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EXISTENCE AND COMPACTNESS OF SOLUTION OF SEMILINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH FINITE DELAY

ІСНУВАННЯ ТА КОМПАКТНІСТЬ РОЗВ'ЯЗКУ НАПІВЛІНІЙНИХ ІНТЕГРО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ЗІ СКІНЧЕННИМ ЗАПІЗНЕННЯМ

We present some existence and uniqueness results for a class of functional integro-differential evolution equations generated by the resolvent operator for which the semigroup is not necessarily compact. It is proved that the set of solutions is compact. Our approach is based on fixed point theory. Finally, some examples are given to illustrate the results.

Наведено деякі результати щодо існування та єдиності розв'язків деякого класу функціональних інтегро-диференціальних еволюційних рівнянь, породжених резольвентним оператором, де напівгрупа необов'язково компактна. Доведено компактність множини розв'язків. Наш підхід ґрунтується на теорії нерухомих точок. Крім того, наведено кілька прикладів, що ілюструють отримані результати.

1. Introduction. Nonlinear evolution equations appeared in many fields of applied mathematics, and also in other branches of science as material science, biological sciences, physics and mechanics. For example, the nonlinear reaction-diffusion equations from heat transfers, Cahn – Hilliard equations from material science, the nonlinear Klein – Gordon equations and nonlinear Schrödinger equations from quantum mechanics and Navier – Stokes and Euler equations from fluid mechanics. See the books [1 – 3, 10, 11].

The study of the existence of mild solutions for integro-differential equations and inclusions in abstract spaces has been done by several authors, see the works [4 – 9, 13 – 15].

In this paper, we consider the following semilinear integro-differential problem:

$$u'(t) = Au(t) + \int_0^t U(t-s)u(s)ds + F\left(t, u_t, \int_0^t \rho(t,s,u_s)ds\right) \quad \text{a.e.} \quad t \in \mathbb{R}_+, \tag{1.1}$$

$$u(t) = \phi(t), \quad t \in J_0,$$

where $J_0 = [-r, 0]$, the operator A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on a real Banach space $(E, \|\cdot\|)$ with domain $D(A)$, $F: \mathbb{R}_+ \times \mathcal{C}(J_0, E) \times E \to E$, is a given function, $\rho : \Delta \times \mathcal{C}(J_0, E) \to E$ is a continuous function, with $\Delta = \{(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+;\ s \geq t\}$, and $\phi \in \mathcal{C}(J_0, E)$. For any $t \in \mathbb{R}^+$, $U(t)$ is a closed linear operator on E, with domain $D(A) \subset D(U(t))$, which is independent of t.

For any function u defined on $J = J_0 \cup \mathbb{R}_+$ and any $t \in \mathbb{R}_+$, we denote by u_t the element of $\mathcal{C}(J_0, E)$ defined by

$$u_t(\theta) = u(t+\theta), \quad \theta \in J_0,$$

where $u_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

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Here we are interested to study the existence of the mild solutions for the above partial functional integro-differential evolution equations where the semigroup is not necessarily compact and we prove the compactness of the set of the solutions.

Using methods and theorems of functional analysis, we propose a set of sufficient conditions to ensure the existence of solutions. Specifically, to establish our main results we have used the theory of the resolvent operator in the sense of Grimmer and Kuratowski's measure of noncompactness.

2. Preliminaries. In this section, we introduce some notations, definitions, and preliminary facts which will be used throughout,

$$BC(J,E)=\left\{u\in\mathcal{C}(J,E)\Big/\sup_{t\in J}\|u(t)\|<\infty\right\}$$

is a Banach space with the following norm:

$$\|u\|_\infty=\sup_{t\in J}\|u(t)\|\quad\forall u\in BC(J,E).$$

$\mathcal{B}$ represents the Banach space $D(A)$ equipped with the graph norm

$$\|u\|_{\mathcal{B}}=\|Au\|_\infty+\|u\|_\infty\quad\forall u\in\mathcal{B}.$$

The notation $\mathcal{C}^1(\mathbb{R}^+,E)$ stands for the Banach space of all functions mapping $\mathbb{R}^+$ into E which are continuously differentiable, and the notation $\mathcal{C}(\mathbb{R}^+,\mathcal{B})$ stands for the space of all functions from $\mathbb{R}^+$ into $\mathcal{B}$ which are continuous.

We recall some knowledge on integro-differential equations and the related resolvent operators.

Definition 2.1 [16]. *A resolvent operator for the problem* (1.1) *is a bounded linear operator valued function*, $R(t)\in\mathcal{L}(E)$ *for* $t\ge 0$, *satisfying the following properties*:

(a) $R(0)=I$ (*the identity map of* E) *and* $\|R(t)\|\le Me^{\beta t}$ *for some constants* $M>0$ *and* $\beta\in\mathbb{R}$.

(b) *For each* $u\in E$, $R(t)u$ *is strongly continuous.*

(c) *For any* $u\in E$, $R(\cdot)u\in\mathcal{C}^1(\mathbb{R}^+,E)\cap\mathcal{C}(\mathbb{R}^+,\mathcal{B})$ *and*

$$R'(t)u=AR(t)u+\int_0^t U(t-s)R(s)u\,ds=R(t)Au+\int_0^t R(t-s)U(s)u\,ds.$$

Theorem 2.1 [16]. *Suppose that* $\phi(0)\in D(A)$. *Then* (1.1) *has a resolvent operator*. *Moreover*, *if* u *is a solution of* (1.1), *then*

$$u(t)=\begin{cases}R(t)\phi(0)+\displaystyle\int_0^t R(t-s)F\left(s,u_s,\int_0^s\rho(s,r,u_r)dr\right)ds, & t\in\mathbb{R}_+,\\ \phi(t), & t\in J_0.\end{cases}$$

In order to prove our main results, we need to recall the important properties of Kuratowski measure of noncompactness.

Definition 2.2 [12]. *Let* (E,d) *be a metric space and* $\mathfrak{B}(E)$ *be the set of all bounded subsets of* E. *The Kuratowski measure of noncompactness* α *is a function defined on* $\mathfrak{B}(E)$ *by*

$$\alpha(D)=\inf\{\epsilon>0/D\ \textit{has a finite cover by sets of diameter less or equal to}\ \epsilon\}\quad\forall D\in\mathfrak{B}(E).$$

Using the above definition, we can prove the following lemma.

Lemma 2.1 [12]. *Let (E,d) be a complete metric space and $D_1,\ D_2$ be a bounded subsets of E. Then*:

(1) $\alpha(D_1)=0$ *if and only if* $\overline{D_1}$ *is compact*;

(2) $\alpha(D_1)=\alpha(\overline{D_1})$;

(3) *for any* $\lambda\in\mathbb{R},\ \alpha(\lambda D_1)=|\lambda|\alpha(D_1)$;

(4) *for any* $u\in E,\ \alpha(\{u\}\cup D_1)=\alpha(D_1)$;

(5) $\alpha(\overline{\mathrm{Conv}}(D_1))=\alpha(D_1)$, *where* $\mathrm{Conv}(D_1)$ *is the convex hull of* D_1;

(6) *if* $D_1\subseteq D_2$, *then* $\alpha(D_1)\le\alpha(D_2)$;

(7) $\alpha(D_1\cup D_2)=\max\{\alpha(D_1),\alpha(D_2)\}$;

(8) $\alpha(D_1+D_2)=\alpha(D_1)+\alpha(D_2)$, *where* $D_1+D_2=\{u+v\in E/u\in D_1$ *and* $v\in D_2\}$.

Theorem 2.2 [17]. *Let E be a Banach space, $V\subset E$ be a bounded open neighborhood of* 0 *and $N:\overline{V}\to E$ be a continuous operator satisfying*:

(1) *the Mönch condition*: *if C is a countable subset of $\overline{V}$ and $C\subset\overline{\mathrm{Conv}}(\{0\}\cup N(C))$, then C is relatively compact*,

(2) *the Leary–Schauder boundary condition*: *$x\neq\gamma N(x)$ for all $x\in\partial V$ and* $0<\gamma<1$.

Then $\mathrm{Fix}(N)=\{x\in E: x=N(x)\}$ *is nonempty.*

Lemma 2.2. *Let E be a Banach space, $V\subset E$ be a bounded open neighborhood of* 0 *and $N:\overline{V}\to E$ be a continuous operator satisfying the Mönch condition. Then* $\mathrm{Fix}(N)$ *is compact*.

Proof. If $\mathrm{Fix}(N)=\varnothing$, it is clair that $\mathrm{Fix}(N)$ is compact. Now, if $\mathrm{Fix}(N)\neq\varnothing$, let $(x_n)_{n\in\mathbb{N}}\subset$ $\subset\mathrm{Fix}(N)$. Then

$$(x_n)_{n\in\mathbb{N}}\subset\overline{\mathrm{Conv}}(\{0\}\cup N(V)).$$

By the Mönch condition, we get $\{x_n: n\in\mathbb{N}\}$ is relatively compact. Thus, there exists a subsequence of $(x_n)_{n\in\mathbb{N}}$ that converges to some $x\in\overline{V}$. By the continuity of N, we conclude that $x=N(x)\in$ $\in\mathrm{Fix}(N)$.

