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A. K. Mukhamedov[1] (Nat. Univ. Uzbekistan, Tashkent)

ON THE RATE OF CONVERGENCE IN THE INVARIANCE PRINCIPLE FOR WEAKLY DEPENDENT RANDOM VARIABLES

ПРО ШВИДКІСТЬ ЗБІЖНОСТІ В ПРИНЦИПІ ІНВАРІАНТНОСТІ ДЛЯ СЛАБКО ЗАЛЕЖНИХ ВИПАДКОВИХ ВЕЛИЧИН

We consider nonstationary sequences of φ-mixing random variables. By using the Levy – Prokhorov distance, we estimate the rate of convergence in the invariance principle for nonstationary φ-mixing random variables. The obtained results extend and generalize several known results for nonstationary φ-mixing random variables.

Розглянуто нестаціонарні послідовності φ-мішаних випадкових величин. За допомогою відстані Леві – Прохорова оцінено швидкість збіжності в принципі інваріантності для нестаціонарних φ-мішаних випадкових величин. Одержані результати розширюють та узагальнюють ряд відомих результатів про нестаціонарні φ-мішані випадкові величини.

1. Introduction. Let $\{\xi_{kn},\ k=1,2,\ldots,k(n),\ n=1,2,\ldots\}$ be a sequence of random variables (r.v.'s) on a probability space $\{\Omega,\Im,P\}$. Let $M_a^b(n)=\sigma\{\xi_{kn}, a\le k\le b\}$, $1\le a\le b\le k(n)$. For each $m\ge 1$ define (see [11])

$$\alpha(m)=\sup_{k,n}\sup_{A\in M_1^k(n),\, B\in M_{k+m}^{k(n)}(n)}|P(A\cap B)-P(A)P(B)|,$$

$$\beta(m)=E\left\{\sup_{k,n}\sup_{A\in M_{k+m}^{k(n)}(n)}\Big|P(A/M_1^k(n))-P(A)\Big|\right\},$$

$$\varphi(m)=\sup_{k,n}\sup_{A\in M_1^k(n),\, B\in M_{k+m}^{k(n)}(n)}|P(B/A)-P(B)|,\quad P(A)>0.$$

The sequence is said to be strongly mixing (s.m.), absolutely regular (a.r.), *uniformly strong mixing* (u.s.m.), if $\alpha(m)\to 0$, $\beta(m)\to 0$ and $\varphi(m)\to 0$ as $m\to\infty$, respectively.

Let

$$S_{kn}=\sum_{j\le k}\xi_{jn},\quad S_n=S_{k(n)n},\quad B_{kn}^2=ES_{kn}^2,\quad B_n^2=B_{k(n)n}^2,\quad S_{0n}=B_{0n}^2=0,$$

$$L_{ns}=B_n^{-s}\sum_{j\le k(n)}E\,|\xi_{jn}|^s,\quad E\xi_{kn}=0,\quad \varphi(0)=1.$$

By $C(\cdot)$ with an index or without it, we denote a positive constants (not always the same in the various formulas) depending only on the values in parentheses, by C an absolute positive constant.

Consider the points $t_{kn}=\dfrac{\max_{1\le i\le k}B_{in}^2}{\max_{1\le i\le k(n)}B_{in}^2}$ in the interval $[0;1]$, order them and construct on the interval $[0;1]$ continuous random polygon $W_n(t)$ with vertices $\left(t_{kn};\dfrac{S_{kn}}{B_n}\right)$. If some t_{kn} are the same, i.e.,

[1] e-mail: muhamedov1955@mail.ru.

$$B_{k_1 n}^2 = B_{k_2 n}^2 = \ldots = B_{k_r n}^2, \quad k_i \neq k_j,$$

then we take any of these points $\left(t_{k_r n}; \dfrac{S_{k_i n}}{B_n}\right)$.

Consider the space $C[0;1]$ of continuous functions on $[0;1]$ equipped with the norm $\|x(t)\| = \sup_{0 \leq t \leq 1} |x(t)|$, which generates σ-algebra $\Im_C$. If a W_n is distribution of the process $\{W_n(t), t \in [0;1]\}$ and W is distribution of the standard Winer process $\{W(t),\ t \in [0;1]\}$, then the weak convergence W_n to W means that

$$\lim_{n \to \infty} P(W_n(t) \in A) = P(W(A))$$

for any Borel set A such that $W(\partial A) = 0$. This fact is usually called the invariance principle (IP). Donsker [8] proved IP for i.i.d. random variables and Yu. V. Prokhorov [16] proved IP for the triangular arrays $\{\xi_{kn},\ k = 1, 2, \ldots, k(n),\ n = 1, 2, \ldots\}$ of independent in each series r.v.'s under Lundeberg's condition:

$$\Lambda_n(\varepsilon) = \frac{1}{B_n^2} \sum_{k=1}^{n} E\{X_{kn}^2;\ |X_{kn}| > \varepsilon B_n\} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad \varepsilon > 0.$$

Under Lundeberg's condition T. M. Zuparov, A. K. Muhamedov [26] and M. Peligrad, S. Utev [15] proved IP for a nonstationary φ-mixing and α-mixing r.v.'s, respectively.

Define $L(P;Q)$ the Levy–Prokhorov distance between the distributions P and Q in $C[0;1]$ (see [3, p. 327])

$$L(P;Q) = \inf\{\varepsilon > 0 \colon P(A) \leq Q(A^\varepsilon) + \varepsilon \ \text{and} \ Q(A) \leq P(A^\varepsilon) + \varepsilon \ \text{for all} \ A \in \Im_C\},$$

where A^ε is a ε-neighborhood of A. Then IP can be written as $L(W_n; W) \to 0$ as $n \to \infty$.

