < \sigma$, $\gamma < +\infty$, we set*

$$I_\omega(z) = \int\limits_{\mathbb{D}} \frac{(1-|w|^2)^\sigma |w|^{2\gamma}}{|1 - z\overline{w}|^{2+\sigma+\omega}} dA(w).$$

Then I_ω is continuous on $\mathbb{D}$ and

$$I_\omega(z) \sim \begin{cases} 1 & \textit{if} \quad \omega < 0, \\ \log \dfrac{1}{1-|z|^2} & \textit{if} \quad \omega = 0, \\ \dfrac{1}{(1-|z|^2)^\omega} & \textit{if} \quad \omega > 0, \end{cases}$$

when $|z| \longrightarrow 1^-$ *where* $\varphi \sim \psi$ *means that there exist* $0 < c_1 < c_2$ *such that we have* $c_1\varphi(z) \leq \leq \psi(z) \leq c_2\varphi(z)$.

Proof. The proof is similar to [3] (Theorem 1.7).

Theorem 1. *For every* $-1 < \alpha,\ a, b < +\infty$ *and* $m \in \mathbb{N}$, *we consider the two integral operators* T *and* S *defined by*

$$Tf(z) = \frac{1}{z^m} \int\limits_{\mathbb{D}} \frac{f(w)(1-|w|^2)^{\alpha-a} w^m}{|w|^{2b}(1-z\overline{w})^{2+\alpha}} d\mu_{a,b}(w),$$

$$Sf(z) = \frac{1}{|z|^m} \int\limits_{\mathbb{D}} \frac{f(w)(1-|w|^2)^{\alpha-a} |w|^{m-2b}}{|1-z\overline{w}|^{2+\alpha}} d\mu_{a,b}(w).$$

Then, for every $1 \leq p < +\infty$, *the following assertions are equivalent*:

(1) T *is bounded on* $L^p(\mathbb{D}, d\mu_{a,b})$,

(2) S *is bounded on* $L^p(\mathbb{D}, d\mu_{a,b})$,

(3) $p(\alpha+1) > a+1$ *and* $\begin{cases} m-2 < 2b \leq m & \textit{if} \quad p = 1, \\ mp-2 < 2b < mp-2+2p & \textit{if} \quad p > 1. \end{cases}$

Proof. (2) $\Longrightarrow$ (1) is obvious.

(1) $\Longrightarrow$ (2) can be deduced using the transformation

$$\Omega_z f(w) = \frac{(1-z\overline{w})^{2+\alpha}|w|^m}{|1-z\overline{w}|^{2+\alpha} w^m} f(w).$$

(2) $\Longrightarrow$ (3). Now assume that S is bounded on $L^p(\mathbb{D}, d\mu_{a,b})$. If we apply S to $f_N(z) = (1 - - |z|^2)^N$ for N large enough, we get

$$\|Sf_N\|^p_{a,b,p} = \int\limits_{\mathbb{D}} (1-|z|^2)^a |z|^{2b-mp} \left(\int\limits_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha+N}|w|^m}{|1-z\overline{w}|^{2+\alpha}} dA(w) \right)^p dA(z)$$

is finite. Thanks to Lemma 2, we obtain $b > \dfrac{mp}{2} - 1$.

To prove the other inequalities, we need $S^\star$ the adjoint operator of S with respect to the inner product $\langle ., . \rangle_{a,b}$. It is given by

$$S^\star g(w) = (1-|w|^2)^{\alpha-a}|w|^{m-2b} \int\limits_{\mathbb{D}} \frac{g(z)}{|z|^m|1-z\overline{w}|^{2+\alpha}} d\mu_{a,b}(z) =$$

$$= (1-|w|^2)^{\alpha-a}|w|^{m-2b} \int\limits_{\mathbb{D}} \frac{g(z)|z|^{2b-m}(1-|z|^2)^a}{|1-z\overline{w}|^{2+\alpha}} dA(z).$$

We distinguish two cases:

1. *Case* $p = 1$: S is bounded on $L^1(\mathbb{D}, d\mu_{a,b})$ gives $S^\star$ is bounded on $L^\infty(\mathbb{D}, d\mu_{a,b})$. By applying $S^\star$ on the constant function $g \equiv 1$, we have

$$\sup_{w\in\mathbb{D}^*} \left(1-|w|^2\right)^{\alpha-a}|w|^{m-2b}\int\limits_{\mathbb{D}} \frac{|z|^{2b-m}(1-|z|^2)^a}{|1-z\overline{w}|^{2+\alpha}}dA(z) < +\infty.$$

Thanks to Lemma 2, we get $m-2b \geq 0$ and $\alpha - a > 0$. The desired inequalities are proved.