Lemma 2.3 [22]. *Let $I=[0,a]$ be a compact interval in $\mathbb{R}$, E be a real Banach space and $B=\{u\in E;\ \|u-u(0)\|<b\}$ with $b\in\mathbb{R}_+$. Assume that f be a function from $I\times B$ into a Banach space F which satisfies the Carathéodory conditions and the condition*: *for any subset X of B*,

$$\alpha(f(T\times X))\le\sup_{t\in T}h(t,\alpha(X))$$

for each closed subset T of I, where $h\colon I\times\mathbb{R}_+\to\mathbb{R}_+$ is a Kamke function. Let K be a bounded strongly measurable function from $I\times I$ into the space of bounded linear mappings from F to E. If V is an equicontinuous set of functions $I\to B$, then

$$\alpha\left(\left\{\int\limits_T K(t,s)f(s,u(s)ds;\ u\in V)\right\}\right)\le\int\limits_T\|K(s,t)\|h(s,\alpha(V(s)))ds.$$

Lemma 2.4 [18]. *Let $f,g\in L^1(\mathbb{R}_+,\mathbb{R}),\ B,u\in C(\mathbb{R}_+,\mathbb{R}_+)$ such that*

$$u(t)\le B(t)+\int\limits_0^t f(s)u(s)ds+\int\limits_0^t f(s)\left(\int\limits_0^s g(\tau)d\tau\right)ds\quad\forall t\in\mathbb{R}_+.$$

If B is nondecreasing function, then

$$u(t) \leq B(t)\left(1 + \int\limits_0^t f(s)\exp\left(\int\limits_0^s (f(\tau) + g(\tau)d\tau\right)ds\right) \quad \forall t \in \mathbb{R}_+.$$

3. Main results.

Definition 3.1. *A function $u \in \mathcal{C}(J,E)$ is said to be a mild solution of the problem* (1.1) *if*

$$u(t) = \begin{cases} R(t)\phi(0) + \int_0^t R(t-s)F\left(s, u_s, \int_0^s \rho(s,r,u_r)dr\right)ds, & t \in \mathbb{R}_+, \\ \phi(t), & t \in J_0. \end{cases} \tag{3.1}$$

In order to give the first result of existence and uniqueness, we shall need the following hypotheses:

$(\mathcal{H}_1)$ The function $F\colon J \times \mathcal{C}(J_0,E) \times E \to E$ is Carathéodory and satisfies the following conditions:

(i) There exists $f_1 \in L^1(\mathbb{R}_+)$ such that

$$\|F(t,\phi,x) - F(t,\psi,y)\| \leq f_1(t)(\|\phi-\psi\|_\infty + \|x-y\|) \quad \forall \phi,\psi \in \mathcal{C}(J_0,E) \quad \text{and} \quad x,y \in E.$$

(ii) The function $g_1 : t \in \mathbb{R}_+ \to g_1(t) = \|F(t,0,0)\| \in \mathbb{R}_+$ belongs to $L^1(\mathbb{R}_+)$.

$(\mathcal{H}_2)$ The function $\rho\colon \Delta \times \mathcal{C}(J_0,E) \to E$ satisfies the following conditions:

(i) There exists $f_2 \in L^1(\mathbb{R}_+)$ such that

$$\|\rho(t,s,\phi) - \rho(t,s,\psi)\| \leq f_2(s)\|\phi-\psi\|_\infty \quad \forall \phi,\psi \in \mathcal{C}(J_0,E) \quad \text{and} \quad (t,s) \in \Delta.$$

(ii) There exists a constant $N \geq 0$ such that

$$\int\limits_0^s \|\rho(s,r,0)\|dr \leq N \quad \forall s \in \mathbb{R}_+.$$

Theorem 3.1. *Assume that the conditions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. Then the problem* (1.1) *has a unique mild solution.*

Proof. We put $g = \max(f_1, f_2)$. Assume that $\beta \geq 0$. For some constant $\lambda > 0$, we introduce the following real vectorial space:

$$BC_*(J,E) = \left\{ u \in \mathcal{C}(J,E) \Big/ \sup_{t \in J}\left(e^{-\beta t} e^{-\lambda \int_0^t g(\xi)d\xi}\|u(t)\|\right) < \infty \right\}.$$

It is clair that $BC(J,E) \subset BC_*(J,E)$. This space, endowed with the norm

$$\|u\|_* = \sup_{t \in J}\left(e^{-\beta t} e^{-\lambda \int_0^t g(\xi)d\xi}\|u(t)\|\right)$$

for all $u \in BC_*(J,E)$, is a Banach space.

We consider now the operator $N\colon BC_*(J,E) \to BC_*(J,E)$ defined by

$$(Nu)(t) = \begin{cases} R(t)\phi(0) + \int_0^t R(t-s)F\left(s, u_s, \int_0^s \rho(s,r,u_r)dr\right)ds, & t \in \mathbb{R}_+, \\ \phi(t), & t \in J_0. \end{cases}$$

It is clair that the fixed points of that operator are solutions of the problem (1.1). Using the Banach fixed point theorem, we prove that N has a unique fixed point. Then we obtain to show that the operator N is well defined and contractive.

***Step* 1:** N is well defined.

In fact, let $u \in BC_*(J,E)$, then, for any $t \in \mathbb{R}_+$, we get

$$\|(Nu)(t)\| \le Me^{\beta t}\|\phi(0)\| +$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}\left\|F\left(s, u_s, \int_0^s \rho(s,r,u_r)dr\right) - F(s,0,0) + F(s,0,0)\right\|ds \le$$

$$\le Me^{\beta t}\|\phi(0)\| + Me^{\beta t}\int_0^t e^{-\beta s}f_1(s)\left(\|u_s\|_\infty + \left\|\int_0^s \rho(s,r,u_r)dr\right\|\right)ds +$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}\|F(s,0,0)\|ds \le$$

$$\le Me^{\beta t}\|\phi(0)\| + Me^{\beta t}\int_0^t e^{-\beta s}f_1(s)\|u_s\|_\infty ds + Me^{\beta t}\int_0^t e^{-\beta s}\|F(s,0,0)\|ds +$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}f_1(s)\left(\int_0^s \|\rho(s,r,u_r) - \rho(s,r,0) + \rho(s,r,0)\|dr\right)ds \le$$

$$\le Me^{\beta t}\|\phi(0)\| + Me^{\beta t}\int_0^t e^{-\beta s}f_1(s)\|u_s\|_\infty ds + Me^{\beta t}\int_0^t e^{-\beta s}f_1(s)\left(\int_0^s f_2(r)\|u_r\|_\infty dr\right)ds +$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}f_1(s)\left(\int_0^s \|\rho(s,r,0)\|dr\right)ds + Me^{\beta t}\int_0^t e^{-\beta s}\|F(s,0,0)\|ds \le$$

$$\le Me^{\beta t}\|\phi(0)\| + Me^{\beta t}\int_0^t e^{-\beta s}g(s)\|u_s\|_\infty ds + Me^{\beta t}\int_0^t e^{-\beta s}f_1(s)\left(\int_0^s g(r)\|u_r\|_\infty dr\right)ds +$$

$$+MNe^{\beta t}\int_0^t e^{-\beta s}f_1(s)ds + Me^{\beta t}\int_0^t e^{-\beta s}\|F(s,0,0)\|ds \le$$

$$\leq Me^{\beta t}\|\phi(0)\| + Me^{\beta t}\int\limits_0^t \Big(g(s)e^{\lambda\int_0^s g(\xi)d\xi}\Big)\Big(e^{-\lambda\int_0^s g(\xi)d\xi}e^{-\beta s}\|u_s\|_\infty\Big)ds+$$

$$+Me^{\beta t}\int\limits_0^t e^{-\beta s}f_1(s)\left(\int\limits_0^s e^{\beta r}\Big(g(r)e^{\lambda\int_0^r g(\xi)d\xi}\Big)\Big(e^{-\lambda\int_0^r g(\xi)d\xi}e^{-\beta r}\|u_r\|_\infty\Big)dr\right)ds+$$

$$+MNe^{\beta t}\int\limits_0^t f_1(s)ds + Me^{\beta t}\int\limits_0^t \|f(s,0,0)\|ds \leq$$

$$\leq Me^{\beta t}\|\phi(0)\| + \frac{M}{\lambda}e^{\beta t}\|u\|_*\Big(e^{\lambda\int_0^t g(\xi)d\xi} - 1\Big)+$$

$$+\frac{M}{\lambda}e^{\beta t}\int\limits_0^t e^{-\beta s}f_1(s)\Big(e^{\beta s}\|u\|_*\Big(e^{\lambda\int_0^s g(\xi)d\xi} - 1\Big)\Big)ds + MNe^{\beta t}\|f_1\|_{L^1} + Me^{\beta t}\|g_1\|_{L^1} \leq$$

$$\leq Me^{\beta t}(\|\phi(0)\| + N\|f_1\|_{L^1} + \|g_1\|_{L^1}) + \frac{M}{\lambda}\|u\|_* e^{\beta t}e^{\lambda\int_0^t g(\xi)d\xi} + \frac{M}{\lambda}e^{\beta t}e^{\lambda\int_0^t g(\xi)d\xi}\|f_1\|_{L^1}.$$

Then

$$\|(Nu)(t)\| \leq Me^{\beta t}(\|\phi(0)\| + N\|f_1\|_{L^1} + \|g_1\|_{L^1}) + \frac{M}{\lambda}e^{\beta t}e^{\lambda\int_0^t g(\xi)d\xi}(\|u\|_* + \|f_1\|_{L^1}).$$

We have proved that, for any $t \in \mathbb{R}_+$,

$$e^{-\beta t}e^{-\lambda\int_0^t g(\xi)d\xi}\|(Nu)(t)\| \leq$$

$$\leq Me^{-\lambda\int_0^t g(\xi)d\xi}\big(\|\phi(0)\| + N\|f_1\|_{L^1} + \|g_1\|_{L^1}\big) + \frac{M}{\lambda}(\|u\|_* + \|f_1\|_{L^1}) \leq$$

$$\leq M\big(\|\phi(0)\| + N\|f_1\|_{L^1} + \|g_1\|_{L^1}\big) + \frac{M}{\lambda}(\|u\|_* + \|f_1\|_{L^1}) < +\infty.$$

On the other hand, since $\phi \in \mathcal{C}(J_0, E)$, then $\|Nu(t)\| < \infty$ for any $t \in J_0$.
Hence,

$$\sup_{t\in J}\Big(e^{-\beta t}e^{-\lambda\int_0^t g(\xi)d\xi}\|(Nu)(t)\|\Big) < +\infty,$$

this means that the function $Nu \in BC_*(J, E)$.