It is known that

$$L(W_n; W) = \max\{\varepsilon \colon P(\|W_n(\cdot) - W(\cdot)\| > \varepsilon)\}. \tag{1}$$

In order to estimate (1) it is enough to estimate $P(\|W_n(\cdot) - W(\cdot)\| > \varepsilon)$. A rate of convergence in the IP was studied in detail when the sequence of r.v.'s are independent. The first estimation in this case was proposed by Yu. V. Prokhorov [16]. He proved that

$$L(W_n; W) = o\Big(L_{n3}^{1/4} \ln^2 L_{n3}\Big), \quad n \to \infty.$$

This latter estimate was improved in i.i.d. case by Heyde [10], Dudley [7], and others. A. A. Borovkov [4] proved that

$$L(W_n; W) = C(s) L_{ns}^{1/(s+1)}, \quad 2 < s \leq 3. \tag{2}$$

It should be noted that in all the above estimates the one probability space method was used. R. M. Dudley [7] and A. A. Borovkov [4] showed that neither method of Prokhorov nor method of Skorokhod can be used to get (2) in the case $s > 5$. J. Komlos, P. Major, G. Tusnady (KMT) [13] proposed method which allowed them in i.i.d. case to prove (1) for all $s > 2$. Modifying the method of KMT, A. I. Sakhanenko [17–21] extends (2) to the general case.

The fact that (2) is the best possible was proved by several authors A. A. Borovkov [4], A. I. Sakhanenko [17–21], T. V. Arak [1], J. Komlos, P. Major and G. Tusnady [14]. I. Berkes, W. Philipp [2], and A. A. Borovkov, A. I. Sakhanenko [5], T. M. Zuparov, A. K. Muhamedov [26, 27] proposed the methods to obtain estimates of Levy–Prokhorov distances for different classes of weakly dependent sequence.

Yoshihara [25] obtained the first result:

$$L(W_n;W)=O\Big(n^{-1/8}\ln^{1/2}n\Big)$$

for a.r. strictly stationary sequence $\{\xi_k,\ k\in N\}$ satisfying

$$\sum_{k=1}^{\infty}k\cdot(\beta(k))^{\delta/(4+\delta)}<\infty,$$

under the existence of an absolute moment of order $4+\delta,\ \delta>0$. Kanagawa [12] obtained the rate of convergence for the u.s.m. and s.m. strictly stationary sequences of r.v.'s.

Using the Prokhorov method, the best estimate in IP is obtained [9] in the stationary case with s.m. conditions, namely,

1) if the coefficients $\alpha(k)$ of s.m. decreases exponentially to zero and

$$0<\sigma=E\xi_1^2+2\sum_{i=2}^{\infty}E\xi_1\xi_i<\infty, \tag{3}$$

then

$$L(W_n;W)=O\Big(n^{-\frac{s-2}{2(s-1)}}\ln^{\frac{2s+1}{6}}n\Big);$$

2) if the coefficients $\alpha(k)$ of s.m. decreases to zero as following:

$$\alpha(k)\le Cn^{-\theta s(s-1)/(s-2)^2},\quad C>0,\quad \theta>1,$$

and condition (3) holds, then

$$L(W_n;W)=O\Big(n^{-\frac{(s-2)(\theta-1)}{6(\theta+1)+2(\theta-1)(s-2)}}\sqrt{\ln n}\Big).$$

For the case u.s.m. S. A. Utev [23] for weak stationary sequences $\{\xi_k,\ k\in N\}$ showed that

$$L(W_n;W)=C(s;g;\sigma)\left(n^{-s/2}\sum_{i=1}^{n}E|\xi_i|^s\right)^{1/(s+1)},\quad 2<s<5,$$

under the conditions (3) and $\phi(k)\le A\cdot k^{-g(s)},\ g(s)>j(u)(j(u)-1),\ u=(2+5s)/2(5-s),$ $j(u)=2\min\{k\in N:2k\ge u\}$.

T. M. Zuparov and A. K. Muhamedov [27] announced the estimate for nonstatsionary u.s.m. sequence

$$L(W_n;W)\le C(s;\theta;K)L_{ns}^{\frac{1}{s+1}}$$

under $2<s<6$ and $\phi(k)\le Ak^{-\theta(s)}$, here $\theta(s)>2s$.

In this paper, using Levy – Prokhorov distance, Bernshtein's method, I. Berkes, W. Philipp [2] approximation theorem's, S. A. Utev's [24] moment inequalities and results of A. I. Sakhanenko [19], we will obtain the best possible rate of convergence in the IP, extend and generalize several known results on a nonstationary φ-mixing random variables.

This paper is organized as follows. The main results will be given in Section 2. In Section 3, we will give auxiliary lemmas, and in Section 4, we will prove the results.

2. Main results.

Theorem 2.1. *Suppose that for any numbers θ and s such that*

$$\theta > \max(4, s, s(s-2)/4), \quad s > 2,$$

the following conditions hold:

$$\varphi(\tau) \leq K\tau^{-\theta}, \quad K > 0,$$

$$E\,|\xi_{kn}|^s < \infty, \quad k = 1, 2, \ldots, k(n), \quad n = 1, 2, \ldots.$$

Then there exist a Wiener process $\{W(t),\ t \in [0;1]\}$ and a constant $C(s;\theta;K)$ such that inequality

$$P(\|W_n(t) - W(t)\| > x) \leq C(s;\theta;K)\frac{L_{ns}}{x^s}$$

holds for all $x > 0$.

Corollary. *Under the conditions of Theorem* 2.1 *the following inequality takes place*:

$$L(W_n; W) \leq C(s;\theta;K)L_{ns}^{\frac{1}{s+1}}.$$

Theorem 2.2. *Under the conditions of Theorem* 2.1 *and $\theta > \max(4, s, 3s(s-2)/4)$ there exist a Wiener process $\{W(t),\ t \in [0;1]\}$ and a constant $C(s;\theta;K)$ such that inequality*

$$E\|W_n(t) - W(t)\|^s \leq C(s;\theta;K)L_{ns}$$

holds.