2. *Case* $p > 1$: Let $q > 1$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Again by applying $S^\star$ on the function f_N for N large enough, we obtain

$$\|S^\star f_N\|^q_{a,b,q} = \int\limits_{\mathbb{D}} (1-|w|^2)^{a+q(\alpha-a)}|w|^{2b+(m-2b)q}\left(\int\limits_{\mathbb{D}} \frac{|z|^{2b-m}(1-|z|^2)^{a+N}}{|1-z\overline{w}|^{2+\alpha}}dA(z)\right)^q dA(w)$$

is finite and hence all inequalities

$$\frac{mp}{2} - 1 < b < \frac{mp}{2} - 1 + p$$

and $p(\alpha+1) > a+1$ hold.

(3) $\Longrightarrow$ (2). We start by the case $p=1$. We assume that $m-2 < 2b \leq m$ and $\alpha > a$. Using Lemma 2, one can prove easily the boundedness of S on $L^1(\mathbb{D}, d\mu_{a,b})$.

Now for $p > 1$, to prove the boundedness of S on $L^p(\mathbb{D}, d\mu_{a,b})$ we will use the Schur test. We set

$$h(z) = \frac{1}{|z|^t(1-|z|^2)^s} \qquad \text{and} \qquad \kappa(z,w) = \frac{(1-|w|^2)^{\alpha-a}|w|^{m-2b}}{|z|^m|1-z\overline{w}|^{2+\alpha}}.$$

Thanks to Lemma 2, if

$$\frac{m}{q} \leq t < \frac{m+2}{q}, \quad 0 < s < \frac{\alpha+1}{q}, \tag{3.1}$$

then

$$\int\limits_{\mathbb{D}} \kappa(z,w)h(w)^q d\mu_{a,b}(w) = \frac{1}{|z|^m}\int\limits_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha-sq}|w|^{m-tq}}{|1-z\overline{w}|^{2+\alpha}}dA(w) \leq$$

$$\leq \frac{c_1}{|z|^m(1-|z|^2)^{sq}} = c_1|z|^{tq-m}h(z)^q \leq c_1 h(z)^q$$

for some positive constant $c_1 > 0$.

Similarly, if

$$\frac{2b-m}{p} \leq t < \frac{2b-m+2}{p}, \qquad \frac{a-\alpha}{p} < s < \frac{a+1}{p} \tag{3.2}$$

then

$$\int\limits_{\mathbb{D}} \kappa(z,w)h(z)^p d\mu_{a,b}(z) = (1-|w|^2)^{\alpha-a}|w|^{m-2b}\int\limits_{\mathbb{D}} \frac{|z|^{2b-m-tp}(1-|z|^2)^{a-sp}}{|1-z\overline{w}|^{2+\alpha}}dA(z) \leq$$

$$\leq c_2\frac{|w|^{m-2b}}{(1-|w|^2)^{sp}} = c_2|w|^{m-2b+tp}h(w)^p \leq c_2 h(w)^p$$

with $c_2 > 0$. Thanks to the hypothesis given in assertion (3), we have

$$\left]\frac{m}{q}, \frac{m+2}{q}\right[\cap \left]\frac{2b-m}{p}, \frac{2b-m+2}{p}\right[\neq \varnothing, \qquad \left]0, \frac{\alpha+1}{q}\right[\cap \left]\frac{a-\alpha}{p}, \frac{a+1}{p}\right[\neq \varnothing.$$

This proves the existence of t and s satisfying (3.1) and (3.2). Thanks to Schur's test, S is bounded on $L^p(\mathbb{D}, d\mu_{a,b})$.

Theorem 1 is proved.