***Step* 2:** N is a contractive mapping.

In fact, let $u, v \in BC_*(J, E)$, then, for any $t \in \mathbb{R}_+$, we get

$$\|(Nu)(t) - (Nv)(t)\| =$$

$$= \left\|\int\limits_0^t R(t-s)F\left(s, u_s, \int\limits_0^s \rho(s,r,u_r)dr\right)ds - \int\limits_0^t R(t-s)F\left(s, v_s, \int\limits_0^s \rho(s,r,v_r)dr\right)ds\right\| =$$

$$= \left\| \int_0^t R(t-s) \left[F\left(s, u_s, \int_0^s \rho(s,r,u_r)dr\right) - F\left(s, v_s, \int_0^s \rho(s,r,v_r)dr\right)\right] ds \right\| \le$$

$$\le \int_0^t \|R(t-s)\| \left\| F\left(s, u_s, \int_0^s \rho(s,r,u_r)dr\right) - F\left(s, v_s, \int_0^s \rho(s,r,v_r)dr\right)\right\| ds \le$$

$$\le Me^{\beta t} \int_0^t e^{-\beta s} f_1(s) \left(\|u_s - v_s\|_\infty + \left\| \int_0^s (\rho(s,r,u_r) - \rho(s,r,v_r))dr \right\| \right) ds \le$$

$$\le Me^{\beta t} \int_0^t e^{-\beta s} f_1(s) \|u_s - v_s\|_\infty ds + Me^{\beta t} \int_0^t e^{-\beta s} f_1(s) \left(\int_0^s f_2(r) \|u_r - v_r\|_\infty dr \right) ds \le$$

$$\le Me^{\beta t} \int_0^t e^{-\beta s} g(s) \|u_s - v_s\|_\infty ds + Me^{\beta t} \int_0^t e^{-\beta s} f_1(s) \left(\int_0^s g(r) \|u_r - v_r\|_\infty dr \right) ds \le$$

$$\le Me^{\beta t} \int_0^t e^{-\beta s} g(s) \|u_s - v_s\|_\infty ds + Me^{\beta t} \int_0^t e^{-\beta s} f_1(s) \left(\int_0^s g(r) \|u_r - v_r\|_\infty dr \right) ds \le$$

$$\le Me^{\beta t} \int_0^t \left(g(s) e^{\lambda \int_0^s g(\xi)d\xi} \right) \left(e^{-\lambda \int_0^s g(\xi)d\xi} e^{-\beta s} \|u_s - v_s\|_\infty \right) ds +$$

$$+ Me^{\beta t} \int_0^t e^{-\beta s} f_1(s) \left(\int_0^s e^{\beta r} \left(g(r) e^{\lambda \int_0^r g(\xi)d\xi} \right) \left(e^{-\lambda \int_0^r g(\xi)d\xi} e^{-\beta r} \|u_r - v_r\|_\infty dr \right) \right) ds \le$$

$$\le Me^{\beta t} \int_0^t \left(e^{\lambda \int_0^s g(\xi)d\xi} \right)' ds \|u - v\|_* + Me^{\beta t} \int_0^t f_1(s) \left(\int_0^s \left(e^{\lambda \int_0^r g(\xi)d\xi} \right)' dr \right) ds \|u - v\|_* \le$$

$$\le \frac{M}{\lambda} \exp\left(\beta t + \int_0^t g(s)ds \right) \|u - v\|_* + \frac{M\|f_1\|_{L^1}}{\lambda} \exp\left(\beta t + \int_0^t g(s)ds \right) \|u - v\|_*.$$

Then

$$\exp\left(-\beta t - \int_0^t g(s)ds \right) \|(Nu)(t) - (Nv)(t)\| \le \left(\frac{M}{\lambda} + \frac{M\|f_1\|_{L^1}}{\lambda} \right) \|u - v\|_*, \quad \text{for all } t \in J.$$

Therefore,

$$\|N(u) - N(v)\|_* \le \frac{M(1 + \|f_1\|_{L^1})}{\lambda} \|u - v\|_* \quad \text{for all } u, v \in BC_*(J, E).$$

Then, for $\lambda > M(1 + \|f_1\|_{L^1})$, N is a contraction. By Banach fixed point theorem, the unique fixed point of N is the unique mild solution in $BC_*(J, E)$ of the problem (1.1).

Theorem 3.1 is proved.

For the next result, we present an application of the Mönch fixed point theorem type to problem (1.1).

Theorem 3.2. *Let E be a separable Banach space and $F\colon \mathbb{R}_+ \times \mathcal{C}(J_0, E) \times E \to E$ be a Carathéodory function, such that $(\mathcal{H}_2)$ is fulfilled, and the following condition holds:*

$(\mathcal{H}_3)$ *$\{T(t)\}_{t\geq 0}$ is operator-norm continuous for $t > 0$.*

$(\mathcal{H}_4)$ *There exist $f \in L^1(\mathbb{R}_+)$ such that $f_3 = e^{-\beta\cdot} f \in L^1(\mathbb{R}_+)$ with $\beta \leq 0$, and*

$$\|F(t, \phi, x)\| \leq f(t)(\|\phi\|_\infty + \|x\| + 1) \quad \text{for all} \quad \phi \in \mathcal{C}(J_0, E), \quad x \in E, \quad \text{a.e. } t \in J.$$

$(\mathcal{H}_5)$ *There exists $g_2 \in L^1(\mathbb{R}_+)$ such that, for all bounded $D_1 \in \mathcal{C}(J_0, E)$, $D_2 \subset E$, we have*

$$\alpha(F(t, D_1, D_2)) \leq g_2(t)\left(\sup_{\theta\in J_0} \alpha(D_1(\theta)) + \alpha(D_2)\right) \quad \text{for a.e. } t \in J,$$

where

$$D_1(\theta) = \{\phi(\theta) : \phi \in D_1\}, \quad \theta \in J_0.$$

$(\mathcal{H}_6)$ *There exists $g_* \in L^1(\mathbb{R}_+)$ such that, for all bounded $D \in \mathcal{C}(J_0, E)$, we have*

$$\alpha(\rho(t, s, D)) \leq g_*(t) \sup_{\theta\in J_0} \alpha(D(\theta)) \quad \text{for a.e. } (t, s) \in \Delta.$$

Then problem (1.1) *has, at least, one mild solution and the solution set is compact.*

Proof. We put $g = \max(f, f_2)$, For some constant $\lambda > 0$, we introduce the following real vectorial space:

$$BC_*(J, E) = \left\{u \in \mathcal{C}(J, E) \Big/ \sup_{t\in J}\left(e^{\beta t} e^{-\lambda\int_0^t g(\xi)d\xi} \|u(t)\|\right) < \infty\right\}.$$

It is clair that $BC(J, E) \subset BC_*(J, E)$. This space, endowed with the norm

$$\|u\|_* = \sup_{t\in J}\left(e^{\beta t} e^{-\lambda\int_0^t g(\xi)d\xi} \|u(t)\|\right)$$

for all $u \in BC_*(J, E)$, is a Banach space.

We consider now the same operator $N\colon BC_*(J, E) \to BC_*(J, E)$ defined in the proof of Theorem 3.1. It is clair that the fixed points of that operator are solutions of the problem (1.1). Using the Mönch fixed point theorem, we prove that N has, at least, one fixed point. Let us prove that the conditions of the Mönch fixed point Theorem 2.2 are satisfied following several steps.

***Step* 1:** N is well defined.

We follow the same manner as in step 1 of the proof of Theorem 3.1.

***Step* 2:** N is continuous.

In fact, let $(u^{(n)})_{n\in\mathbb{N}}$ be a sequence in $BC_*(I,W)$ such that $u^{(n)} \to u$ in $BC_*(I,W)$. Then

$$\|(Nu^{(n)})(t)-(Nu)(t)\| = \left\| \int_0^t R(t-s)F\left(s, u_s^{(n)}, \int_0^s \rho(s,r,u_r^{(n)})dr\right)ds - \right.$$

$$\left. - \int_0^t R(t-s)F\left(s, u_s, \int_0^s \rho(s,r,u_r)dr\right)ds \right\| =$$

$$= \left\| \int_0^t R(t-s)\left[F\left(s, u_s^{(n)}, \int_0^s \rho(s,r,u_r^{(n)})dr\right) - F\left(s, u_s, \int_0^s \rho(s,r,u_r)dr\right)\right]ds \right\| \le$$

$$\le \int_0^t \|R(t-s)\| \left\| F\left(s, u_s^{(n)}, \int_0^s \rho(s,r,u_r^{(n)})dr\right) - F\left(s, u_s, \int_0^s \rho(s,r,u_r)dr\right)\right\| ds. \qquad (3.2)$$

The sequence $(F_n)_{n\in\mathbb{N}}$, defined by $F_n\colon t\in\mathbb{R}_+ \to F_n(t) = F\left(t, u_t^{(n)}, \int_0^t \rho(t,r,u_r^{(n)})\right)$, satisfies the conditions of the Lebesgue dominated convergence theorem. In fact, since F is a Carathéodory function and $\mathcal{C}(J_0,E)\times E$ is separable, then F is measurable. For any $n\in\mathbb{N}$, the function $h_n\colon t\in\mathbb{R}_+ \to h_n(t) = \left(t, u_t^{(n)}, \int_0^t \rho(t,r,u_r^{(n)})\right)$ is also measurable because $u^{(n)}$ and ρ are continuous. As F_n is the composite function of two measurable functions h_n and F, it follows that F_n is measurable too.

