DOI: 10.37863/umzh.v74i9.6237

UDC 517.5

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ON LAPPAN'S FIVE-VALUED THEOREM FOR φ-NORMAL FUNCTIONS IN SEVERAL VARIABLES [2]

ПРО П'ЯТИЗНАЧНУ ТЕОРЕМУ ЛАППАНА ДЛЯ φ-НОРМАЛЬНИХ ФУНКЦІЙ КІЛЬКОХ ЗМІННИХ

Let $\mathbb{U}^m \subset \mathbb{C}^m$ be a unit ball centered at the origin and let $\mathbb{P}^n$ be an n-dimensional complex projective space with the metric $E_{\mathbb{P}^n}$. Also, let $\varphi\colon [0,1) \to (0,\infty)$ be a smoothly increasing function. A holomorphic mapping $f\colon \mathbb{U}^m \to \mathbb{P}^n$ is called *φ-normal* if $(\varphi(\|z\|))^{-1}(E_{\mathbb{P}^n}(f(z), df(z))(\xi))$ is bounded above for $z \in \mathbb{U}^m$ and $\xi \in \mathbb{C}^m$ such that $\|\xi\| = 1$, where $df(z)$ is the map from $T_z\big(\mathbb{U}^m\big)$ to $T_{f(z)}\big(\mathbb{P}^n\big)$ induced by f. For $n = 1$, f is called a φ-normal function. We present an extension of Lappan's five-valued theorem to the class of φ-normal functions.

Нехай $\mathbb{U}^m \subset \mathbb{C}^m$ — одинична куля з центром у початку координат, $\mathbb{P}^n$ — n-вимірний комплексний проективний простір з метрикою $E_{\mathbb{P}^n}$, а $\varphi\colon [0,1) \to (0,\infty)$ — плавно зростаюча функція. Голоморфне відображення $f\colon \mathbb{U}^m \to \mathbb{P}^n$ називається *φ-нормальним*, якщо $(\varphi(\|z\|))^{-1}(E_{\mathbb{P}^n}(f(z), df(z))(\xi))$ обмежено зверху для $z \in \mathbb{U}^m$ і $\xi \in \mathbb{C}^m$ так, що $\|\xi\| = 1$, де $df(z)$ — відображення з $T_z\big(\mathbb{U}^m\big)$ у $T_{f(z)}\big(\mathbb{P}^n\big)$, індуковане f. При $n = 1$ f називається φ-нормальною функцією. Встановлено розширення п'ятизначної теореми Лаппана на клас φ-нормальних функцій.

1. Introduction and main results. A meromorphic function f on a planar domain $D \subset \mathbb{C}$ is said to be *normal* in D if the family $\{f \circ \tau : \tau \in \mathcal{T}\}$ is normal in D, where $\mathcal{T}$ is the set of all conformal self maps of D. A well-known result of Lehto and Virtanen [6] gives the following characterization of normal functions: A meromorphic function f on the unit disc $\mathbb{D} \subset \mathbb{C}$ is normal if and only if $\sup_{z\in\mathbb{D}}\big(1 - |z|^2\big) f^{\#}(z) < \infty$, where $f^{\#} := \dfrac{|f'(z)|}{1 + |f(z)|^2}$ is the spherical derivative of f. Answering a question posed by Pommerenke [7] (Problem 3.2), Lappan gave the following known *five-valued* theorem.

Result 1.1 ([5], Theorem 1). *Let S be any set consisting of five distinct values in $\mathbb{C} \cup \{\infty\}$. If f is a meromorphic function on the unit disc $\mathbb{D}$ such that*

$$\sup\Big\{\big(1 - |z|^2\big) f^{\#}(z) : z \in f^{-1}(S)\Big\} < \infty,$$

then f is a normal function.

Lappan commented on the sharpness of the number *five* in the above result, he showed that *five* cannot be replaced by *three* and that there are a few cases where *five* cannot be replaced by *four* (see [5], Theorems 3 and 4).

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[2] The work on the initial stage of the research reported in this paper was done when the author was a postdoctoral fellow at the Indian Institute of Science. The support by the UGC of that phase of the work, through a Dr. D. S. Kothari Postdoctoral Fellowship (Grant no. No.F.4-2/2006 (BSR)/MA/19-20/0022), is gratefully acknowledged.