Remark. S. A. Utev [24] proved convergence of $E\|W_n(t) - W(t)\|^s$ to zero. The inequality in Theorem 2.2 for nonstationary sequence of φ-mixing random variables is obtained the first time.

Concerning the existence of the sequences which satisfy the conditions of Theorems 2.1 and 2.2, we can say the following:

R. C. Bradley [6] proved in the Theorem 3.3 that if $X := (X_k,\ k \in Z)$ is a (not necessarily stationary) Markov chain and $\varphi(n) < 1/2$ for some $n \geq 1$, then $\varphi(n) \to 0$ at least exponentially fast as $n \to \infty$.

From $X := (X_k,\ k \in Z)$ strictly stationary sequence of Markov chain we constructed nonstationary sequence $\xi := (\xi_{kn}, 1 \leq k \leq n)$ following: $\xi_{2k-1n} = -X_{2k-1},\ 1 \leq 2k-1 \leq n$, and $\xi_{2kn} = X_{2k},\ 1 \leq 2k \leq n$, for every series. As $X := (X_k,\ k \in Z)$ strictly stationary sequence are satisfying φ-mixing condition with exponentially fast as $n \to \infty$, then $\xi := (\xi_{kn},\ 1 \leq k \leq n)$ sequence are also satisfying φ-mixing condition with exponentially fast as $n \to \infty$. In addition, if $E\,|X_k|^s,\ s > 2$, then $\xi := (\xi_{kn},\ 1 \leq k \leq n)$ nonstationary sequence satisfies the conditions of the main theorems.

3. Auxiliary lemmas.

Lemma 3.1 (see [11]). *Let r.v.'s ξ and η be measurable with respect to σ-algebras M_1^k and $M_{k+\tau}^{k(n)}$, respectively, where $k \geq 1,\ k+\tau \leq k(n)$. If $E|\xi|^p < \infty$ and $E|\eta|^q < \infty$ for $p > 1,\ q > 1$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then*

$$|E\xi \cdot \eta - E\xi \cdot E\eta| \leq 2\varphi^{\frac{1}{p}}(\tau) E^{\frac{1}{p}} |\xi|^p E^{\frac{1}{q}} |\eta|^q.$$

Lemma 3.2 (see [2]). *Let $\{(S_k, \sigma_k),\ k \geq 1\}$ be a sequence of complete separable metric spaces. Let $\{X_k,\ k \geq 1\}$ be a sequence of random variables with values in S_k and let $\{B_k,\ k \geq 1\}$ be a sequence of σ-fields such that X_k is B_k-measurable. Suppose that, for some $\varphi_k \geq 0$,*

$$|P(AB) - P(A)P(B)| \leq \varphi_k P(A)$$

for all $B \in B_k,\ A \in \bigcup_{j<k} B_j$. Then without changing its distribution we can redefine the sequence $\{X_k,\ k \geq 1\}$ on a richer probability space together with a sequence $\{Y_k,\ k \geq 1\}$ of independent random variables such that Y_k has the same distribution as X_k and

$$P(\sigma_k(X_k, Y_k) \geq 6\varphi_k) \leq 6\varphi_k, \quad k = 1, 2, \ldots .$$

Lemma 3.3 (see [24]). *Let $\{X_k,\ k \geq 1\}$ the sequence of random variables satisfying u.s.m. condition and $\varphi(p) < \dfrac{1}{4}$. Then there exists a constant $C(\varphi(p))$, depending only on $\varphi(p)$, such that for all $t \geq 1$ and all $1 \leq q \leq t$ the following inequality takes place*:

$$E \max_{1\leq k\leq n} \left| \sum_{j=1}^{k} X_j \right|^t \leq (C(\varphi(p)t))^t \left\{ p^t E \max_{1\leq k\leq n} |X_k|^t + \max_{1\leq k\leq n} \left(E \left| \sum_{j=1}^{k} X_j \right|^q \right)^{\frac{t}{q}} \right\}.$$

Lemma 3.4 (see [19]). *Let $\{X_k,\ k \geq 1\}$ be a sequence of independent random variables such that $EX_k = 0,\ \sum_{k=1}^{n} EX_k^2 = 1$. Suppose that $t_0 = 0,\ t_k = \sum_{i=1}^{k} EX_i^2,\ k = 1, 2, \ldots, n,\ L_{ns} = \sum_{i=1}^{n} E|X|^s$. Let $S(t)$ be continuous random polygon with vertices $\left(t_k, S(t_k) = \sum_{j=1}^{k} X_j\right)$. Then, for any numbers $s \geq 2$ and $b \geq 1$, there exists a Wiener process $\{W(t),\ t \in [0,1]\}$ such that inequality*

$$P(\|S(t) - W(t)\| \geq C_1 sbx) \leq \left(\frac{L_{ns}}{bx}\right)^b + P\left(\max_{1\leq i\leq n} |X_i| > x\right)$$

is true for all $x > 0$.

We introduce the following notation:

$$\xi_{jn}(x) = \xi_{jn} I\{|\xi_{jn}| \leq CxB_n\} - E\xi_{jn} I\{|\xi_{jn}| \leq CxB_n\}, \quad \bar{\xi}_{jn}(x) = \xi_{jn} - \xi_{jn}(x),$$

where $x > 0$ an arbitrary real number,

$$S_{kn}(b) = \sum_{j=b+1}^{b+k} \xi_{jn}, \quad S_{kn}(b, x) = \sum_{j=b+1}^{b+k} \xi_{jn}(x), \quad \bar{S}_{kn}(b, x) = \sum_{j=b+1}^{b+k} \bar{\xi}_{jn}(x),$$

$$S_n(x) = S_{k(n)n}(0,x),$$

$$B^2_{kn}(b) = ES^2_{kn}(b), \quad B^2_{kn}(b,x) = ES^2_{kn}(b,x), \quad B^2_n(x) = ES^2_n(x), \quad \varphi_t = \sum_{i=0}^{k(n)+1} \varphi^{1/t}(i),$$

