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HOMOGENEITY-BASED EXPONENTIAL STABILITY ANALYSIS FOR CONFORMABLE FRACTIONAL-ORDER SYSTEMS

АНАЛІЗ ЕКСПОНЕНЦІАЛЬНОЇ СТАБІЛЬНОСТІ НА ОСНОВІ ОДНОРІДНОСТІ ДЛЯ КОНФОРМНИХ СИСТЕМ ДРОБОВОГО ПОРЯДКУ

We study the exponential stability of homogeneous fractional time-varying systems, and the existence of Lyapunov homogeneous function for the conformable fractional homogeneous systems. We also prove that local and global behaviors are similar. A numerical example is given to illustrate the efficiency of the obtained results.

Досліджено експоненціальну стійкість однорідних дробових систем, що змінюються з часом, і встановлено існування однорідної функції Ляпунова для конформних дробових однорідних систем. Крім того, доведено, що локальна та глобальна поведінка таких систем є однаковими. Для ілюстрації ефективності отриманих результатів наведено числовий приклад.

1. Introduction. Fractional calculus is a generalization of classical differential. These fractional operators effectively model certain real world phenomena, especially when the dynamics are affected by inherent constraints in the system. Indeed, the fractional differential equations are used to modelling many phenomena in biology, physics [2, 11, 27, 30]. We can find many definitions for fractional derivatives and fractional integrals, such as Riemann – Liouville, Caputo, Hadamard, Riesz, Grnwald – Letnikov, Marchaud, etc. (see, for example, [18, 22] and the references therein).

Another basic concept, defined by Khalil et al. [17], called the conformable fractional derivative whose most properties agree with Newton derivative and can be utilized to solve local fractal-type differential equations more effectively, later developed by T. Abdeljawad [4]. Moreover, Abdeljawad gave the fractional chain rule, the fractional integration by parts formulas, the fractional power series expansion, and the fractional Laplace transforms definition. Then in a short time, many papers provided mathematical models in the structure of which conformable fractional derivatives have been used [1, 3, 5 – 7, 9, 10, 12, 20, 26, 29]. Since the stability of such systems is very important, Rezazadeh et al. [28] studied the stability of conformable fractional linear differential equations systems for the first time. As the case of classical differential the second method or the direct method of Lyapunov plays an important role in the study of the stability of systems because it does not require any knowledge of the form of the solutions, therefore it is ideal for dealing with nonlinear systems. The method uses a complementary function called the Lyapunov function to describe the asymptotic behavior of solutions of differential equations. The Lyapunov function method is known to be a tool used in stability analysis [21, 25].

In the classical derivatives, homogeneous vector fields play a prominent role in various aspects of nonlinear systems and in control theory which has been introduced by Rothschild and Stein [24]. Homogeneous systems offer many desirable properties. Due to homogeneity, asymptotic stability of the origin implies global asymptotic stability as well as the existence of a C^1 Lyapunov function

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which is also homogeneous [23]. Many approaches in homogeneous system design rely on this theory [8, 13 – 16, 19, 23]. It is therefore interesting to study the homogeneity in the conformable fractional derivative system. In this paper, we introduce the homogeneity in the conformable fractional derivative system and study the stability of this type of system as well as the existence of a homogeneous Lyapunov function.

This paper is organized as follows. The preliminary is given in Section 2. The property for some class of homogeneous time-varying system is presented in Section 3. The existence of homogeneous Lyapunov function is given in Section 4. In Section 5, an example is presented to illustrate the results.

2. Preliminary. Prior to presenting the main results, we recall and present some definitions and theorems which will be used intensively in our study.

Definition 2.1. *Given a function f defined on $[a,\infty)$. Then the conformable fractional derivative of a function f of order $0<\alpha\le 1$ at $t>0$ was defined by*

$$T_\alpha f(t) := \lim_{\varepsilon\to 0}\frac{f\big(t+\varepsilon t^{1-\alpha}\big)-f(t)}{\varepsilon}.$$

If f is α-differentiable in some $(0,a)$, $a>0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define $f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t)$.