Theorem 2. *For every* $1 < p < +\infty$ *and* $-1 < a,\ b < +\infty$, *the topological dual of* $\mathcal{A}^p_{a,b}(\mathbb{D}^*)$ *is the space* $\mathcal{A}^q_{a,b}(\mathbb{D}^*)$ *under the integral pairing*

$$\langle f, g\rangle_{a,b} = \int\limits_{\mathbb{D}} f(z)\overline{g(z)}d\mu_{a,b}(z) \quad \forall f \in \mathcal{A}^p_{a,b}(\mathbb{D}^*), \quad g \in \mathcal{A}^q_{a,b}(\mathbb{D}^*),$$

where q *is the conjugate exponent of* p.

Proof. Thanks to Hölder inequality, every function $g \in \mathcal{A}^q_{a,b}(\mathbb{D}^*)$ defines a bounded linear form on $\mathcal{A}^p_{a,b}(\mathbb{D}^*)$ via the above integral pairing. Conversely, let G be a bounded linear form on $\mathcal{A}^p_{a,b}(\mathbb{D}^*)$. Then thanks to Hahn – Banach extension theorem, one can extend G to a bounded linear form on $L^p(\mathbb{D}, d\mu_{a,b})$ (still denoted by G) with the same norm. By duality, there exists $\psi \in L^q(\mathbb{D}, d\mu_{a,b})$ such that

$$G(f) = \langle f, \psi\rangle_{a,b} \quad \forall f \in \mathcal{A}^p_{a,b}(\mathbb{D}^*).$$

Claim that if $m = m_{p,b}$ given in (1.2), then, thanks to Theorem 1, $\mathbb{P}_{a,m}$ maps continuously $L^p(\mathbb{D}, d\mu_{a,b})$ onto $\mathcal{A}^p_{a,b}(\mathbb{D}^*)$ and $\mathbb{P}_{a,m}f = f$ for every $f \in \mathcal{A}^p_{a,b}(\mathbb{D}^*)$. It follows that

$$G(f) = \langle f, \psi\rangle_{a,b} = \langle \mathbb{P}_{a,m}f, \psi\rangle_{a,b} = \langle f, \mathbb{P}^\star_{a,m}\psi\rangle_{a,b} \quad \forall f \in \mathcal{A}^p_{a,b}(\mathbb{D}^*).$$

If we set $g = \mathbb{P}^\star_{a,m}\psi$ then $g \in \mathcal{A}^q_{a,b}(\mathbb{D}^*)$ and $G(f) = \langle f, g\rangle$ for every $f \in \mathcal{A}^p_{a,b}(\mathbb{D}^*)$.

Theorem 2 is proved.

4. Inequalities on $\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$. The aim here is to extend the two famous Hardy – Littlewood and Fejér – Riesz inequalities to our new spaces, these inequalities were proved firstly on Hardy spaces, then on Bergman spaces with some applications. In our case we give only one application on Toeplitz operators. To reach this aim, for a holomorphic function f on $\mathbb{D}^*$ and $0 < r < 1$, we consider the main value on the circle:

$$M_p(r, f) := \left(\frac{1}{2\pi}\int\limits_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}},$$

$$M_\infty(r, f) := \sup_{\theta\in[0,2\pi]} \left|f(re^{i\theta})\right|.$$

We set

$$\mathscr{I}(r) = \mathscr{I}_{\alpha,\beta,p}(r) := \frac{2r^{pm_{p,\beta}}}{\mathscr{B}(\alpha+1, \beta+1)} \int\limits_r^1 t^{2\beta - pm_{p,\beta}+1}(1-t^2)^\alpha dt.$$

4.1. Hardy – Littlewood inequality. To prove the Hardy – Littlewood inequality on $\mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$, we need to prove firstly the following lemma.

Lemma 3. *For every $p > 1$ and $f \in \mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$, we have*

$$M_p(r,f) \le \frac{\|f\|_{\alpha,\beta,p}}{\mathscr{J}(r)^{\frac{1}{p}}}.$$

In particular,

$$M_p(r,f) \le \frac{\kappa_1\|f\|_{\alpha,\beta,p}}{r^{\max(\frac{2\beta}{p},\nu_f)}(1-r^2)^{\frac{\alpha+1}{p}}},$$

where $\kappa_1 = \big((\alpha+1)\mathscr{B}(\alpha+1,\beta+1)\big)^{\frac{1}{p}}$.