$$L_{ns} = B_n^{-s} \sum_{j \le k(n)} E|\xi_{jn}|^s, \quad L_{nsx}(a,b) = B_n^{-s} \sum_{j=a+1}^{b} E|\xi_{jn}(x)|^s, \quad s > 2,$$

$$\bar{\varphi}_t = \sum_{i=0}^{k(n)+1} (i+1)\varphi^{1/t}(i).$$

We define the positive integers m_i using the algorithm

$$m_0 = 0, \quad m_{i+1} = \min\left\{ m : m_i < m < n : E\left(\sum_{k=m_i+1}^{m+1} \xi_{kn}(x)\right)^2 > h(n)\right\}$$

for $i = 1, 2, \ldots, M-1$, where $M-1$ is the last, for which we can define m_{M-1}, i.e.,

$$E\left(\sum_{j=m_{M-1}+1}^{k(n)} \xi_{jn}(x)\right)^2 < h(n),$$

where $h(n)$ is a sequence of positive numbers.

By η_j and $\eta_j(x)$, respectively, we denote the amount

$$\eta_j = \sum_{i=m_{j-1}+1}^{m_j} \xi_{in}, \quad \eta_j(x) = \sum_{i=m_{j-1}+1}^{m_j} \xi_{in}(x).$$

We describe the positive integers l_i using the mentioned algorithm

$$l_0 = 0, \quad l_{i+1} = \min\left\{ l : l_i < l < M : E\left(\sum_{k=l_i+1}^{l+1} \eta_k(x)\right)^2 > T(n)\right\}$$

for $i = 1, 2, \ldots, N-1$, where $M-1$ is the last, for which we can define l_{N-1}, i.e.,

$$E\left(\sum_{j=l_{N-1}+1}^{M} \eta_j(x)\right)^2 < T(n),$$

where $T(n)$ is a sequence of positive numbers. $T(n)$ and $h(n)$ will be selected later.

By ψ_j and $\psi_j(x)$, respectively, we denote the amount

$$\psi_j = \sum_{i=l_{j-1}+1}^{l_j-1} \eta_i, \quad \psi_j(x) = \sum_{i=l_{j-1}+1}^{l_j-1} \eta_i(x).$$

Lemma 3.5. *The following inequalities are true*:

$$\left|B_{kn}^2(b) - B_{kn}^2(b,x)\right| \le C(\varphi_s) B_n^2 x^{2-s} L_{ns}(b), \tag{4}$$

$$\max_{1\le k\le N}\left|\sum_{j=1}^{k}(D\psi_j - D\psi_j(x))\right| \le C(\varphi_s) B_n^2 x^{2-s} L_{ns}, \tag{5}$$

$$\max_{1\le k\le N}\left|B_{m_k}^2 - \sum_{j=1}^{k} D\psi_j(x)\right| \le C(\varphi_2) N\cdot h(n), \tag{6}$$

$$E\psi_j^2(x) \le T(n) + \theta\cdot h(n), \quad |\theta| \le C(\varphi_2), \tag{7}$$

$$M \le C(\bar\varphi_2)\frac{B_n^2(x)}{h(n)}, \quad N \le C(\bar\varphi_2)\frac{B_n^2(x)}{T(n)}. \tag{8}$$

Proof. It is obvious that

$$\left|B_{kn}^2(b) - B_{kn}^2(b,x)\right| = \left|E\left(\sum_{j=b+1}^{b+k}\left(\xi_{jn}(x) + \bar\xi_{jn}(x)\right)\right)^2 - E\left(\sum_{j=b+1}^{b+k}\xi_{jn}(x)\right)^2\right| \le$$

$$\le \left|\sum_{b+1\le i\ne j\le b+k} E\xi_{in}(x)\bar\xi_{jn}(x)\right| + \left|\sum_{b+1\le i\ne j\le b+k} E\bar\xi_{in}(x)\xi_{jn}(x)\right| +$$

$$+\left|\sum_{b+1\le i\ne j\le b+k} E\bar\xi_{in}(x)\bar\xi_{jn}(x)\right|.$$

We estimate first term on the right-hand side of the inequality. Another term will be estimated analogously. Due to Lemma 3.1 and the Hölder inequality, we have

$$\left|\sum_{b+1\le i\ne j\le b+k} E\xi_{in}(x)\bar\xi_{jn}(x)\right| \le$$

$$\le \sum_{b+1\le i\ne j\le b+k} \varphi^{1/s}(|j-i|) E^{1/s}|\xi_{in}(x)|^s E^{(s-1)/s}\left|\bar\xi_{jn}(x)\right|^{s(s-1)} \le$$

$$\le C\left(\sum_{i=0}^{k(n)}\varphi^{1/s}(i)\right) B_n^2 x^{2-s} L_{ks}(b) \le C(\varphi_s) B_n^2 x^{2-s} L_{ks}(b).$$

Inequality (4) is proved. Inequality (5) can be estimated analogously.

Now, we prove inequality (6). For this, the difference $\left|B_n^2(x) - \sum_{j=1}^N D\psi_j(x)\right|$ will be estimated when $k = N$, and other cases will be proved analogously. It is obvious that $B_n^2(x) = E\left(\sum_{j=1}^N (\psi_j(x) + \eta_{l_j}(x))\right)^2$. By Lemma 3.1, we have

$$\left|B_n^2(x) - \sum_{j=1}^N E\psi_j^2(x)\right| = \left|E\left(\sum_{j=1}^N (\psi_j(x) + \eta_{l_j}(x))\right)^2 - \sum_{j=1}^N E\psi_j^2(x)\right| \leq$$

$$\leq \left|2 \sum_{1\leq j\leq l\leq N} E(\psi_j(x) + \eta_{l_j}(x))(\psi_k(x) + \eta_{l_k}(x))\right| \leq$$

$$\leq 2\left|\sum_{j=1}^N E(\psi_j(x) + \eta_{l_j}(x))\left(\sum_{l=j+1}^N E(\psi_k(x) + \eta_{l_k}(x))\right)\right| \leq$$

$$\leq 2\left|\sum_{j=1}^N E\left(\sum_{i=1}^{l_j} \eta_i(x)\right)\left(\sum_{i=l_j+1}^{l_M} \eta_i(x)\right)\right| \leq$$

$$\leq 2\sum_{i=1}^{k(n)} (i+1)\varphi^{1/2}(i) N \cdot h(n) \leq C(\bar{\varphi}_2) N \cdot h(n).$$