The main aim of this work is to study the five-valued theorem of Lappan, in higher-dimensional setting, for φ-normal functions, we shall formulate this work in Theorem 1.1. The notion of φ-normal functions on the unit disc $\mathbb{D} \subset \mathbb{C}$ was introduced by Aulaskari and Rättyä [1]. Here, φ represents a smoothly increasing function, by definition: An increasing function $\varphi\colon [0,1) \to (0,\infty)$ is called *smoothly increasing* if

$$1 \leq \varphi(r)(1-r) \to \infty \quad \text{as} \quad r \to 1^-,$$

and

$$\mathcal{R}_a(z) := \frac{\varphi(|a + z/\varphi(|a|)|)}{\varphi(|a|)} \to 1 \quad \text{as} \quad |a| \to 1^-$$

uniformly on compact subsets of $\mathbb{C}$. For given such a function φ, a meromorphic function f on $\mathbb{D}$ is called *φ-normal* if

$$\|f\|_{\mathcal{N}^\varphi} := \sup_{z\in\mathbb{D}} \frac{f^\#(z)}{\varphi(|z|)} < \infty.$$

The class of all φ-normal functions is denoted by $\mathcal{N}^\varphi$.

Aulaskari and Rättyä established the following five-valued theorem, analogous to Lappan's five-valued theorem, for φ-normal function.

Result 1.2 ([1], Theorem 9). *Let f be a meromorphic function on the unit disc $\mathbb{D} \subset \mathbb{C}$ and let $\varphi\colon [0,1) \to (0,\infty)$ be smoothly increasing. Then $f \in \mathcal{N}^\varphi$ if and only if there exists a set $S \subset \mathbb{C} \cup \{\infty\}$ consists of five distinct values such that*

$$\sup\Big\{ f^\#(z)/\varphi(|z|) \colon z \in f^{-1}(S) \Big\} < \infty.$$

T. V. Tan and N. V. Thin [8] established the following result wherein the set E, appeared in the statement of Result 1.2, consists of only **four** distinct values and still yields the same conclusion as in Result 1.2.

Result 1.3 ([8], Theorem 4). *Let f be a meromorphic function on the unit disc $\mathbb{D}$ and let $\varphi\colon [0,1) \to (0,\infty)$ be smoothly increasing. Assume that there is a subset $S \subset \mathbb{C} \cup \{\infty\}$ containing four distinct values such that*

$$\sup_{z\in f^{-1}(S)} \frac{f^\#(z)}{\varphi(|z|)} < \infty \qquad \text{and} \qquad \sup_{z\in f^{-1}(S\setminus\{\infty\})} (f')^\#(z) < \infty.$$

Then f is φ-normal.

Hu and Thin, recently, extended the concept of φ-normal function to higher dimensional settings in [4]. Let $\mathbb{U}^m := \big\{ z \in \mathbb{C}^m \colon \|z\| < 1 \big\}$ be the unit ball centered at the origin, and $\mathbb{P}^n$ be the n-dimensional complex projective space. Let $\operatorname{Hol}\big(\mathbb{U}^m, \mathbb{P}^n\big)$ denote the set of holomorphic mappings from $\mathbb{U}^m$ to $\mathbb{P}^n$. When $n = 1$, $\operatorname{Hol}\big(\mathbb{U}^m, \mathbb{P}^1\big)$ is just the set of meromorphic functions on $\mathbb{U}^m$. According to Hu – Thin: Let $\varphi\colon [0,1) \to (0,\infty)$ be an increasing function that satisfies the following properties:

$$\varphi(r)(1-r) \geq 1 \quad \text{for all} \quad r \in [0,1) \tag{1.1}$$

and

$$\mathcal{R}_a(z) := \frac{\varphi(\|a + z/\varphi(\|a\|)\|)}{\varphi(\|a\|)} \to 1 \quad \text{as} \quad \|a\| \to 1^- \tag{1.2}$$

uniformly on compact subsets of $\mathbb{C}^m$. Here $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{C}^m$. For given such a function φ, an element $f \in \mathrm{Hol}\left(\mathbb{U}^m, \mathbb{P}^n\right)$ is called *φ-normal* if