Proof. Let $f \in \mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$ and $0 < r < 1$. We set $F(z) = z^m f(z)$ with $m = m_{p,\beta}$. As F is holomorphic on $\mathbb{D}$, then we obtain

$$\|f\|^p_{\alpha,\beta,p} = \frac{1}{\pi\mathscr{B}(\alpha+1,\beta+1)}\int_0^1\int_0^{2\pi}|f(te^{i\theta})|^p(1-t^2)^\alpha t^{2\beta+1}dtd\theta =$$

$$= \frac{2}{\mathscr{B}(\alpha+1,\beta+1)}\int_0^1 M_p^p(t,f)(1-t^2)^\alpha t^{2\beta+1}dt =$$

$$= \frac{2}{\mathscr{B}(\alpha+1,\beta+1)}\int_0^1 M_p^p(t,F)(1-t^2)^\alpha t^{2\beta-pm+1}dt \ge$$

$$\ge \frac{2}{\mathscr{B}(\alpha+1,\beta+1)}M_p^p(r,F)\int_r^1(1-t^2)^\alpha t^{2\beta-pm+1}dt = M_p^p(r,f)\mathscr{J}(r)$$

and the first inequality is proved. The particular case can be deduced from the following inequality:

$$\mathscr{J}(r) = \frac{r^{pm}}{\mathscr{B}(\alpha+1,\beta+1)}\int_{r^2}^1(1-t)^\alpha t^{\beta-pm/2}dt \ge$$

$$\ge \begin{cases} \dfrac{r^{2\beta}(1-r^2)^{\alpha+1}}{(\alpha+1)\mathscr{B}(\alpha+1,\beta+1)} & \text{if} \quad 2\beta \ge pm, \\ \dfrac{r^{pm}(1-r^2)^{\alpha+1}}{(\alpha+1)\mathscr{B}(\alpha+1,\beta+1)} & \text{if} \quad 2\beta < pm \end{cases} \ge$$

$$\ge \frac{r^{\max(2\beta,pm)}(1-r^2)^{\alpha+1}}{(\alpha+1)\mathscr{B}(\alpha+1,\beta+1)}.$$

Lemma 3 is proved.

Now we can prove the Hardy – Littlewood inequality on meromorphic Bergman spaces.

Theorem 3. *For every $1 < p \le \tau \le \infty$, there exists a positive constant κ such that, for every $f \in \mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$, we have*

$$M_\tau(r,f) \le \frac{\kappa\|f\|_{\alpha,\beta,p}}{r^{\max\left(\frac{2\beta}{p},m\right)}(1-r^2)^{\frac{\alpha+2}{p}-\frac{1}{\tau}}},$$

where $m = m_{p,\beta}$.

The Hardy – Littlewood inequality is proved in [6] for classical Bergman spaces $(\beta = 0)$.

Proof. The case $\tau = p$ is simply the previous lemma. Let we start by the case $\tau = \infty$. Let $f \in \mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$ and $0 < r < 1$. We set $F(z) = z^m f(z)$. Again as F is holomorphic on $\mathbb{D}$, then, thanks to the Cauchy formula, we have

$$F(re^{i\theta}) = \frac{s}{2\pi}\int_0^{2\pi} \frac{s^m e^{imt} f(se^{it})}{se^{it} - re^{i\theta}} e^{it} dt,$$

where $s = \dfrac{1+r}{2}$. Applying Hölder's inequality $\left(\dfrac{1}{p} + \dfrac{1}{q} = 1\right)$ and Lemma 3, we obtain

$$r^m |f(re^{i\theta})| \leq \left(\frac{1}{2\pi}\int_0^{2\pi} s^{pm} |f(se^{it})|^p dt\right)^{\frac{1}{p}} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{s^q}{|se^{it} - re^{i\theta}|^q} dt\right)^{\frac{1}{q}} \leq$$