Proof of inequality (7). By the definitions of random variables $\psi_j(x)$ and $\eta_{ij}(x)$, we obtain $E\eta_{m_j+1n}^2(x) \leq h(n)$ and

$$E\psi_j^2(x) \leq T(n) < E\big(\psi_j(x) + \eta_{l_j}(x)\big)^2 \leq E\psi_j^2(x) + 2E\psi_j(x)\eta_{l_j}(x) +$$

$$+ E\eta_{l_j}^2(x) \leq T(n) + 2E\left(\sum_{i=l_{j-1}+1}^{l_j} \eta_i(x)\right)\eta_{l_j}(x) + E\eta_{l_j}^2(x) \leq$$

$$\leq T(n) + 2\sum_{i=1}^N \varphi^{1/2}(i) E^{1/2}\eta_{l_i}^2(x) E^{1/2}\eta_{l_j+1}^2(x) + E\eta_{l_j+1}^2(x) \leq$$

$$\leq T(n) + C(\varphi_2) h(n).$$

Relations (4) and (5) imply that

$$B_n^2(x) \geq \sum_{i=1}^N D\psi_j(x) - C(\varphi_2) N \cdot h(n) \geq \sum_{i=1}^{N-1} D\psi_j(x) - C(\varphi_2) N \cdot h(n) \geq$$

$$\geq (N-1)\cdot T(n) - C(\varphi_2) N \cdot h(n).$$

Hence, we obtained second inequality (8). Since $h(n) = o(T(n))$, first inequality (8) estimated analogously this. Consequently, Lemma 3.5 is proved.

4. Proofs of theorems. ***Proof of Theorem* 2.1.** Denote by $W_{nx}(t)$, the random polygon with vertices $\left(t_{kn}; \dfrac{S_k(x)}{B_n}\right)$. Polygon with vertices $\left(t_{m_k n}; \dfrac{S_{m_k}(x)}{B_n}\right)$ denoted by $\overline{W}_{nx}(t)$. Denote by $\overline{\overline{W}}$ and $\widehat{W}_{nx}(t)$ the random polygons with vertices $\left(t_{m_k n}; \dfrac{\sum_{j=1}^{k} \psi_j(x)}{B_n}\right)$ and $\left(t_{m_k n}; \dfrac{\sum_{j=1}^{k} \widehat{\psi}_j(x)}{B_n}\right)$, respectively, where $\widehat{\psi}_j(x),\ j = 1, 2, \ldots, N$, are independent r.v.'s marginal distributions of which coincide with the distributions of r.v.'s $\psi_j(x)$. Polygon with vertices

$$\left(\frac{\sqrt{\sum_{j=1}^{k} D\psi_j(x)}}{\sqrt{\sum_{j=1}^{N} D\psi_j(x)}}; \frac{\sum_{j=1}^{k} \widehat{\psi}_j(x)}{\sqrt{\sum_{j=1}^{N} D\psi_j(x)}}\right)$$

denoted by $\widetilde{W}_{nx}(t)$.

It is obvious that

$$P(\|W_n(t) - W(t)\| > x) \le P\Big(\|W_n(t) - W_{nx}(t)\| > \frac{x}{6}\Big) +$$

$$+P\Big(\big\|W_{nx}(t) - \overline{W}_{nx}(t)\big\| > \frac{x}{6}\Big) + P\Big(\Big\|\overline{W}_{nx}(t) - \overline{\overline{W}}_{nx}(t)\Big\| > \frac{x}{6}\Big) +$$

$$+P\Big(\Big\|\overline{\overline{W}}_{nx}(t) - \widehat{W}_{nx}(t)\Big\| > \frac{x}{6}\Big) + P\Big(\Big\|\widehat{W}_{nx}(t) - \widetilde{W}_{nx}(t)\Big\| > \frac{x}{6}\Big) +$$

$$+P\Big(\Big\|\widetilde{W}_{nx}(t) - W(t)\Big\| > \frac{x}{6}\Big) = \sum_{i=1}^{6} P_i. \tag{9}$$

Now to prove Theorem 2.1, we estimate each terms on the right-hand side of (9). Without loss of generality, we assume that $L_{ns} < 1$. Let $T(n) = C(s, \theta, K) B_n^2 x^{\frac{2(t-s)}{t-2}} L_{ns}^{\frac{2}{t-2}}, \quad t > s$. Then

$$N \le C(s, \theta, K) \frac{B_n^2(x)}{T(n)} \ll C(s, \theta, K) x^{-\frac{2(t-s)}{t-2}} L_{ns}^{-\frac{2}{t-2}}.$$

Estimate P_1. It is apparent that

$$P_1 = P\Big(\|W_n(t) - W_{nx}(t)\| > \frac{x}{6}\Big) \le P\Big(\max_{k \le k(n)} |\xi_{kn}| > C_1 B_n x\Big) \le C \frac{L_{ns}}{x^s}.$$