$$\|f\|_{\mathcal{N}^\varphi} := \sup_{\|\xi\|=1,\ z\in\mathbb{U}^m} \frac{E_{\mathbb{P}^n}(f(z), df(z)(\xi))}{\varphi(\|z\|)} < \infty, \tag{1.3}$$

where $E_{\mathbb{P}^n}(f(z), df(z)(\xi))$ is the norm, associated with the Fubini–Study metric $ds^2_{\mathbb{P}^n}$ on $\mathbb{P}^n$, at $f(z)$ in the direction of the vector $df(z)(\xi) \in T_{f(z)}\left(\mathbb{P}^n\right)$, where $T_{f(z)}\left(\mathbb{P}^n\right)$ is the holomorphic tangent space to $\mathbb{P}^n$ at $f(z)$. Note here that $df(z)$ is the mapping from $T_z\left(\mathbb{U}^m\right)$ to $T_{f(z)}\left(\mathbb{P}^n\right)$ induced by f. We shall defer the explanation of $E_{\mathbb{P}^n}(.,.)$ to the Section 2.

Hu and Thin found the following analogue of Lappan's five-valued theorem.

Result 1.4 ([4], Theorem 2.5). *Let $f \in \mathrm{Hol}\left(\mathbb{U}^m, \mathbb{P}^1\right)$ and let $\varphi\colon [0,1) \to \mathbb{R}^+$ be an increasing function satisfying* (1.1) *and* (1.2)*. Then $f \in \mathcal{N}^\varphi$ if and only if there exists a set S of five distinct values in $\mathbb{C} \cup \{\infty\}$ such that*

$$\sup_{\|\xi\|=1,\ z\in f^{-1}(S)} \frac{E_{\mathbb{P}^1}(f(z), df(z)(\xi))}{\varphi(\|z\|)} < \infty.$$

Now the following natural question arises:

Whether the cardinality of S in the statement of Result 1.4 can be reduced and yet yield the same conclusion?

Motivated by the Result 1.3 we establish the following theorem wherein we reduce the cardinality of S **but** we impose extra conditions, analogous to the conditions in the statement of Result 1.3.

Theorem 1.1. *Let $f \in \mathrm{Hol}\left(\mathbb{U}^m, \mathbb{P}^1\right)$ and let $\varphi\colon [0,1) \to \mathbb{R}^+$ be an increasing function satisfying* (1.1) *and* (1.2)*. Suppose that there exists a set S with four distinct points in $\mathbb{C} \cup \{\infty\}$ such that*

$$\sup_{\|\xi\|=1,\ z\in f^{-1}(S)} \frac{E_{\mathbb{P}^1}(f(z), df(z)(\xi))}{\varphi(\|z\|)} < \infty \tag{1.4}$$

and

$$\sup_{\|\xi\|=1,\ z\in f^{-1}(S\setminus\{\infty\})} \frac{E_{\mathbb{P}^1}\left(\sum_{l=1}^m f_{z_l}(z), d\sum_{l=1}^m f_{z_l}(z)(\xi)\right)}{(\varphi(\|z\|))^2} < \infty, \tag{1.5}$$

where $f_{z_l}(z) = \dfrac{\partial f}{\partial z_l}$, $l = 1, \ldots, m$. Then $f \in \mathcal{N}^\varphi$.

We end this section with a brief explanation of some common notations.

1.1. Some notations. We fix the following notation, which we shall use without any further clarification.

As in the discussion above, $\|\cdot\|$ will denote the Euclidean norm. Expressions like "unit vector" will be with reference to this norm.

We shall denote the standard Hermitian product in $\mathbb{C}^n$ by $\langle\cdot,\cdot\rangle$.

2. Basic notions. This section is devoted to elaborating upon concepts and terms that mentioned in Section 1, and to introducing certain notions that we shall need in our proofs.

The n-dimensional complex projective space $\mathbb{P}^n$ is the space $\big(\mathbb{C}^{n+1}\setminus\{0\}\big)/\sim$, where $(a_0,\ldots,a_n)\sim(b_0,\ldots,b_n)$ if and only if $(a_0,\ldots,a_n)=\lambda(b_0,\ldots,b_n)$ for some $\lambda\in\mathbb{C}\setminus\{0\}$. In other words, $\mathbb{P}^n$ is the set of all complex lines, passing through the origin in $\mathbb{C}^{n+1}$. A point in $\mathbb{P}^n$ is an equivalence class of some $(a_0,\ldots,a_n)\in\mathbb{C}^m\setminus\{0\}$ which we denote by $[a_0:\cdots:a_n]$, and it is called the homogeneous coordinate of the point. When $n=1$, the projective space $\mathbb{P}^1$ is identified with the extended complex plane $\mathbb{C}\cup\{\infty\}$.