Considering the following time-varying conformable fractional derivative nonlinear system:

$$T_\alpha x(t) = f(t,x(t)), \quad x(t_0)=x_0, \tag{2.1}$$

where $\alpha\in(0,1]$, $x(t)\in\mathbb{R}^n$ is the state vector, $f\colon \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is a continuous function and $f(t,0)=0$. Suppose that the function f is smooth enough to guarantee the existence of a global solution $x(t)=x(t,t_0,x_0)$ of system (2.1) for each initial condition (t_0,x_0).

2.1. Notion of stability.

Definition 2.2. *The origin of the system* (2.1) *is said to be fractional exponentially stable if*

$$\|x(t)\|\le M\|x_0\|E_\alpha(-s,t-t_0), \quad t\ge t_0,$$

where $0<\alpha\le s, M>0$, and $E_\alpha(s,t)=\exp\left(s\dfrac{t^\alpha}{\alpha}\right)$.

Definition 2.3. *The origin of the system* (2.1) *is said to be stable, if, for $\varepsilon>0$, there exists $\delta>0$ such that the origin of the system* (2.1) *satisfies $\|x(t)\|<\varepsilon$ for all $t>t_0$ when $\|x_0\|<\delta$. The origin of the system* (2.1) *is asymptotically stable if it is stable and it satisfies $\lim_{t\to+\infty}x(t)=0$.*

2.2. Homogeneity.

Definition 2.4. *For any $r=(r_1,\ldots,r_n)\in\mathbb{R}^n$ with $r_i>0$, $i\in\{1,\ldots,n\}$, and $\lambda>0$, the dilation vector of $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$ associated with weight r is defined as*

$$\Delta_\lambda(x)=(\lambda^{r_1}x_1,\ldots,\lambda^{r_n}x_n).$$

The homogeneous norm of $x\in\mathbb{R}^n$ associated with weight r is defined as

$$\|x\|_r=\left(\sum_{i=1}^n |x|^{\frac{\varrho}{r_i}}\right)^{\frac{1}{\varrho}}, \qquad \varrho=\prod_{i=1}^n r_i.$$

An important property is that $\|\Delta_\lambda(x)\|_r = \lambda\|x\|_r$.

The homogeneous norm is not a standard norm, because the triangle inequality is not satisfied. However, there exist $\overline{\sigma} > 0$ and $\underline{\sigma} > 0$ such that

$$\underline{\sigma}\|x\|_r \leq \|x\| \leq \overline{\sigma}\|x\|_r.$$

Definition 2.5. (1) *A continuous function* $h\colon R^n \longrightarrow \mathbb{R}$ *is* r*-homogeneous of degree* k *if* $h(\Delta_\lambda(x)) = \lambda^k h(x)$ *for all* $\lambda > 0$ *and* $x \in \mathbb{R}^n$.

(2) *We say that a continuous function* $f : \mathbb{R}^n \longrightarrow R^n$ *is* r*-homogeneous of degree* k *if each* f_i, $i \in \{1, \ldots, n\}$, *is* r*-homogeneous of degree* $k + r_i$, *i.e.*, $f(\Delta_\lambda(x)) = \lambda^k \Delta_\lambda(f(x))$ *for all* $\lambda > 0$ *and* $x \in \mathbb{R}^n$.

(3) *The system* (2.1) *is* r*-homogeneous of degree* k *if the vector field* f *is* r*-homogeneous of degree* k.