Estimate P_2. By virtue of the Chebyshev inequality, Lemmas 3.3 and 3.5 for $q = 2,\ t > s$, we have

$$P_2 = P\Big(\big\|W_{nx}(t) - \overline{W}_{nx}(t)\big\| > \frac{x}{6}\Big) \le$$

$$\le \sum_{j \le N} P\Big(\max_{m_{j-1} \le k \le m_j} |S_{kn}(x) - S_{m_{j-1}n}(x)| > C \frac{x B_n}{12}\Big) \le$$

$$\le C \frac{1}{x^t B_n^t} \sum_{j \le N} E \max_{m_{j-1} \le k \le m_j} |S_{kn}(x) - S_{m_{j-1}n}(x)|^t \le$$

$$\leq C(t,\theta,K)\left[\frac{L_{nt}(x)}{x^t}+\frac{1}{x^t}\left(\frac{T(n)}{B_n^2}\right)^{\frac{t-2}{2}}\right]\leq C(s,\theta,K)\frac{L_{ns}}{x^s}.$$

Estimate P_3. It is obvious that

$$P_3=P\Big(\Big\|\overline{W}_{nx}(t)-\overline{\overline{W}}_{nx}(t)\Big\|>\frac{x}{6}\Big)\leq P\left(\max_{k\leq N}\left|\sum_{j\leq k}\frac{\eta_{m_j}(x)}{B_n}\right|>\frac{x}{6}\right).$$

Now estimate the P_3, analogously P_2 we obtain

$$P_3=P\Big(\Big\|\overline{W}_{nx}(t)-\overline{\overline{W}}_{nx}(t)\Big\|>\frac{x}{6}\Big)\leq C(s,\theta,K)\frac{L_{ns}}{x^s}.$$

Estimate P_4. It is obvious that

$$P\Big(\Big\|\overline{\overline{W}}_{nx}(t)-\widehat{W}_{nx}(t)\Big\|>\frac{x}{6}\Big)\leq P\left(\max_{k\leq N}\left|\sum_{j\leq k}\left(\frac{\psi_j(x)}{B_n}-\frac{\widehat{\psi}_j(x)}{B_n}\right)\right|>\frac{x}{6}\right).$$

Using the Berkes – Philipp approximation theorem (see Lemma 3.2), Lemmas 3.3 and 3.4, we get

$$P_4\leq\sum_{j\leq N}P\left(\left|\frac{\psi_j(x)}{B_n}-\frac{\widehat{\psi}_j(x)}{B_n}\right|>\frac{x}{6N}\right)\leq$$

$$\leq\sum_{j\leq N}P\left(\left|\frac{\psi_j(x)}{B_n}-\frac{\widehat{\psi}_j(x)}{B_n}\right|>6\varphi(p)\right)\leq 6N\varphi(p)$$

when $\dfrac{x}{6N\varphi(p)}>6$ or $36N\varphi(p)\leq x$, where $p=\min_{j\leq N}(m_j-m_{j-1})$. To obtain the estimation $P_4\leq C(s,\theta,K)\dfrac{L_{ns}(x)}{x^s}$, we find p from condition

$$N\varphi(p)\leq Cx,$$

$$N\varphi(p)\leq C\frac{L_{ns}}{x^s}.$$

From this and due to Lemma 3.5, we have

$$N\varphi(p)\leq nKp^{-\theta}\leq C(s,\theta,K)x^{-\frac{2(t-s)}{t-2}}L_{ns}^{-\frac{2}{t-2}}p^{-\theta}\leq C(s,\theta,K)\min\left(x,\frac{L_{ns}}{x^s}\right).$$

Then

$$p\geq C(s,\theta,K)\left(\max\left(x^{-\frac{3t-2(s+1)}{t-2}}L_{ns}^{-\frac{2}{t-2}};x^{\frac{t(s-2)}{t-2}}L_{ns}^{-\frac{t}{t-2}}\right)\right)^{\frac{1}{\theta}}.$$

Estimate P_5. It is clear that

$$P\Big(\Big\|\widehat{W}_{nx}(t)-\widetilde{W}_{nx}(t)\Big\|>\frac{x}{5}\Big)\leq$$

$$\leq P\left(\max_{k\leq N}\left|\left(1-\frac{B_n}{\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}\right)\sum_{j\leq k}\left(\frac{\widehat{\psi}_j}{B_n}\right)\right|>\frac{x}{5}\right)\leq$$

$$\leq P\left(\max_{k\leq N}\left|\sum_{j\leq k}\left(\frac{\widehat{\psi}_j(x)}{\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}\right)\right|>\frac{xB_n\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}{5\left(B_n-\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}\right)}\right)\leq$$

$$\leq C\left|\frac{B_n-\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}{xB_n}\right|^t E\left(\max_{k\leq N}\left|\sum_{j\leq k}\left(\frac{\widehat{\psi}_j(x)}{\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}\right)\right|^t\right).$$

Hence, by Lemma 3.3, we obtain

$$E\left(\max_{k\leq N}\left|\sum_{j\leq k}\left(\frac{\widehat{\psi}_j(x)}{\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}\right)\right|^t\right)\leq C(t,\theta,K). \tag{10}$$

As

$$\frac{B_n-\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}{B_n}=\frac{B_n^2-\sum_{j\leq N} D\widehat{\psi}_j(x)}{B_n\left(B_n+\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}\right)}$$

and $D\widehat{\psi}_j(x)=D\psi_j(x)$, from Lemma 3.5 we have $\sum_{j\leq N} D\psi_j(x)=B_n^2(1+o(1))$. As a result, estimation of $B_n^2-\sum_{j\leq N} D\hat{\psi}_j(x)$ will be enough. Let $h(n)=T(n)x^{\frac{t-s}{t}}L_{ns}^{\frac{1}{t}}$, Lemma 3.5 implies that

$$\left|\frac{B_n^2-\sum_{j\leq N} D\psi_j(x)}{xB_n^2}\right|\leq C(\varphi_2)\left(\frac{Nh(n)+B_n^2x^{2-s}L_{ns}}{xB_n^2}\right)=$$

$$=C(\varphi_2)\left(\frac{h(n)}{xT(n)}+x^{1-s}L_{ns}\right)\leq C(t,\varphi_2)\left(x^{-\frac{s}{t}}L_{ns}^{\frac{1}{t}}+x^{1-s}L_{ns}\right). \tag{11}$$