$$\leq s^m M_p(s,f) \left(\frac{\kappa_2}{\left(\dfrac{1-r}{1+r}\right)^{q-1}}\right)^{\frac{1}{q}} \leq$$

$$\leq s^m \frac{\kappa_1 \|f\|_{\alpha,\beta,p}}{s^{\max(\frac{2\beta}{p}, m)} (1-s^2)^{\frac{\alpha+1}{p}}} \frac{\kappa_3}{(1-r^2)^{1-\frac{1}{q}}} \leq$$

$$\leq \frac{\kappa_4 \|f\|_{\alpha,\beta,p}}{r^{\max(\frac{2\beta}{p} - m, 0)} (1-r^2)^{\frac{\alpha+2}{p}}}.$$

It follows that

$$M_\infty(r,f) \leq \frac{\kappa_4 \|f\|_{\alpha,\beta,p}}{r^{\max(\frac{2\beta}{p}, m)} (1-r^2)^{\frac{\alpha+2}{p}}}.$$

Let now $p < \tau < \infty$. We have

$$M_\tau(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p |f(re^{it})|^{\tau-p} dt\right)^{\frac{1}{\tau}} \leq$$

$$\leq M_\infty^{1-\frac{p}{\tau}}(r,f) M_p^{\frac{p}{\tau}}(r,f) \leq$$

$$\leq \left(\frac{\kappa_4 \|f\|_{\alpha,\beta,p}}{r^{\max(\frac{2\beta}{p}, m)} (1-r^2)^{\frac{\alpha+2}{p}}}\right)^{1-\frac{p}{\tau}} \left(\frac{\kappa_1 \|f\|_{\alpha,\beta,p}}{r^{\max(\frac{2\beta}{p}, m)} (1-r^2)^{\frac{\alpha+1}{p}}}\right)^{\frac{p}{\tau}} =$$

$$= \frac{\kappa \|f\|_{\alpha,\beta,p}}{r^{\max(\frac{2\beta}{p}, m)} (1-r^2)^{\frac{\alpha+2}{p} - \frac{1}{\tau}}}.$$

Theorem 3 is proved.

4.2. Fejér – Riesz inequality. The aim here is to prove a generalization of the following lemma to meromorphic Bergman spaces.

Lemma 4 (see [2]). *Let g be a holomorphic function in the Hardy space $H^p(\mathbb{D})$. Then, for any $\xi \in \mathbb{C}$ with $|\xi| = 1$, we have*

$$\int_{-1}^{1} |g(t\xi)|^p dt \leq \frac{1}{2}\|g\|_{H^p}^p := \frac{1}{2}\int_{0}^{2\pi} |g(e^{i\theta})|^p d\theta.$$

Theorem 4 (Fejér – Riesz inequality). *For every $f \in \mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$ and $\xi \in \mathbb{C}$ with $|\xi| = 1$, we have*

$$\int_{-1}^{1} \big|f(t\xi)\big|^p \mathscr{J}\big(|t|\big)dt \leq \pi\|f\|^p_{\alpha,\beta,p}.$$

Claim that if $\beta = 0$ then we find the Zhu result [7].

Proof. Let $f \in \mathcal{A}^p_{\alpha,\beta}(\mathbb{D}^*)$ and $F(z) = z^m f(z)$ where $m = m_{p,\beta}$. If we set $F_r(z) = F(rz)$ for $0 < r < 1$, Then $F_r \in H^p(\mathbb{D})$ and, thanks to Lemma 4, for every $\xi \in \mathbb{C},\ |\xi| = 1$,

$$\int_{-1}^{1} |F_r(t\xi)|^p dt \leq \frac{1}{2}\int_{0}^{2\pi} |F_r(e^{i\theta})|^p d\theta,$$

that is,

$$\int_{-1}^{1} |F(rt\xi)|^p dt \leq \frac{1}{2}\int_{0}^{2\pi} |F(re^{i\theta})|^p d\theta. \tag{4.1}$$