Let us also revisit to the notion of holomorphic mappings alluded to in Section 1. Let $D\subset\mathbb{C}^m$ be a domain, and $f\colon D\to\mathbb{P}^n$ be a holomorphic mapping. Fixing a system of homogeneous coordinates on $\mathbb{P}^n$, for each $a\in D$, we have a holomorphic map $\widetilde{f}(z):=(f_0(z),\ldots,f_n(z))$ on some neighborhood U of a such that $\{z\in U\,|\,f_0(z)=\ldots=f_n(z)=0\}=\varnothing$ and $f(z)=[f_0(z)\colon\cdots:f_n(z)]$ for each $z\in U$. We shall call any such holomorphic map $\widetilde{f}\colon U\to\mathbb{C}^{n+1}$ a *reduced representation* of f on U. A holomorphic mapping $f\colon D\to\mathbb{P}^1$ is called a *meromorphic function* on D.

We now recall the meaning of the object $E_{\mathbb{P}^n}$ that alluded to in (1.3). First, we see it in the general case: Let M be a complete Hermitian complex manifold of dimension n with a Hermitian metric

$$ds_M^2=\sum_{i,k=1}^{n}h_{i\,k}(p)dz_id\bar{z}_k,$$

where $z=(z_1,\ldots,z_n)$ are local coordinates in a neighborhood of a point $p\in M$. Then ds_M^2 reduces to a norm on the holomorphic tangent space $T_p(M)$ of M at p in the direction of the vector $\xi\in T_p(M)$

$$E_M(p,\xi)=\big(ds_M^2(\xi,\bar{\xi})\big)^{1/2},\quad \xi\in T_p(M),$$

which further defines the distance between points $p,q\in M$ by

$$d_M(p,q):=\inf_{\gamma}\int\limits_0^1 E_M\big(\gamma(t),\gamma'(t)\big)\,dt,$$

where the infimum is taken over all parametric curves $\gamma\colon[0,1]\to M$ satisfying $\gamma(0)=p$ and $\gamma(1)=q$. When $M=\mathbb{P}^n$, the Hermitian metric has a nice description, let us recall briefly: Fixing a system of homogeneous coordinates $\xi=[\xi_0\colon\ldots:\xi_n]$, the Hermitian metric is given in homogeneous coordinates by

$$ds^2_{\mathbb{P}^n}=\frac{\langle d\xi,d\xi\rangle\langle\xi,\xi\rangle-|\langle\xi,d\xi\rangle|^2}{\langle\xi,\xi\rangle^2}.$$

This is called the Fubini – Study metric in homogeneous coordinates on $\mathbb{P}^n$. In the local coordinates on the open sets $U_l:=\{[\xi_0\colon\ldots:\xi_n]\in\mathbb{P}^n:\xi_l\neq0\}$, $l=0,\ldots,n$, of the standard covering of $\mathbb{P}^n$, we can rewrite the expression. Suppose that the coordinates are $\zeta_1=\xi_1/\xi_0,\ldots,\zeta_n=\xi_n/\xi_0$, with $\zeta=(\zeta_1,\ldots,\zeta_n)$ in the open set $U_0\cong\mathbb{C}^n$. Then

$$ds^2_{\mathbb{P}^n}=\frac{\big(1+\|\zeta\|^2\big)\|d\zeta\|^2-|\langle\zeta,d\zeta\rangle|^2}{\big(1+\|\zeta\|^2\big)^2}.$$

When $n = 1,$ this is the well-known spherical metric on $\mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$, and in this case for a holomorphic map $f\colon \mathbb{C} \to \mathbb{P}^n$, $E_{\mathbb{P}^1}(f, df)$ is the spherical derivative $f^{\#}$ of f, which is defined as $f^{\#} := \dfrac{|f'|}{1+|f|^2}$. We could also derive the spherical derivative from the differential form of type $(1,1)$ associated with the Fubini – Study metric. Recently, Dovbush [2] proved a version of Zalcman's lemma in several variables using the spherical derivative derived from the associated differential form.