3. Main results.

Lemma 3.1. *Let* $x(t)$ *be a solution of the* r*-homogeneous system* (2.1) *with the degree* k *for an initial condition* $x_0 \in \mathbb{R}^n$. *Then* $y(t) = \Delta_\lambda\big(x(\lambda^{\frac{k}{\alpha}} t)\big)$ *for all* $t \geq t_0$ *and* $\lambda > 0$ *with the initial condition* $y_0 = \Delta_\lambda(x_0)$ *is a solution of the following modification system*:

$$T_\alpha y(t) = f\big(\lambda^{\frac{k}{\alpha}} t, y(t)\big), \quad t \geq t_0. \tag{3.1}$$

Proof. For $i = 1, \ldots, n,\ y_i(t) = \lambda^{r_i} x_i(\lambda^{\frac{k}{\alpha}} t)$ for all $t \geq t_0$. We have

$$\begin{aligned}
T_\alpha y_i(t) &= T_\alpha\big[\lambda^{r_i} x_i(\lambda^{\frac{k}{\alpha}} t)\big] = \lambda^{r_i} T_\alpha\big[x_i(\lambda^{\frac{k}{\alpha}} t)\big] \\
&= \lambda^{r_i} \lim_{\varepsilon\to 0} \frac{x\Big(\lambda^{\frac{k}{\alpha}}(t + \varepsilon t^{1-\alpha})\Big) - x(\lambda^{\frac{k}{\alpha}} t)}{\varepsilon} \\
&= \lambda^{r_i} \lim_{\varepsilon\to 0} \frac{x\Big(\lambda^{\frac{k}{\alpha}} t + \varepsilon \lambda^{\frac{k}{\alpha}} t^{1-\alpha}\Big) - x(\lambda^{\frac{k}{\alpha}} t)}{\varepsilon} \\
&= \lambda^{r_i} \lim_{\varepsilon\to 0} \frac{x\Big(\lambda^{\frac{k}{\alpha}} t + \varepsilon \lambda^{k} (\lambda^{\frac{k}{\alpha}} t)^{1-\alpha}\Big) - x(\lambda^{\frac{k}{\alpha}} t)}{\varepsilon} \\
&= \lambda^{r_i + k} \lim_{\eta\to 0} \frac{x\Big(\lambda^{\frac{k}{\alpha}} t + \eta (\lambda^{\frac{k}{\alpha}} t)^{1-\alpha}\Big) - x(\lambda^{\frac{k}{\alpha}} t)}{\eta} \\
&= \lambda^{r_i + k} T_\alpha x_i(\lambda^{\frac{k}{\alpha}} t), \quad \eta = \varepsilon\lambda^k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
T_\alpha y(t) &= \Big(T_\alpha y_1(t), \ldots, T_\alpha y_n(t)\Big)^{\mathrm{T}} \\
&= \Big(\lambda^{r_1 + k}\, T_\alpha x_1(\lambda^{\frac{k}{\alpha}} t), \ldots, \lambda^{r_n + k}\, T_\alpha x_n(\lambda^{\frac{k}{\alpha}} t)\Big)^{\mathrm{T}} \\
&= \Big(\lambda^{r_1 + k} f_1\big(\lambda^{\frac{k}{\alpha}} t, x(\lambda^{\frac{k}{\alpha}} t)\big), \ldots, \lambda^{r_n + k} f_n\big(\lambda^{\frac{k}{\alpha}} t, x(\lambda^{\frac{k}{\alpha}} t)\big)\Big)^{\mathrm{T}}
\end{aligned}$$

$$= \lambda^k \Big(\lambda^{r_1} f_1\big(\lambda^{\frac{k}{\alpha}} t, x(\lambda^{\frac{k}{\alpha}} t)\big), \ldots, \lambda^{r_n} f_n(\lambda^{\frac{k}{\alpha}} t, x(\lambda^{\frac{k}{\alpha}} t))\Big)^{\mathrm{T}}$$

$$= \lambda^k \Delta_\lambda \Big(f\big(\lambda^{\frac{k}{\alpha}} t, x(\lambda^{\frac{k}{\alpha}} t)\big)\Big) = f\Big(\lambda^{\frac{k}{\alpha}} t, \Delta_\lambda\big(x(\lambda^{\frac{k}{\alpha}} t)\big)\Big) = f\big(\lambda^{\frac{k}{\alpha}} t, y(t)\big).$$

Then $y(t) = \Delta_\lambda\big(x(\lambda^{\frac{k}{\alpha}} t)\big)$ is a solution of system (3.1).