Since the sequence $(u^{(n)})_{n\in\mathbb{N}}$ converges to u in $BC_*(J,E)$ and F is a Carathéodory function, then, for any $t\in\mathbb{R}_+$,

$$F_n(t) = F\left(t, u_t^{(n)}, \int_0^t \rho(t,r,u_r^{(n)})\right) \longrightarrow F\left(t, u_t, \int_0^t \rho(t,r,u_r)dr\right).$$

Since the sequence $(u^{(n)})_{n\in\mathbb{N}}$ is convergent, then $(u^{(n)})_{n\in\mathbb{N}}$ is bounded by some positive constant M_1. Let $n\in\mathbb{N}$. By the Definition 2.1 and using the hypothesis $(\mathcal{H}_4)$ and $(\mathcal{H}_2)$, we get, for any $t\in\mathbb{R}_+$,

$$\|F_n(t)\| \le f(t)\left(\|u_t^{(n)}\|_\infty + \left\|\int_0^t \rho(t,r,u_r^{(n)})dr\right\| + 1\right) \le$$

$$\le f(t)\left(\|u_t^{(n)}\|_\infty + \int_0^t \left\|\rho(t,r,u_r^{(n)}) - \rho(t,r,0) + \rho(t,r,0)\right\|dr + 1\right) \le$$

$$\le f(t)\left(\|u_t^{(n)}\|_\infty + \int_0^t f_2(r)\|u_r^{(n)}\|_\infty dr + \int_0^t \|\rho(t,r,0)\|dr + 1\right) \le$$

$$\leq f(t)\Big(\|u^{(n)}\|_\infty + \|f_2\|_{L^1}\|u^{(n)}\|_\infty dr + N + 1\Big) \leq$$

$$\leq f(t)(M_1 + \|f_2\|_{L^1}M_1 + N + 1) = Kf(t),$$

with $K = M_1 + \|f_2\|_{L^1}M_1 + N + 1$ is a positive constant. Since f is measurable, then Kf is measurable. By the Lebesgue dominated convergence theorem, we obtain that the right-hand side of the inequality (3.2) tends to 0 as n approaches to $+\infty$, this implies that $\|(Nu^{(n)})(t) - (Nu)(t)\| \to$ $\to 0$ as $n \to +\infty$. Thus, N is continuous.

***Step* 3:** N maps bounded sets of $BC_*(J,E)$ into bounded sets of $BC_*(J,E)$.

In fact, let $d > 0$ and $B_d = \{u \in BC_*(J,E)/\|u\|_* \leq d\}$, we show that $N(B_d)$ is bounded. By the Definition 2.1 and using the hypothesis $(\mathcal{H}_4)$ and $(\mathcal{H}_2)$, we get, for any $t \in \mathbb{R}_+$,

$$\|(Nu)(t)\| \leq Me^{\beta t}\|\phi(0)\| + Me^{\beta t}\int_0^t e^{-\beta s}\left\|F\left(s, u_s, \int_0^s \rho(s,r,u_r)dr\right)\right\|ds \leq$$

$$\leq Me^{\beta t}\|\phi(0)\| + Me^{\beta t}\int_0^t e^{-\beta s}f(s)\left(\|u_s\|_\infty + \left\|\int_0^s \rho(s,r,u_r)dr\right\| + 1\right)ds \leq$$

$$\leq Me^{\beta t}\|\phi(0)\| + Me^{\beta t}\int_0^t e^{-\beta s}f(s)\|u_s\|_\infty ds +$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}f(s)\left(\int_0^s \|\rho(s,r,u_r) - \rho(s,r,0) + \rho(s,r,0)\|dr\right)ds + Me^{\beta t}\int_0^t e^{-\beta s}f(s)ds \leq$$

$$\leq Me^{\beta t}\|\phi(0)\| + Me^{\beta t}\int_0^t e^{-\beta s}f(s)\|u_s\|_\infty ds + Me^{\beta t}\int_0^t e^{-\beta s}f(s)\left(\int_0^s f_2(r)\|u_r\|_\infty dr\right)ds +$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}f(s)\left(\int_0^s \|\rho(s,r,0)\|dr\right)ds + Me^{-\beta t}\|f\|_{L^1} \leq$$

$$\leq Me^{\beta t}\|\phi(0)\| + Me^{\beta t}\int_0^t e^{-\beta s}g(s)\|u_s\|_\infty ds + Me^{\beta t}\int_0^t e^{-\beta s}f(s)\left(\int_0^s g(r)\|u_r\|_\infty dr\right)ds +$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}f(s)\left(\int_0^s \|\rho(s,r,0)\|dr\right)ds + Me^{-\beta t}\|f\|_{L^1} \leq$$

$$\leq Me^{\beta t}\|\phi(0)\| + Me^{\beta t}\int_0^t e^{-2\beta s}\Big(g(s)e^{\lambda\int_0^s g(\xi)d\xi}\Big)e^{\beta s}e^{-\lambda\int_0^s g(\xi)d\xi}\|u_s\|_\infty ds +$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}f(s)\left(\int_0^s e^{-\beta r}\Big(g(r)e^{\lambda\int_0^r g(\xi)d\xi}\Big)\Big(e^{\beta r}e^{-\lambda\int_0^r g(\xi)d\xi}\|u_r\|_\infty\Big)dr\right)ds+$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}f(s)\left(\int_0^s \|\rho(s,r,0)\|dr\right)ds+Me^{-\beta t}\|f\|_{L^1}\le$$

$$\le Me^{\beta t}\|\phi(0)\|+\frac{M}{\lambda}e^{-\beta t}\|u\|_*\Big(e^{\lambda\int_0^t g(\xi)d\xi}-1\Big)+$$

$$+\frac{M}{\lambda}e^{\beta t}\int_0^t e^{-\beta s}f(s)\Big(e^{-\beta s}\|u\|_*\Big(e^{\lambda\int_0^s g(\xi)d\xi}-1\Big)\Big)ds+MNe^{-\beta t}\|f\|_{L^1}+Me^{-\beta t}\|f\|_{L^1}\le$$

$$\le M\|\phi(0)\|+Me^{-\beta t}(N+1)\|f\|_{L^1}+\frac{M}{\lambda}e^{-\beta t}e^{\lambda\int_0^t g(\xi)d\xi}\|u\|_*(1+\|f\|_{L^1}),$$

that is,

$$e^{\beta t}e^{-\lambda\int_0^t g(\xi)d\xi}\|(Nu)(t)\|\le M(\|\phi(0)\|+(N+1)\|f\|_{L^1})+\frac{M}{\lambda}d(1+\|f\|_{L^1})=l_1.$$

On the other hand, since $\phi\in\mathcal{C}(J_0,E)$, we have $\|Nu(t)\|\le\sup_{t\in J_0}\|\phi(t)\|=l_2$ for any $t\in J_0$, with l_1 and $l_2\in\mathbb{R}_+$. Hence,

$$\sup_{t\in J}\Big(e^{\beta t}e^{-\lambda\int_0^t g(\xi)d\xi}\|(Nu)(t)\|\Big)\le l_1+l_2=l.$$

This means that $\|Nu\|_*\le l$, which proves that $N(B_d)\subset B_l$. Thus $N(B_d)$ is bounded.

***Step* 4:** N maps bounded sets of $BC_*(J,E)$ into equicontinuous sets of $BC_*(J,E)$.

In fact, let $d>0$ and $B_d=\{u\in BC_*(J,E)/\|u\|_*\le d\}$. We show that $N(B_d)$ is equicontinuous. Let $t_1,t_2\in[a,b]$ with $t_1<t_2$ and $[a,b]$ is a compact interval in $\mathbb{R}_+$. By Definition 2.1 and using the hypothesis $(\mathcal{H}_4)$, $(\mathcal{H}_3)$ and $(\mathcal{H}_2)$, we get, for any $u\in B_d$,

$$\|(Nu)(t_1)-(Nu)(t_2)\|\le$$

$$\le\|(R(t_1)-R(t_2))\phi(0)\|+\left\|\int_0^{t_1}R(t_1-s)F\left(s,u_s,\int_0^s\rho(s,r,u_r)dr\right)ds-\right.$$

$$\left.-\int_0^{t_2}R(t_2-s)F\left(s,u_s,\int_0^s\rho(s,r,u_r)dr\right)ds\right\|\le$$

$$\le\|(R(t_1)-R(t_2))\phi(0)\|+\int_0^{t_1}\left\|(R(t_1-s)-R(t_2-s))F\left(s,u_s,\int_0^s\rho(s,r,u_r)dr\right)\right\|ds+$$

$$+\int_{t_1}^{t_2}\left\|R(t_2-s)F\left(s,u_s,\int_0^s\rho(s,r,u_r)dr\right)\right\|ds\le$$

$$\leq \|R(t_1) - R(t_2)\|_{\mathcal{L}(E)} \|\phi(0)\| +$$

$$+ \int\limits_0^{t_1} \|R(t_1 - s) - R(t_2 - s)\|_{\mathcal{L}(E)} f(s) \left(\|u_s\|_\infty + \left\| \int\limits_0^s \rho(s, r, u_r) dr \right\| + 1 \right) ds +$$

$$+ \int\limits_{t_1}^{t_2} M e^{\beta(t_2 - s)} f(s) \left(\|u_s\|_\infty + \left\| \int\limits_0^s \rho(s, r, u_r) dr \right\| + 1 \right) ds \leq$$

$$\leq \|R(t_1) - R(t_2)\|_{\mathcal{L}(E)} \|\phi(0)\| + \int\limits_0^{t_1} \|R(t_1 - s) - R(t_2 - s)\|_{\mathcal{L}(E)} f(s) \times$$

$$\times \left(d + \int\limits_0^s \|\rho(s, r, u_r) - \rho(s, r, 0) + \rho(s, r, 0)\| dr + 1 \right) ds +$$

$$+ \int\limits_{t_1}^{t_2} M e^{\beta(t_2 - s)} f(s) \left(d + \int\limits_0^s \|\rho(s, r, u_r) - \rho(s, r, 0) + \rho(s, r, 0)\| dr + 1 \right) ds \leq$$

$$\leq \|R(t_1) - R(t_2)\|_{\mathcal{L}(E)} \|\phi(0)\| +$$

$$+ \int\limits_0^{t_1} \|R(t_1 - s) - R(t_2 - s)\|_{\mathcal{L}(E)} f(s) \left(d + \int\limits_0^s f_2(r) \|u_r\|_\infty dr + N + 1 \right) ds +$$

$$+ \int\limits_{t_1}^{t_2} M e^{\beta(t_2 - s)} f(s) \left(d + \int\limits_0^s f_2(r) \|u_r\|_\infty dr + N + 1 \right) ds.$$