Thanks to inequality (4.1) and Fubini theorem, we have

$$\|f\|^p_{\alpha,\beta,p} = \frac{1}{\pi\mathscr{B}(\alpha+1,\beta+1)}\int_0^1\int_0^{2\pi} |f(re^{i\theta})|^p r^{2\beta+1}(1-r^2)^\alpha dr d\theta =$$

$$= \frac{1}{\pi\mathscr{B}(\alpha+1,\beta+1)}\int_0^1\left(\int_0^{2\pi} |F(re^{i\theta})|^p d\theta\right) r^{2\beta-pm+1}(1-r^2)^\alpha dr \geq$$

$$\geq \frac{2}{\pi\mathscr{B}(\alpha+1,\beta+1)}\int_0^1\left(\int_{-1}^{1} |F(rt\xi)|^p dt\right) r^{2\beta-pm+1}(1-r^2)^\alpha dr =$$

$$= \frac{2}{\pi\mathscr{B}(\alpha+1,\beta+1)}\int_0^1\left(\int_{-r}^{r} |F(s\xi)|^p ds\right) r^{2\beta-pm}(1-r^2)^\alpha dr =$$

$$= \frac{2}{\pi\mathscr{B}(\alpha+1,\beta+1)}\int_{-1}^1 |F(s\xi)|^p\left(\int_{|s|}^{1} r^{2\beta-pm}(1-r^2)^\alpha dr\right) ds =$$

$$= \frac{2}{\pi\mathscr{B}(\alpha+1,\beta+1)}\int_{-1}^1 |f(s\xi)|^p\left(|s|^{pm}\int_{|s|}^{1} r^{2\beta-pm}(1-r^2)^\alpha dr\right) ds \geq$$

$$\geq \frac{1}{\pi}\int\limits_{-1}^{1} |f(s\xi)|^p \mathscr{I}\big(|s|\big)ds.$$

Theorem 4 is proved.

As an application of the Fejér – Riesz inequality on the Toeplitz operators, we have the following result.

Theorem 5. *For every $\xi \in \mathbb{D}^*$, if we consider the Toeplitz operator $\mathcal{T}$ defined by*

$$\mathcal{T}f(z) = \int\limits_{-1}^{1} f(\xi x)\mathbb{K}_{\alpha,\beta}(z,\xi x)\mathscr{I}_{\alpha,\beta,2}(|x|)dx,$$

then $\mathcal{T}$ is a positive bounded linear operator on $\mathcal{A}_{\alpha,\beta}(\mathbb{D}^*)$.

When $\beta = 0$, this result is due to Andreev [1] proved in a restricted case.

Proof. Thanks to Fubini theorem, for every $f \in \mathcal{A}_{\alpha,\beta}(\mathbb{D}^*)$ one has

$$\langle \mathcal{T}f, f\rangle_{\alpha,\beta} = \int\limits_{\mathbb{D}} \mathcal{T}f(z)\overline{f(z)}d\mu_{\alpha,\beta}(z) =$$

$$= \int\limits_{\mathbb{D}}\left(\int\limits_{-1}^{1} f(\xi x)\mathbb{K}_{\alpha,\beta}(z,\xi x)\mathscr{I}_{\alpha,\beta,2}(|x|)dx\right)\overline{f(z)}d\mu_{\alpha,\beta}(z) =$$

$$= \int\limits_{-1}^{1} f(\xi x)\overline{\int\limits_{\mathbb{D}} \mathbb{K}_{\alpha,\beta}(\xi x, z)f(z)d\mu_{\alpha,\beta}(z)}\,\mathscr{I}_{\alpha,\beta,2}\big(|x|\big)dx =$$

$$= \int\limits_{-1}^{1} f(\xi x)\overline{f(\xi x)}\mathscr{I}_{\alpha,\beta,2}(|x|)dx \leq \pi\|f\|^2_{\alpha,\beta,2}.$$

The last inequality is the Fejér – Riesz one in the particular case $p = 2$.

This proves that the operator $\mathcal{T}$ is positive and thus it is self-adjoint and bounded with norm $\|\mathcal{T}\| \leq \pi$. Indeed,

$$\|\mathcal{T}\| = \sup\big\{|\langle \mathcal{T}f, f\rangle_{\alpha,\beta}|;\ \|f\|_{\alpha,\beta,2} = 1\big\} \leq \pi.$$

Theorem 5 is proved.

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