3. Essential lemmas. One of the well-known result in the theory of normality is the rescaling lemma of Zalcman. We shall use the following Zalcman type rescaling result in order to prove Theorem 1.1. Loosely speaking, it says that a non-φ-normal map can be rescaled at a small scale to obtain a non-constant holomorphic map $g\colon \mathbb{C} \to \mathbb{P}^n$ in the limit.

Lemma 3.1 ([4], Theorem 2.4). *A holomorphic mapping $f \in \mathrm{Hol}\left(\mathbb{U}^m, \mathbb{P}^n\right)$ is not φ-normal if and only if there exist*

(a) *a compact subset $K_0 \subset \mathbb{U}^m$;*

(b) *points $z_j \in \mathbb{U}^m$ such that $\|z_j\| \to 1^-$;*

(c) *points $z_j^* \in K_0$ such that $w_j := z_j + z_j^*/\varphi(\|z_j\|) \in \mathbb{U}^m$ for j sufficiently large;*

(d) *positive numbers ρ_j with $\rho_j \to 0^+$;*

(e) *Euclidean unit vectors $\xi_j \in \mathbb{C}^m$ with $\|\xi_j\| = 1$*

such that the sequence $g_j(\zeta) := f(w_j + (\rho_j/\varphi(\|z_j\|))\xi_j\zeta)$, where $\zeta \in \mathbb{C}$ satisfies

$$(w_j + (\rho_j/\varphi(\|z_j\|))\xi_j\zeta) \in \mathbb{U}^m,$$

converges uniformly on compact subsets of $\mathbb{C}$ to a non-constant holomorphic mapping $g\colon \mathbb{C} \to \mathbb{P}^n$.

We shall also use the first and the second fundamental theorem of Nevanlinna theory in the proof of Theorem 1.1. We refer the monograph by Hayman [3] for detailed study of Nevanlinna theory.

First fundamental theorem. *Let $f(z)$ be a non-constant meromorphic in the complex plane $\mathbb{C}$. Then*

$$N(r, 1/f) \leq T(r, f) + O(1),$$

where $T(r,f)$ and $\overline{N}(r,f)$ are the characteristic function and the counting function, respectively, of Nevanlinna theory.

Second fundamental theorem. *Let $f(z)$ be a non-constant meromorphic in the complex plane $\mathbb{C}$. If $a_k \in \mathbb{C} \cup \{\infty\}$, $k = 1, \ldots, q$, $q \geq 3$, are distinct complex numbers, then*

$$(q-2)T(r,f) \leq \sum_{k=1}^{q} \overline{N}\left(r, \frac{1}{f - a_k}\right) + o(T(r,f)),$$

where $T(r,f)$ and $\overline{N}(r,f)$ are the characteristic function and the counting function (ignoring multiplicities), respectively, of Nevanlinna theory.

4. Proof of Theorem 1.1. Suppose, on the contrary, that, under conditions (1.4) and (1.5), $f \notin \mathcal{N}^{\varphi}$. Then by Lemma 3.1, there exist

(a) a compact subset $K_0 \subset \mathbb{U}^m$;

(b) points $z_j \in \mathbb{U}^m$ such that $\|z_j\| \to 1^-$;

(c) points $z_j^* \in K_0$ such that $w_j := z_j + z_j^*/\varphi(\|z_j\|) \in \mathbb{U}^m$ for $j >> 1$;

(d) positive numbers ρ_j with $\rho_j \to 0^+$;

(e) Euclidean vectors $\xi_j \in \mathbb{C}^m$ with $\|\xi_j\| = 1$

such that the sequence $g_j(\zeta) := f(w_j + (\rho_j/\varphi(\|z_j\|))\xi_j\zeta)$, where $\zeta \in \mathbb{C}$ satisfies

$$(w_j + (\rho_j/\varphi(\|z_j\|))\xi_j\zeta) \in \mathbb{U}^m,$$

converges uniformly on compact subsets of $\mathbb{C}$ to a non-constant holomorphic mapping $g\colon \mathbb{C} \to \mathbb{P}^n$. From the proof of Result 1.4 (see [4], Proof of Theorem 2.5) we have:

($\star$) For all $a \in S$ each zero of $g(\zeta) - a$ has multiplicity at least 2.