***Remark* 3.1.** (1) Let $y(t)$ be a solution of the r-homogeneous system (3.1) with the degree k for an initial condition $y_0 \in \mathbb{R}^n$. For $\lambda > 0$, the system (2.1) has a solution $x(t) = \Delta_{\lambda^{-1}}\big(y(\lambda^{-\frac{k}{\alpha}} t)\big)$ for all $t \geq t_0$ with the initial condition $x_0 = \Delta_{\lambda^{-1}}(y_0)$.

(2) An advantage of homogeneous systems described by fractional nonautonomous system is that any of its solution can be obtained from another solution under the dilation rescaling and a suitable time reparametrization.

(3) For the conformable fractional derivative

$$T_\alpha x(t) = f(x(t)), \quad x(t_0) = x_0. \tag{3.2}$$

If $x(t)$ is a solution of the r-homogeneous system (3.2) with the degree k for an initial condition $x_0 \in \mathbb{R}^n$, then $y(t) = \Delta_\lambda\big(x(\lambda^{\frac{k}{\alpha}} t)\big)$ for $\lambda > 0$, and $t \geq t_0$ is also a solution of (3.2) with the initial condition $y_0 = \Delta_\lambda(x_0)$.

Theorem 3.1. *If the system* (2.1) *and the system* (3.1) *are r-homogeneous with degree k, then the system* (2.1) *is globally exponentially stable if and only if the system* (3.1) *is globally exponentially stable.*

Proof. Let $x(t, t_0, x_0)$ is a solution of system (2.1) which is globally exponentially stable. So,

$$\|x(t, t_0, x_0)\|_r \leq M \|x_0\|_r E_\alpha(-s, t - t_0).$$

From Lemma 3.1 we $y(t, t_0, y_0) = \Delta_\lambda\big(x(\lambda^{\frac{k}{\alpha}} t, \lambda^{\frac{k}{\alpha}} t_0, x_0)\big)$ with $y_0 = \Delta_\lambda(x_0)$ is a solution of the system (3.1). We get

$$\begin{aligned}
\big\|y(t, t_0, y_0)\big\|_r &= \lambda \big\|x(\lambda^{\frac{k}{\alpha}} t, \lambda^{\frac{k}{\alpha}} t_0, x_0)\big\|_r \\
&\leq M\lambda \|x_0\|_r E_\alpha(-s, \lambda^{\frac{k}{\alpha}} t - \lambda^{\frac{k}{\alpha}} t_0) \leq M \|y_0\|_r \exp\left(-s \frac{(\lambda^{\frac{k}{\alpha}} t - \lambda^{\frac{k}{\alpha}} t_0)^\alpha}{\alpha}\right) \\
&\leq M \|y_0\|_r \exp\left(-s\lambda^k \frac{(t - t_0)^\alpha}{\alpha}\right) \leq M \|y_0\|_r E_\alpha(-s\lambda^k, t - t_0).
\end{aligned}$$

It follows that the system (3.1) is globally exponentially stable.

Inversely, using the same technique, we proved that if the system (3.1) is globally exponentially stable, then the system (2.1) is also globally exponentially stable.