It follows that

$$P_5=P\left(\left\|\widehat{W}_{nx}(t)-\widetilde{W}_{nx}(t)\right\|>\frac{x}{6}\right)\leq C(t,\varphi_2)\left(x^{-\frac{s}{t}}L_{ns}^{\frac{1}{t}}+x^{1-s}L_{ns}\right)^t\leq$$

$$\leq C(t,\varphi_2)\left(\frac{L_{ns}}{x^s}+\left(x\frac{L_{ns}}{x^s}\right)^t\right). \tag{12}$$

It is obvious that if $0 < x \leq 1$, then $P_5 \leq C(t, \varphi_2)\dfrac{L_{ns}}{x^s}$. Now let $x \geq 1$, then to obtain estimation of $P_5 \leq C(t, \varphi_2)\dfrac{L_{ns}}{x^s}$, second term of inequality (12) should satisfy the condition $\left(x\dfrac{L_{ns}}{x^s}\right)^t \leq \dfrac{L_{ns}}{x^s}$. This inequality holds, if $x \geq L_{ns}^{\frac{t-1}{s(t-1)-t}}$. Hence, inequality $P_4 \leq C(t, \varphi_2)\dfrac{L_{ns}}{x^s}$ holds for all $x > 0$.

Estimate P_6. Using Lemma 3.4, we have

$$P_6 = P\left(\left\|\widetilde{W}_{nx}(t) - W(t)\right\| > \frac{x}{6}\right) \leq C\left(\frac{1}{x}\right)^t \left(\sum_{j \leq N} E\left|\frac{\widehat{\psi}_j(x)}{\sqrt{\sum_{j \leq N} D\widehat{\psi}_j(x)}}\right|^t\right).$$

Now we estimate $\sum_{j \leq N} E\left|\widehat{\psi}_j(x)\right|^t$. Since $\widehat{\psi}_j(x)$ are independent r.v.'s marginal distributions of which coincide with the distributions of r.v.'s $\psi_j(x)$, by Lemmas 3.3 and 3.5, we find

$$\sum_{j \leq N} E\left|\psi_j(x)\right|^t \leq \sum_{j \leq N}\left(\sum_{i=l_{j-1}}^{l_j} E|\eta_i(x)|^t + (D\psi_j(x))^{t/2}\right) \leq$$

$$\leq C(t)\left(\sum_{i=1}^{k(n)} E|\xi_{in}(x)|^t + N(T(n))^{t/2}\right). \tag{13}$$

Hence, from Lemma 3.5 and the definition of $T(n)$, we get

$$P_6 = P\left(\left\|\widetilde{W}_{nx}(t) - W(t)\right\| > \frac{x}{5}\right) \leq C(t, \varphi)\left(\frac{1}{x^t}L_{nt} + \frac{1}{x^t}\left(\frac{T(n)}{B_n^2}\right)^{\frac{t-2}{2}}\right) \leq C(t, \varphi)\frac{L_{ns}}{x^s}. \tag{14}$$

We will demonstrate the possibility of dividing above mentioned isolated groups, namely, when $n \to \infty$, the conditions B_n^2, $T(n)$, $h(n) \to \infty$, $T(n) = o(B_n^2)$, $h(n) = o(T(n))$, $L_{ns} \to 0$ should be satisfied and we will explain the necessity of curtailing in order to prove Theorem 2.1. The conditions are clear in the stationary case. In this case, the following asymptotical relations will be valid, i.e., $L_{ns} \approx n^{-\frac{s-2}{2}}$ for $s > 2$, $T(n) \approx n^{\frac{t-s}{t-2}}$ for some t, $t > s$, and $h(n) \approx n^{\frac{2t^2-(3s-2)t+2s-4}{2t(t-2)}}$ for some t, $t > t_0 = \dfrac{3s - 2 + \sqrt{9s^2 - 28s + 36}}{4} > s$, $p \gg n^{\frac{t(s-2)}{2\theta(t-2)}}$, $N \ll n^{\frac{t(s-2)}{s(t-2)}}$ and $\theta > \max\left(4, s, \dfrac{s(s-2)}{4}\right)$.

To obtain necessary estimation of P_2 and P_4, it will be demanded the availability of a moment of t which is bigger than s. That is why, curtailing is necessary.

Theorem 2.1 is proved.

As it was mentioned above, Levy – Prokhorov distance between the distributions W_n and W were determined in (1). Through selecting $\varepsilon = x = L_{ns}^{\frac{1}{s+1}}$ in relation (1) and Theorem 2.1, respectively, a proof of corollary can be obtained.

***Proof of Theorem* 2.2.** The method of the proof of Theorem 2.2 remains the same as of Theorem 2.1. Here we only list those places in which we make the appropriate changes.

As in the proof of Theorem 2.1, the following inequality is valid:

$$E\|W_n(t)-W(t)\|^s \le E\|W_n(t)-W_{nx}(t)\|^s + E\left\|W_{nx}(t)-\overline{W}_{nx}(t)\right\|^s +$$

$$+E\left\|\overline{W}_{nx}(t)-\overline{\overline{W}}_{nx}(t)\right\|^s + E\left\|\overline{\overline{W}}_{nx}(t)-\widehat{W}_{nx}(t)\right\|^s +$$

$$+E\left\|\widehat{W}_{nx}(t)-\widetilde{W}_{nx}(t)\right\|^s + E\left\|\widetilde{W}_{nx}(t)-W(t)\right\|^s = \sum_{i=1}^{6} E_i. \qquad (15)$$