Then

$$\|(Nu)(t_1) - (Nu)(t_2)\| \leq \|R(t_1) - R(t_2)\|_{\mathcal{L}(E)} \|\phi(0)\| +$$

$$+ \int\limits_0^{t_1} \|R(t_1 - s) - R(t_2 - s)\|_{\mathcal{L}(E)} f(s) (d + d\|f_2\|_{L^1} + N + 1) ds +$$

$$+ \int\limits_{t_1}^{t_2} M e^{\beta(t_2 - s)} f(s) (d + d\|f_2\|_{L^1} + N + 1) ds.$$

By hypothesis $(\mathcal{H}_2)$, we have $\|R(t_1) - R(t_2)\|_{\mathcal{L}(E)}$ tends to 0 as $t_1 \to t_2$. This leads to the right-hand side of the above inequality tends to 0 as $t_1 \to t_2$ independently of u. Thus, $\|(Nu)(t_1) - (Nu)(t_2)\| \to 0$ as $t_1 \to t_2$.

We denote by $\omega^T(u, \epsilon)$ the modulus of continuity of $u \in E$ on the interval $[0, T]$, i.e.,

$$\omega^T(u, \epsilon) = \sup \big\{ \|u(t) - u(s)\|;\ t, s \in [0, T] \ \text{ and } \ |t - s| \leq \epsilon \big\}.$$

For any $K \subset E$, we put

$$\omega^T(K,\epsilon) = \sup\{\omega^T(u,\epsilon);\ u \in K\} \quad \text{and} \quad \omega_0^T(K) = \lim_{\epsilon\to 0}\omega^T(K,\epsilon).$$

Let us consider the measure of noncompactness μ defined on the family of bounded subsets of $BC_*(J,E)$ by

$$\mu(K) = \omega_0^T(K) + \sup_{t\in J}\left(\exp\left(-\beta t - \lambda\int\limits_0^t G(\xi)d\xi\right)\alpha(K(t))\right) + \lim_{t\to\infty}\sup_{u\in K}\|u(t)\|$$

for any K bounded subset of $BC_*(J,E)$, where $\lambda > 0$ and $G = \max\{g_2, g_*\}$.

***Step* 5:** We prove that the Mönch condition is satisfied. Let K be a bounded countable subset of $BC_*(J,E)$ such that $K \subset \overline{\mathrm{Conv}}(\{0\}\cup N(K)\})$. Suppose that $K \subset B_d = \{u \in BC_*(J,E)/\|u\|_* \leq \leq d\}$, where $d > 0$. We have to show that K is relatively compact. To do this, it suffices to prove that $\mu(K) = 0$.

This will be given in several claims:

***Claim* 1:** $\omega_0^T(K) = 0$.

In fact, using the properties of the function $\omega_0^T(\cdot)$ (see [21]), and the fact that $N(B_d)$ is equicontinuous, we get

$$\omega_0^T(K) \leq \omega_0^T\left(\overline{\mathrm{Conv}}(\{0\}\cup N(K))\right) = \omega_0^T(N(K)) = 0.$$

Hence, we infer that $\omega_0^T(K) = 0$.

***Claim* 2:** $\sup\limits_{t\in J}\left(\exp\left(-\beta t - \lambda\int_0^t G(\xi)d\xi\right)\alpha(K(t))\right) = 0.$

In fact, let $t \in \mathbb{R}_+$. We put $K(t) = \{u(t); u \in K\}$ and $K_t = \{u_t; u \in K\}$. By hypothesis $(\mathcal{H}_5)$, $(\mathcal{H}_6)$ and applying the Lemma 2.3, we get

$$\alpha(K(t)) \leq \alpha(N(K(t))) = \alpha\left\{R(t)\phi(0) + \int\limits_0^t R(t-s)F\left(s, K_s, \int\limits_0^s \rho(s,r,K_r)dr\right)ds\right\} \leq$$

$$\leq \alpha\left\{\int\limits_0^t R(t-s)F\left(s, K_s, \int\limits_0^s \rho(s,r,K_r)dr\right)ds\right\} \leq$$

$$\leq \int\limits_0^t \|R(t-s)\|g_2(s)\left(\sup_{\theta\in J_0}(\alpha(K_s(\theta))) + \alpha\left(\int\limits_0^s \rho(s,r,K_r)dr\right)\right)ds \leq$$

$$\leq \int\limits_0^t Me^{\beta(t-s)}g_2(s)\left(\sup_{\theta\in J_0}(\alpha(K_s(\theta))) + \alpha\left(\int\limits_0^s \rho(s,r,K_r)dr\right)\right)ds \leq$$

$$\leq Me^{\beta t}\int\limits_0^t e^{-\beta s}g_2(s)\left(\sup_{\theta\in J_0}(\alpha(K_s(\theta)))\right)ds +$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}g_2(s)\left(\int_0^s g_*(r)\left(\sup_{\theta\in J_0}(\alpha(K_r(\theta)))dr\right)\right)ds\le$$

$$\le Me^{\beta t}\int_0^t e^{-\beta s}G(s)\left(\sup_{\theta\in J_0}(\alpha(K_s(\theta)))\right)ds+$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}g_2(s)\left(\int_0^s G(r)\left(\sup_{\theta\in J_0}(\alpha(K_r(\theta)))dr\right)\right)ds\le$$

$$\le Me^{\beta t}\int_0^t G(s)e^{\lambda\int_0^t G(\xi)d\xi}e^{-\beta s}e^{-\lambda\int_0^s G(\xi)d\xi}\left(\sup_{\theta\in J_0}(\alpha(K_s(\theta)))\right)ds+$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}g_2(s)\left(\int_0^s e^{\beta r}G(r)e^{\lambda\int_0^t G(\xi)d\xi}e^{-\beta s}e^{-\lambda\int_0^r G(\xi)d\xi}\left(\sup_{\theta\in J_0}(\alpha(K_r(\theta)))dr\right)\right)ds\le$$

$$\le Me^{\beta t}\sup_{t\in J}\left(\exp\left(-\beta t-\lambda\int_0^t G(\xi)d\xi\right)\alpha(K(t))\right)\int_0^t G(s)e^{\lambda\int_0^s G(\xi)d\xi}ds+$$

$$+Me^{\beta t}\sup_{t\in J}\left(\exp\left(-\beta t-\lambda\int_0^t G(\xi)d\xi\right)\alpha(K(t))\right)\int_0^t e^{-\beta s}g_2(s)\left(\int_0^s e^{\beta r}G(r)e^{\lambda\int_0^r G(\xi)d\xi}dr\right)ds\le$$

$$\le\frac{M}{\lambda}\exp\left(\beta t+\lambda\int_0^t G(\xi)d\xi\right)\sup_{t\in J}\left(\exp\left(-\beta t-\lambda\int_0^t G(\xi)d\xi\right)\alpha(K(t))\right)+$$

$$+\frac{M}{\lambda}\exp\left(\beta t+\lambda\int_0^t G(\xi)d\xi\right)\sup_{t\in J}\left(\exp\left(-\beta t-\lambda\int_0^t G(\xi)d\xi\right)\alpha(K(t))\right)\|g_2\|_{L^1}.$$

The above inequality reduces to

$$\alpha(K(t))\le\frac{M}{\lambda}(1+\|g_2\|_{L^1})\exp\left(\beta t+\lambda\int_0^t G(\xi)d\xi\right)\sup_{t\in J}\left(\exp\left(-\beta t-\lambda\int_0^t G(\xi)d\xi\right)\alpha(K(t))\right).$$

This proves that for any $t\in\mathbb{R}_+$, we have

$$\exp\left(-\beta t-\lambda\int_0^t G(\xi)d\xi\right)\alpha(K(t))\le\frac{M}{\lambda}(1+\|g_2\|_{L^1})\sup_{t\in J}\left(\exp\left(-\beta t-\lambda\int_0^t G(\xi)d\xi\right)\alpha(K(t))\right).$$

That means

$$\sup_{t\in J}\left(\exp\left(-\beta t-\lambda\int_0^t G(\xi)d\xi\right)\alpha(K(t))\right)\leq$$

$$\leq\frac{M}{\lambda}(1+\|g_2\|_{L^1})\sup_{t\in J}\left(\exp\left(-\beta t-\lambda\int_0^t G(\xi)d\xi\right)\alpha(K(t))\right).$$

Taking $\lambda>M(1+\|g_2\|_{L^1})$, we obtain $0<\dfrac{M}{\lambda}(1+\|g_2\|_{L^1})<1$, and therefore

$$\sup_{t\in J}\left(\exp\left(-\beta t-\lambda\int_0^t G(\xi)d\xi\right)\alpha(K(t))\right)=0.$$