We now aim for the following *claim: For* $a \in S \setminus \{\infty\}$, *each zero of* $g(\zeta) - a$ *has multiplicity at least* 3.

Suppose that $a_0 \in S \setminus \{\infty\}$ and $\zeta_0 \in \mathbb{C}$ such that $g(\zeta_0) - a_0 = 0$. By Hurwitz's theorem, there exists a sequence $\zeta_{0\,j} \to \zeta_0$ such that $g_j(\zeta_{0\,j}) = f(w_j + (\rho_j/\varphi(\|z_j\|))\xi_j\zeta_{0\,j}) = a_0$ for all j sufficiently large. We write $w_j^0 := w_j + (\rho_j/\varphi(\|z_j\|))\xi_j\zeta_{0\,j}$. From ($\star$), we have $g'(\zeta_0) = 0$, and condition (1.5) shows that there exists a constant $M > 0$ such that, for $j \in \mathbb{N}$,

$$\frac{1}{\Big(\varphi\big(\|w_j^0\|\big)\Big)^2} E_{\mathbb{P}^1}\left(\sum_{l=1}^{m} f_{z_l}\big(w_j^0\big), d\sum_{l=1}^{m} f_{z_l}\big(w_j^0\big)(\xi_j)\right) \le M. \tag{4.1}$$

Set $\xi_j := (\xi_{1\,j}, \ldots, \xi_{m\,j})$, then, for $j \in \mathbb{N}$,

$$g_j'(\zeta) = \frac{\rho_j}{\varphi(\|z_j\|)} \sum_{l=1}^{m} \xi_{l\,j} f_{z_l}(w_j + (\rho_j/\varphi(\|z_j\|))\xi_j\zeta).$$

Note that

$$E_{\mathbb{P}^1}\big(g_j'(\zeta_{0\,j}), dg_j'(\zeta_{0\,j})(1)\big) =$$

$$= \frac{\rho_j^2}{(\varphi(\|z_j\|))^2} \sum_{l=1}^{m} |\xi_{l\,j}| E_{\mathbb{P}^1}\big(f_{z_l}\big(w_j^0\big), df_{z_l}\big(w_j^0\big)(\xi_j)\big) \le$$

$$\le \frac{\rho_j^2}{(\varphi(\|z_j\|))^2} E_{\mathbb{P}^1}\left(\sum_{l=1}^{m} f_{z_l}\big(w_j^0\big), d\sum_{l=1}^{m} f_{z_l}\big(w_j^0\big)(\xi_j)\right) \le$$

$$\le \rho_j^2 \frac{(\varphi(\|w_j^0\|))^2}{(\varphi(\|z_j\|))^2} M. \tag{4.2}$$

The last inequality follows from (4.1). Hence by (1.2) and (4.2), we get

$$E_{\mathbb{P}^1}\big(g'(\zeta_0), dg'(\zeta_0)(1)\big) = \lim_{j\to\infty} E_{\mathbb{P}^1}\big(g_j'(\zeta_{0\,j}), dg_j'(\zeta_{0\,j})(1)\big) = 0.$$

Therefore, $g''(\zeta_0) = 0$. Hence, for any $a \in S \setminus \{\infty\}$, each zero of $g(\zeta) - a$ has multiplicity at least 3. We write $S = \{a_1, a_2, a_3, a_4\}$, where a_1, a_2, a_3 are finite and a_4 is either finite or infinite. By the first and second fundamental theorem of Nevanlinna theory, we have

$$2T(r, g) \le \sum_{k=1}^{4} \overline{N}(r, 1/g - a_k) + o(T(r, g)) \le$$

$$\leq \frac{1}{3}\sum_{k=1}^{3} N(r, 1/g - a_k) + \frac{1}{2} N(r, 1/g - a_4) + o(T(r,g)) \leq$$

$$\leq \frac{3}{2} T(r,g) + o(T(r,g)),$$

for all $r \in [1, \infty)$ excluding a set of finite Lebesgue measure. This shows that g is constant, which is a contradiction.

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Received 21.07.20