Now, to prove Theorem 2.2, we estimate each term on the right-hand side of (15) and we take $x = L_{ns}^{1/s}$. Then we have

$$T(n) = B_n^2 L_{ns}^{\frac{2t}{s(t-2)}}, \quad h(n) = T(n)L_{ns}^{1/s} = B_n^2 L_{ns}^{\frac{3t-2}{s(t-2)}}, \quad N = \frac{B_n^2}{T(n)} = L_{ns}^{-\frac{2t}{s(t-2)}}, \quad \frac{h(n)}{T(n)} = L_{ns}^{\frac{1}{s}}.$$

Estimate E_1. It is obvious that

$$E_1 = E\|W_n(t)-W_{nx}(t)\|^s \le E\left(\max_{k\le k(n)} |\xi_{kn}|^s / B_n^s\right) \le L_{ns}.$$

Estimate E_2. Based on moment inequality, Lemmas 3.3 (for $q=2,\ t>s$) and 3.5, the definition of $T(n)$, the following inequality takes place:

$$E_2 = E\left\|W_{nx}(t)-\overline{W}_{nx}(t)\right\|^s \le E^{s/t}\left\|W_{nx}(t)-\overline{W}_{nx}(t)\right\|^t \le$$

$$\le \left(\sum_{j\le N} E\left(\max_{m_{j-1}\le k\le m_j} \left|S_{kn}(x)-S_{m_{j-1}n}(x)\right|^t / B_n^t\right)\right)^{s/t} \le$$

$$\le C\left(\sum_{j\le N} E\left(\frac{1}{B_n^t}\max_{m_{j-1}\le k\le m_j} \left|S_{kn}(x)-S_{m_{j-1}n}(x)\right|^t\right)\right)^{s/t} \le$$

$$\le C(t,\theta,K)\left(L_{nt}(x) + \left(\frac{T(n)}{B_n^2}\right)^{\frac{t-2}{2}}\right)^{s/t} \le C(s,\theta,K)L_{ns}. \qquad (16)$$

Estimate E_3. It is obvious that

$$E_3 = E\left\|\overline{W}_{nx}(t)-\overline{\overline{W}}_{nx}(t)\right\|^s \le E\max_{k\le N}\left|\sum_{j\le k}\frac{\eta_{m_j}(x)}{B_n}\right|^s.$$

Now estimate the E_3, analogously E_2 we obtain

$$E_3 = E\left\|\overline{W}_{nx}(t)-\overline{\overline{W}}_{nx}(t)\right\|^s \le C(s,\theta,K)L_{ns}.$$

Estimate E_4. By Lemmas 3.2, 3.3, and 3.5 and replicating a paper [24], E_4 can be estimated as follows:

$$E_4 \le E\left(\max_{k\le N}\left|\sum_{j\le k}\left(\frac{\psi_j(x)}{B_n}-\frac{\widehat{\psi}_j(x)}{B_n}\right)\right|^s\right) \le N^s \max_{j\le N} E\left|\frac{\psi_j(x)}{B_n}-\frac{\widehat{\psi}_j(x)}{B_n}\right|^s \le$$

$$\leq N^s\left((6\varphi(p))^s + \max_{j\leq N}\left(E\left|\frac{\psi_j(x)}{B_n} - \frac{\widehat{\psi}_j(x)}{B_n}\right|^s,\ 6\varphi(p) < \left|\frac{\psi_j(x)}{B_n} - \frac{\widehat{\psi}_j(x)}{B_n}\right| \leq 1\right)\right) +$$

$$+ N^s \max_{j\leq N} E\left|\frac{\psi_j(x)}{B_n} - \frac{\widehat{\psi}_j(x)}{B_n}\right|^t \leq$$

$$\leq CN^s\left(\varphi^s(p) + \max_{j\leq N} P\left(\left|\frac{\psi_j(x)}{B_n} - \frac{\widehat{\psi}_j(x)}{B_n}\right| \geq 6\varphi(p)\right) + \left(\frac{T(n)}{B_n^2}\right)^{t/2}\right) \leq L_{ns}.$$

In this case, mixing coefficients decreases be as $N^s\varphi(p) \leq L_{ns}$. In its turn, $N^s\varphi(p) \leq L_{ns}^{-\frac{2t}{t-2}} p^{-\theta} \leq L_{ns} \Rightarrow p \geq L_{ns}^{-\frac{3t-2}{\theta(t-2)}}$ for $\theta > \max\left(4, s, \dfrac{3s(s-2)}{4}\right)$.

Estimate $\boldsymbol{E_5}$**.** It is obvious that

$$E\left\|\widehat{W}_{nx}(t) - \widetilde{W}_{nx}(t)\right\|^s \leq$$

$$\leq C\left(\frac{B_n - \sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}{B_n}\right)^s E\left(\max_{k\leq N}\left|\sum_{j\leq k}\left(\frac{\widehat{\psi}_j(x)}{\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}\right)\right|^s\right).$$

By Lemma 3.5 and inequalities (10), (11), we get

$$E\left\|\widehat{W}_{nx}(t) - \widetilde{W}_{nx}(t)\right\|^s \leq C(s,\varphi_2)\left|\frac{B_n - \sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}{B_n}\right|^s \leq \left(\frac{h(n)}{T(n)} + x^{2-s}L_{ns}\right)^s \leq L_{ns}.$$

Estimate $\boldsymbol{E_6}$**.** Due to moment inequality and analogous estimates for (13), (14) and (16), by Lemmas 3.3 and 3.4, we have

$$E\left\|\widetilde{W}_{nx}(t) - W(t)\right\|^s \leq E^{s/t}\left\|\widetilde{W}_{nx}(t) - W(t)\right\|^t \leq \left(\sum_{j\leq N} E\left|\frac{\widehat{\psi}_j(x)}{\sqrt{\sum_{j\leq N} D\widehat{\psi}_j(x)}}\right|^t\right)^{s/t} \leq$$

$$\leq C(t,K,\theta)\left(L_{nt}(x) + \left(\frac{T(n)}{B_n^2}\right)^{\frac{t-2}{2}}\right)^{s/t} \leq C(t,K,\theta)L_{ns}.$$

Theorem 2.2 is proved.

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