***Claim* 3:** $\sup_{u\in K}\|u(t)\|\to 0$ as $t\to+\infty$.

In fact, let $u\in K\subset B_d$. According to the step 3, we get, for any $t\in\mathbb{R}_+$,

$$\|u(t)\|=\left\|R(t)\phi(0)+\int_0^t R(t-s)F\left(s,u_s,\int_0^s\rho(s,r,u_r)dr\right)ds\right\|\leq$$

$$\leq Me^{\beta t}\|\phi(0)\|+Me^{\beta t}\int_0^t e^{-\beta s}\left\|F\left(s,u_s,\int_0^s\rho(s,r,u_r)dr\right)\right\|ds\leq$$

$$\leq Me^{\beta t}\|\phi(0)\|+Me^{\beta t}\int_0^t e^{-\beta s}f(s)\left(\|u_s\|_\infty+\left\|\int_0^s\rho(s,r,u_r)dr\right\|+1\right)ds\leq$$

$$\leq Me^{\beta t}\|\phi(0)\|+Me^{\beta t}\int_0^t e^{-\beta s}f(s)\|u_s\|_\infty ds+$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}f(s)\left(\int_0^s\|\rho(s,r,u_r)-\rho(s,r,0)+\rho(s,r,0)\|dr\right)ds+Me^{\beta t}\int_0^t e^{-\beta s}f(s)ds\leq$$

$$\leq Me^{\beta t}\|\phi(0)\|+Me^{\beta t}\int_0^t e^{-\beta s}f(s)ds+Me^{\beta t}\int_0^t e^{-\beta s}f(s)\left(\int_0^s\|\rho(s,r,0)\|dr\right)ds+$$

$$+Me^{\beta t}\int_0^t e^{-\beta s}f(s)\left(\|u_s\|_\infty+\int_0^s f_2(r)\|u_r\|_\infty dr\right)ds\leq$$

$$\leq Me^{\beta t}\|\phi(0)\|+M(1+N)e^{\beta t}\int_0^t e^{-\beta s}f(s)ds+$$

$$+Me^{\beta t}\int\limits_0^t e^{-\beta s}f(s)\left(\|u_s\|_\infty+\int\limits_0^s f_2(r)\|u_r\|_\infty dr\right)ds.$$

Then

$$e^{-\beta t}\|u(t)\|\leq M\|\phi(0)\|+M(1+N)\int\limits_0^t e^{-\beta s}f(s)ds+$$

$$+M\int\limits_0^t e^{-\beta s}f(s)\left(\|u_s\|_\infty+\int\limits_0^s f_2(r)\|u_r\|_\infty dr\right)ds\leq$$

$$\leq M\|\phi(0)\|+M(1+N)\int\limits_0^t e^{-\beta s}f(s)ds+M\int\limits_0^t f(s)e^{-\beta s}\|u_s\|_\infty ds+$$

$$+Me^{-\beta t}\int\limits_0^t f(s)\left(\int\limits_0^s f_2(r)e^{-\beta r}\|u_r\|_\infty dr\right)ds.$$

Putting

$$B(t)=M\|\phi(0)\|+M(1+N)\int\limits_0^t e^{-\beta s}f(s)ds,$$

we get

$$\|u(t)\|\leq B(t)+M\int\limits_0^t f(s)\left(e^{-\beta s}\|u_s\|_\infty+\int\limits_0^s f_2(r)e^{-\beta r}\|u_r\|_\infty dr\right)ds. \tag{3.3}$$

Set

$$V(t)=\sup_{s\in[0,t]}e^{\beta s}\|u(s)\|.$$

Thus, the inequality (3.3) implies

$$V(t)\leq B(t)+M\int\limits_0^t f(s)\left(V(s)+\int\limits_0^s f_2(r)V(r)dr\right)ds.$$

Applying the Lemma 2.4, we obtain

$$V(t)\leq B(t)\left(1+\int\limits_0^t f(s)\exp\left(\int\limits_0^t (f(\tau)+f_2(\tau))d\tau\right)ds\right).$$

Hence

$$e^{-\beta t}\|u(t)\| \leq B(t)\left(1+\int\limits_0^t f(s)\exp\left(\int\limits_0^t (f(\tau)+f_2(\tau))d\tau\right)ds\right).$$

Therefore,

$$\|u(t)\| \leq e^{\beta t}B(t)\left(1+\int\limits_0^t f(s)\exp\left(\int\limits_0^t (f(\tau)+f_2(\tau))d\tau\right)ds\right).$$

Since $\beta < 0$ and by the condition $e^{-\beta\cdot}f \in L^1(\mathbb{R}_+)$ cited in $(\mathcal{H}_4)$, we conclude that the right-hand side of the above inequality tends to 0 as t tends to $+\infty$, and, therefore, $\sup_{u\in K}\|u(t)\| \to 0$ as $t \to +\infty$.

By Claims 1, 2, and 3, we obtain $\mu(K) = 0$. Thus, we find that K is relatively compact.

***Step* 6:** A priori bounds. We prove now the existence of V a bounded open subset of $BC_*(J,E)$ containing 0 and satisfying the Leary – Schauder boundary condition: $u \neq \gamma N(u)$ for all $u \in \partial V$ and $0 < \gamma < 1$.

Let $u \in BC_*(J,E)$ and $u = \gamma Nu$ for some $\gamma \in]0,1[$. Then we get

$$\|u(t)\| \leq Me^{\beta t}\|\phi(0)\| + Me^{\beta t}\int\limits_0^t \left(g(s)e^{\lambda\int_0^s g(\xi)d\xi}\right)\left(e^{-\lambda\int_0^s g(\xi)d\xi}e^{-\beta s}\|u_s\|_\infty\right)ds+$$

$$+Me^{\beta t}\int\limits_0^t e^{-\beta s}f_1(s)\left(\int\limits_0^s e^{\beta r}\left(g(r)e^{\lambda\int_0^r g(\xi)d\xi}\right)\left(e^{-\lambda\int_0^r g(\xi)d\xi}e^{-\beta r}\|u_r\|_\infty\right)dr\right)ds+$$

$$+MNe^{\beta t}\int\limits_0^t f_1(s)ds + Me^{\beta t}\int\limits_0^t \|f(s,0,0)\|ds.$$

Then

$$e^{\beta t-\lambda\int\limits_0^t g(s)ds}\|u(t)\| \leq M\|\phi(0)\| + M\int\limits_0^t g(s)\left(e^{-\lambda\int_0^s g(\xi)d\xi}e^{\beta s}\|u_s\|_\infty\right)ds+$$

$$+M\int\limits_0^t e^{-\beta s}f_1(s)\left(\int\limits_0^s e^{-\beta r}g(r)\left(e^{-\lambda\int_0^r g(\xi)d\xi}e^{\beta r}\|u_r\|_\infty\right)dr\right)ds+$$

$$+MN\int\limits_0^t f_1(s)ds + M\int\limits_0^t \|f(s,0,0)\|ds.$$

Since we have

$$u_s(\theta) = \begin{cases} \phi(s+\theta), & \text{if} \quad s+\theta \in [-r,0], \\ u(s+\theta), & \text{if} \quad s+\theta \geq 0, \end{cases}$$

and, hence,

$$e^{\beta t-\lambda\int_0^t g(s)ds}\|u(t)\| \le M\|\phi(0)\| + M\int\limits_0^t g(s)\|\phi\|_\infty ds+$$

$$+\int\limits_0^t g(s)\left(e^{-\lambda\int_0^s g(\xi)d\xi}e^{\beta s}\sup_{r\in[0,s]}\|u(r)\|\right)ds+$$

$$+M\|\phi\|_\infty\int\limits_0^t e^{-\beta s}f_1(s)\int\limits_0^s g(r)drds+$$

$$+M\int\limits_0^t e^{-\beta s}f_1(s)\left(\int\limits_0^s g(r)\left(e^{-\lambda\int_0^r g(\xi)d\xi}e^{\beta r}\sup_{\xi\in[0,r]}\|u(\xi)\|\right)dr\right)ds+$$

$$+MN\int\limits_0^t f_1(s)ds + M\int\limits_0^t \|f(s,0,0)\|ds.$$

Set

$$V_*(t) = \sup_{s\in[0,t]} e^{\beta s-\lambda\int\limits_0^s g(\tau)d\tau}\|u(t)\|.$$

Then

$$V_*(t) \le M\|\phi(0)\| + M\int\limits_0^t g(s)\|\phi\|_\infty ds+$$

$$+M\|\phi\|_\infty\int\limits_0^t e^{-\beta s}f_1(s)\int\limits_0^s g(r)drds + MN\int\limits_0^t f_1(s)ds+$$

$$+M\int\limits_0^t \|f(s,0,0)\|ds + \int\limits_0^t g(s)V_*(s)ds + M\int\limits_0^t e^{-\beta s}f_1(s)\left(\int\limits_0^s g(r)V_*(r)dr\right)ds.$$

Applying the Lemma 2.4, we obtain

$$V_*(t) \le B_*(t)\left(1+\int\limits_0^t f(s)\exp\left(\int\limits_0^t (g(\tau)+f_3(\tau))d\tau\right)ds\right),$$

where

$$B_*(t) = M\|\phi(0)\| + M\int\limits_0^t g(s)\|\phi\|_\infty ds + M\|\phi\|_\infty\int\limits_0^t e^{-\beta s}f_1(s)\int\limits_0^s g(r)drds + MN\int\limits_0^t f_1(s)ds.$$

Consequently, we have

$$\|u\|_* \leq B_\infty(1 + \|f\|_\infty \exp(\|g\|_\infty + \|f_2\|_\infty)) := M_*.$$

Set $V = \{u \in BC_*(J, E) : \|u\|_* < M_* + 1\}$. So, V is a bounded open neighborhood of 0 and the operator $N : \overline{V} \to BC(J, E)$ satisfies the conditions of Mönch fixed point Theorem 2.2. Hence N has at least a fixed point $u \in V$ which is a solution to problem (1.1). It is clear that $\mathrm{Fix}(N) \subset V$. By Lemma 2.2, $\mathrm{Fix}(N)$ is also compact.

Theorem 3.2 is proved.

4. Applications. In this part, we give some applications on our results of this paper, for this we assume that we have a bounded domain G of $\mathbb{R}^d$ with a smooth boundary $\Gamma = \partial G = \Gamma_0 \cup \Gamma_1,\ \Gamma_0 \cap \Gamma_1 = \varnothing$. Let us consider the following internally damped wave equation:

$$\begin{aligned}
&\partial_{tt} u - \Delta u + a(x)\partial_t u(x,t) = 0, \quad (x,t) \in G \times (0,\infty),\\
&u(x,t) = 0, \quad (x,t) \in \Gamma_0 \times (0,\infty),\\
&\frac{\partial u}{\partial \nu} + u = 0, \quad (x,t) \in \Gamma_1 \times (0,\infty),\\
&u(x,t) = u_0(x,t), \quad u_t(x,t) = u_1(x,t), \quad (x,t) \in G \times [-r,0],
\end{aligned} \tag{4.1}$$

where $a : G \to \mathbb{R}_+$ be a positive continuous function and $u_0 \in \mathcal{C}\big([-r,0], L^2(G)\big),\ u_1 \in L^2(G)$. By putting $v = \partial_t u$, we write the system (4.1) into the following problem:

$$\begin{aligned}
&\partial_t u - v = 0, \quad x \in G, \quad t > 0,\\
&\partial_t v - \Delta u + a(x)v = 0, \quad x \in G, \quad t > 0,\\
&u(x,t) = 0, \quad x \in \Gamma_0, \quad t > 0,\\
&\frac{\partial u}{\partial \nu} + u = 0, \quad x \in \Gamma_1, \quad t > 0,\\
&u(x,t) = u_0(x,t), \quad v(x,t) = u_1(x,t), \quad x \in G, \quad \in [-r,0].
\end{aligned} \tag{4.2}$$

The operator $A_d = -\Delta$ is strict positive and auto-adjoint in $H = L^2(G),\ D(A_d) = H_0^1(G)$. We shall use the semigroup method to demonstrate the global existence and uniqueness of solution, for this purpose we rewrite the system (4.2) as an evolution equation for

$$\begin{aligned}
&\mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t), \quad t > 0,\\
&\mathcal{U}(t) = \mathcal{U}_0(t), \quad t \in [-r,0],
\end{aligned}$$

where $\mathcal{U}(\cdot) = \begin{pmatrix} u(\cdot,\cdot) \\ v(\cdot,\cdot) \end{pmatrix}$, $\mathcal{U}_0(t) = \begin{pmatrix} u_0(\cdot,t) \\ u_1(\cdot,t) \end{pmatrix}$ and $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{A}\begin{pmatrix} u(\cdot,t) \\ v(\cdot,t) \end{pmatrix} = \begin{pmatrix} v \\ -A_d u(\cdot,t) - a(\cdot)v \end{pmatrix}$$

with domain

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{H} \colon u \in H^2(G) \cap H^1_{\Gamma_0}(G), \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1 \right\}$$

in the Hilbert space $\mathcal{H} = H^1_{\Gamma_0}(G) \times L^2(G)$, where $H^1_{\Gamma_0}(G) := \{u \in H^1(G) \colon u = 0 \text{ on } \Gamma_0\}$. We equipped $\mathcal{H}$ with the scalar product

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = \int\limits_G (\nabla u_1 \nabla u_2 + v_1 v_2) dx + \int\limits_{\Gamma_1} u_1 u_2 d\Gamma,$$

and the norm defined by

$$\|(u,v)\|^2_{\mathcal{H}} = \int\limits_G (|\nabla u|^2 + v^2) dx + \int\limits_{\Gamma_1} u^2 d\Gamma.$$

4.1. Well-posedness.

Proposition 4.1. *$\mathcal{A}$ is m-dissipative in the Hilbert space $\mathcal{H}$.*

Proof. Let $\mathcal{U} \in D(\mathcal{A})$, then

$$\langle \mathcal{AU}, \mathcal{U} \rangle_{\mathcal{H}} = \int\limits_G (\nabla u \nabla v + (\Delta u - a(x)v)v \, dx + \int\limits_{\Gamma_1} uv d\Gamma =$$

$$= \int\limits_G \nabla u \nabla v \, dx + \int\limits_G v \Delta u dx - \int\limits_G a(x) v^2 dx + \int\limits_{\Gamma_1} uv d\Gamma.$$

By Green formula, we obtain

$$\langle \mathcal{AU}, \mathcal{U} \rangle_{\mathcal{H}} = - \int\limits_G a(x) v^2 dx \le 0.$$

Hence $\mathcal{A}$ is dissipative.

Now, we show that $R(I - \mathcal{A}) = \mathcal{H}$. For any $f = (f_1, f_2) \in H^1_{\Gamma_0}(G) \times L^2(G)$, we consider the following problem:

$$u - v = f_1, \quad x \in G, \quad t > 0,$$

$$v - \Delta u + a(x) v = f_2, \quad x \in G, t > 0,$$

$$u(x,t) = 0, \quad x \in \Gamma_0, \quad t > 0,$$

$$\frac{\partial u}{\partial \nu} + u = 0, \quad x \in \Gamma_1, \quad t > 0.$$

From the first and the second equation of the above system, we get $-\Delta u + (1+a)u = f_2 + (1+a) f_1$.

We associate this problem with the following bilinear form on $H^1_{\Gamma_0}(G) \times H^1_{\Gamma_0}(G)$:

$$B(u,v) = \int\limits_G \nabla u \nabla v \, dx + \int\limits_G (1+a) uv \, dx + \int\limits_{\Gamma_1} uv \, dx.$$

By the Hölder inequality, there exists $C > 0$ such that

$$|B(u,v)| \leq C\|u\|_{H^1_{\Gamma_0}(G)}\|v\|_{H^1_{\Gamma_0}(G)} \qquad \text{and} \qquad |B(u,u)| \geq \|u\|^2_{H^1_{\Gamma_0}(G)}.$$

Then, by the Lax – Milgram theorem, there is a unique solution $u \in H^1_{\Gamma_0}(G)$ such that

$$\int\limits_G \nabla u \nabla v\, dx + \int\limits_G (1+a)uv\, dx + \int\limits_{\Gamma_1} uv\, dx = \int\limits_G (f_2 + (1+a)f_1)v\, dx \quad \forall v \in H^1_{\Gamma_0}(G).$$

Now, the $\phi \in C_0^\infty(G)$, then $\int_{\Gamma_1} u\phi dx = 0.$

Therefore,

$$\int\limits_G \nabla u \nabla \phi dx + \int\limits_\Omega (1+a)u\phi dx = \int\limits_G (f_2 + (1+a)f_1)\phi dx \quad \forall \phi \in C_0^\infty(G).$$

Then

$$-\int\limits_G \phi \Delta u dx + \int\limits_G (1+a)u\phi dx = \int\limits_G (f_2 + (1+a)f_1)\phi dx \quad \forall \phi \in C_0^\infty(G).$$

Hence $-\Delta u + (1+a)u = f_2 + (1+a)f_1$. Since $f_2 + (1+a)f_1 \in L^2(G)$, then

$$-\Delta u + (1+a)u \in L^2(G) \Longrightarrow \Delta u = (1+a)u - f_2 - (1+a)f_1 \in L^2(G).$$

Hence $u \in H^1(G)$ and $\Delta u \in L^2(G)$. Using the fact that G is smooth, thus by regularity theorem, $u \in H^2(G)$. So, $u \in H^2(G) \cap H^1_{\Gamma_0}(G)$.

Now, we want to show that

$$\frac{\partial u}{\partial \nu} + u = 0 \quad \text{on} \quad \Gamma_1. \tag{4.3}$$

Indeed, for every $v \in H^1_{\Gamma_0}(G)$, we have

$$\int\limits_G \nabla u \nabla v\, dx + \int\limits_G (1+a)uv\, dx + \int\limits_{\Gamma_1} uv\, dx = \int\limits_G (f_2 + (1+a)f_1)v\, dx.$$

Since $u \in H^2(G)$, we can apply Green's formula and we get

$$-\int\limits_G \Delta uv\, dx + \int\limits_\Gamma \frac{\partial u}{\partial \nu} v\, dx + \int\limits_G (1+a)uv\, dx + \int\limits_{\Gamma_1} uv\, dx =$$

$$= \int\limits_G (f_2 + (1+a)f_1)v\, dx - \int\limits_G (\Delta uv + (1+a)uv)\, dx + \int\limits_{\Gamma_1} \left(\frac{\partial u}{\partial \nu} + u\right) v\, dx =$$

$$= \int\limits_G (f_2 + (1+a)f_1)v\, dx.$$

Therefore,

$$\int\limits_{\Gamma_1}\left(\frac{\partial u}{\partial \nu}+u\right)v\,dx=0 \quad \text{for all} \quad v\in H^1_{\Gamma_0}(G).$$

Thus, the condition (4.3) holds. Taking $v=u-f_1$, we find that $\mathcal{U}=(u,v)\in D(\mathcal{A})$ is the solution of the equation $(I+\mathcal{A})\mathcal{U}=f$. Consequently, $\mathcal{A}$ is m-dissipative.

Proposition 4.1 is proved.

As an application of Theorem 3.1, we consider the following nonlinear wave equation with finite delay:

$$\begin{aligned}
&\partial_{tt}u-\Delta u+a(x)\partial_t u(x,t)=\int\limits_0^t g(t-s)\big[\Delta u(\cdot,s)+a(x)\partial_s u(\cdot,s)\big]ds+\\
&\qquad +F\left(t,u_t,\int\limits_0^t \rho(t,s,u_s)ds\right), \quad (x,t)\in G\times(0,\infty),\\
&u(x,t)=0, \quad (x,t)\in\Gamma_0\times(0,\infty),\\
&\frac{\partial u}{\partial \nu}+u=0, \quad (x,t)\in\Gamma_1\times(0,\infty),\\
&u(x,t)=u_0(x,t), \quad u_t(x,t)=u_1(x,t), \quad (x,t)\in G\times[-r,0],
\end{aligned} \tag{4.4}$$

where $F\colon\mathbb{R}_+\times\mathcal{C}(J_0,L^2(G))\times L^2(G)$ is a Carathéodory function and $g\in C^1_b(\mathbb{R}_+,\mathbb{R})=\{f\in\mathcal{C}(\mathbb{R}_+,\mathbb{R})\colon\|f'\|_\infty<\infty\}$. We introduce the following hypotheses:

$(\overline{\mathcal{H}}_1)$ There exists $f_1\in L^2(\mathbb{R}_+)$ such that

$$\|F(t,\phi,u)-F(t,\psi,v)\|_{L^2(G)}\le f_1(t)(\|\phi-\psi\|_\infty+\|u-v\|_{L^2(G)})$$

$$\forall\phi,\psi\in\mathcal{C}(J_0,L^2(G)), \quad u,v\in L^2(G).$$

$(\overline{\mathcal{H}}_2)$ The function $g_1:t\in\mathbb{R}_+\to g_1(t)=\|F(t,0,0)\|\in\mathbb{R}_+$ belongs to $L^2(\mathbb{R}_+)$.

$(\overline{\mathcal{H}}_3)$ The function $\rho\colon\Delta\times\mathcal{C}(J_0,L^2(G))\to L^2(G)$ satisfies the following conditions:

(i) There exists $f_2\in L^2(\mathbb{R}_+)$ such that

$$\|\rho(t,s,\phi)-\rho(t,s,\psi)\|\le f_2(s)\|\phi-\psi\|_\infty \quad \forall\phi,\psi\in\mathcal{C}(J_0,L^2(G)) \quad \text{and} \quad (t,s)\in\Delta.$$

(ii) There exists a constant $N\ge 0$, such that

$$\int\limits_0^s\|\rho(s,r,0)\|dr\le N \quad \forall s\in\mathbb{R}_+.$$

We transform the system (4.4) into the following form:

$$\begin{aligned}
&\mathcal{U}'(t)-\int\limits_0^t B(t-s)\mathcal{A}\mathcal{U}(s)ds=\mathcal{A}\mathcal{U}(t)+F_*(t,\mathcal{U}_t), \quad t>0,\\
&\mathcal{U}(t)=\mathcal{U}_0(t), \quad t\in[-r,0],
\end{aligned} \tag{4.5}$$

where $F_*: \mathbb{R}_+ \times \mathcal{C}(J_0, \mathcal{H}) \to \mathcal{H}$ be a function given by

$$F_*(t, \mathcal{U}_t) = \begin{pmatrix} 0 \\ F\Big(t, u_t, \int_0^t \rho(t,s,u_s)ds\Big) \end{pmatrix}.$$

Let $B(t): D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ be the operator defined by

$$B(t)\mathcal{U} = g(t)\mathcal{A}\mathcal{U}, \quad \mathcal{U} \in D(\mathcal{A}).$$

Theorem 4.1. *Assume that the conditions* $(\overline{\mathcal{H}}_1)$ – $(\overline{\mathcal{H}}_3)$ *are satisfied. Then the problem* (4.5) *has a unique solution.*

Proof. From Proposition 4.1, the operator $\mathcal{A}$ is m-dissipative, then by the Lumer – Phillips theorem [19, 20], $\mathcal{A}$ generates a strongly continuous semigroup of contractions on $\mathcal{H}$ and closed. This implies that $B(t)$ is a closed operator. For every $\Phi = (\phi, \psi)$, $\overline{\Phi} = (\overline{\phi}, \overline{\psi}) \in \mathcal{C}(J_0, \mathcal{H})$, we get

$$\left\|F_*(t,\Phi) - F_*\big(t,\overline{\Phi}\big)\right\|_{\mathcal{H}}^2 = \left\|F\Big(t,\phi,\int_0^t \rho(t,s,\phi)ds\Big) - F\Big(t,\overline{\phi},\int_0^t \rho\big(t,s,\overline{\phi}\big)ds\Big)\right\|_{L^2(G)}^2 \leq$$

$$\leq f_1^2(t)\left\|\int_0^t \rho(t,s,\phi)ds - \int_0^t \rho\big(t,s,\overline{\phi}\big)ds\right\|_{L^2(G)}^2 + f_1^2(t)\left\|\phi - \overline{\phi}\right\|_\infty^2 \leq$$

$$\leq f_1^2(t)\|g\|_{L^2}^2\left\|\phi - \overline{\phi}\right\|_\infty^2 + f_1^2(t)\left\|\phi - \overline{\phi}\right\|_\infty^2.$$

Then

$$\left\|F_*(t,\Phi) - F_*\big(t,\overline{\Phi}\big)\right\|_{\mathcal{H}} \leq f_1(t)(\|g\|_{L^2}^2 + 1)^{\frac{1}{2}}\left\|\phi - \overline{\phi}\right\|_\infty.$$

Hence

$$\left\|F_*(t,\Phi) - F_*\big(t,\overline{\Phi}\big)\right\|_{\mathcal{H}}^2 \leq f_1(t)(\|g\|_{L^2} + 1)^{\frac{1}{2}}\left\|\Phi - \overline{\Phi}\right\|_\infty \quad \text{for all} \quad \Phi, \overline{\Phi} \in \mathcal{C}(J_0, \mathcal{H}),$$

where

$$\left\|\Phi - \overline{\Phi}\right\|_\infty = \sup_{t\in[-r,0]}\left\|\Phi(\cdot,t) - \overline{\Phi}(\cdot,t)\right\|_{\mathcal{H}}.$$

Applying Theorem 3.1, we conclude that the problem (4.5) has a unique mild solution.

Theorem 4.1 is proved.

In the end of this example, we give some functions satisfying the conditions of Theorem 4.1:

Let $h \in C_b(\mathbb{R}_+, \mathbb{R})$, we define $F_1, F_2: \mathbb{R}_+ \times \mathcal{C}(\mathbb{R}_+, L^2(G)) \to L^2(G)H$ by

$$F_1(t,\phi) = h(t)\sin(\phi(-r,x)) \quad \forall x \in G$$

and

$$F_2(t,\phi) = h(t)\sin(\phi(-r,x)) + \int_0^t h(t-s)\cos(s + \phi(-r,x))ds \quad \forall x \in G.$$

4.2. Semilinear parabolic problem. As an application of the Theorem 3.2, we consider the following problem:

$$\begin{aligned} &\partial_t u(x,t) + \mathbb{A}_*(x,D)u(x,t) = -\int\limits_0^t g(t-s)\mathbb{A}_*(x,D)u(x,s)ds + \\ &\quad + F\left(t, u_t, \int\limits_0^t \rho(t,s,u_s)ds\right), \quad (x,t) \in G \times (0,\infty), \\ &D^\nu u(x,t) = 0, \quad (x,t) \in \Gamma \times (0,\infty), \quad |\nu| \le m, \\ &u(x,t) = u_0(x,t), \quad (x,t) \in G \times [-r,0], \end{aligned} \tag{4.6}$$

where $G \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary $\partial G = \Gamma$, $\mathbb{A}_*(x,D) = \sum_{|\nu|\le 2m} a_\nu(x) D^\nu u$ is a strong elliptic operator with coefficients $a_\nu \in C^{2m}(\overline{G})$, $F : \mathbb{R}_+ \times \mathcal{C}([-r,0], L^2(G)) \times L^2(G) \to L^2(G)$, is a given function, $\rho\colon \Delta \times \mathcal{C}([-r,0], L^2(G)) \to L^2(G)$ is a continuous function and $g \in C_b^1(\mathbb{R}_+, \mathbb{R})$. We define the operator $A\colon D(A) \subset L^2(G) \to L^2(G)$ by

$$Au = -\mathbb{A}_*(\cdot, D)u \quad \forall u \in D(A) = H^{2m}(G) \cap H_0^m(G).$$

In [19] (Theorem 7.3.7) we have the following theorem.

Theorem 4.2. *Under the assumption that $\mathbb{A}_*$ is a strong elliptic operator with smooth coefficients, then the operator A generates an analytic semigroup on L^2. Moreover, the semigroups $(S(t))_{t\ge 0}$ associated to A is equicontinuous.*

For every $t \in \mathbb{R}_+$, we define $u(t) = u(\cdot, t)$. Hence the problem (4.6) can be rewritten as follows:

$$\begin{aligned} &u'(t) - Au = \int\limits_0^t g(t-s)Au(s)ds + F\left(t, u_t, \int\limits_0^t \rho(t,s,u_s)ds\right), \quad t \in \mathbb{R}_+, \\ &u(t) = u_0(t), \quad t \in [-r,0]. \end{aligned} \tag{4.7}$$

If we assume that F and ρ satisfied the conditions $(\mathcal{H}_4)$ – $(\mathcal{H}_6)$ of the Theorem 3.2, the problem (4.7) has at least one mild solution.

For examples if F is compact and $\rho(t,s,\cdot)$ is Lipschitz function, then $(\mathcal{H}_4)$ – $(\mathcal{H}_6)$ hold.

For examples if ρ is compact and $F(t,\cdot,\cdot)$ is Lipschitz function, then $(\mathcal{H}_4)$ – $(\mathcal{H}_6)$ hold.

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