Theorem 3.2. *We suppose that the r-homogeneous system* (2.1) *is locally exponentially stable in $B_{\rho,r} = \big\{x \in \mathbb{R}^n, \|x\|_r \leq \rho\big\}$ with $0 < \rho < \infty$. Then the system* (2.1) *is globally exponentially stable.*

Proof. For all $x_0 \in B_{\rho,r}$ and $t \geq t_0$, we have

$$\|x(t, t_0, x_0)\| \leq M \|x_0\|_r E_\alpha(-s, t - t_0) \quad \forall t \geq t_0.$$

Let $\overline{x}_0 \notin B_{\rho,r}$, then there exists $x_0 \in B_{\rho,r}$ such that $\|x_0\|_r = \rho$ and $\overline{x}_0 = \Delta_\lambda(x_0)$ with $\lambda = \|\overline{x}_0\|_r \rho^{-1}$. By using Lemma 3.1, we obtain that

$$\overline{x}(t, t_0, \overline{x}_0) = \Delta_\lambda\big(x(\lambda^{\frac{k}{\alpha}} t, \lambda^{\frac{k}{\alpha}} t_0, x_0)\big)$$

is a solution of system (2.1). Moreover, we get

$$\begin{aligned}
\big\|\overline{x}(t, t_0, \overline{x}_0)\big\|_r &= \big\|\Delta_\lambda(x(\lambda^{\frac{k}{\alpha}} t, \lambda^{\frac{k}{\alpha}} t_0, x_0))\big\|_r = \lambda\big\|x(\lambda^{\frac{k}{\alpha}} t, \lambda^{\frac{k}{\alpha}} t_0, x_0)\big\|_r \\
&\le M\lambda\|x_0\|_r E_\alpha(-s, \lambda^{\frac{k}{\alpha}} t - \lambda^{\frac{k}{\alpha}} t_0) \le M\|\overline{x}_0\|_r \exp\left(-s\frac{(\lambda^{\frac{k}{\alpha}} t - \lambda^{\frac{k}{\alpha}} t_0)^\alpha}{\alpha}\right) \\
&\le M\|\overline{x}_0\|_r \exp\left(-s\lambda^k \frac{(t-t_0)^\alpha}{\alpha}\right) \le M\|\overline{x}_0\|_r E_\alpha(-s\lambda^k, t - t_0).
\end{aligned}$$

Therefore, for all $t \ge t_0$, the system (2.1) is globally exponentially stable.

Theorem 3.3. *If the r-homogeneous system* (3.1) *is locally exponentially stable for all $x_0 \in B_{\rho,r}$ and for a fixed $0 < \rho < \infty$, then the system* (3.1) *is globally exponentially stable.*

Proof. We suppose that the system (3.1) is locally exponentially stable. Using the Theorem 3.1, the system (2.1) is locally exponentially stable. It follows that the system (2.1) is globally exponentially stable. Therefore, the system (3.1) is globally exponentially stable.

In [6], the authors present the following definition of local fractional derivative using kernels.

Definition 3.1. *Let $\kappa\colon [a,b] \to \mathbb{R}$ be a continuous nonnegative map such that $\kappa(t) \neq 0$, whenever $t > a$. Given a function $f\colon [a,b] \to \mathbb{R}$ and $\alpha \in (0,1)$ is a real, we say that f is α-differentiable at $t > a$, with respect to kernel κ, if the limit*

$$f^{(\alpha)}(t) := \lim_{\varepsilon \to 0} \frac{f\big(t + \varepsilon\kappa(t)^{1-\alpha}\big) - f(t)}{\varepsilon}$$

exists. The α-derivative at $t = a$ is defined by

$$f^{(\alpha)}(a) := \lim_{t \to a^+} f^{(\alpha)}(t),$$

if the limit exists.

Let us consider the system

$$x^{(\alpha)}(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{3.3}$$

where $\alpha \in (0,1]$, $x(t) \in \mathbb{R}^n$ is the state vector, $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and $f(t,0) = 0$. Suppose that the function f is smooth enough to guarantee the existence of a global solution $x(t) = x(t, t_0, x_0)$ of system (3.3) for each initial condition (t_0, x_0).

***Assumption* 3.1.** We assume that κ is a positive homogeneous function of degree ι:

$$\kappa(\lambda t) = \lambda^\iota \kappa(t) \quad \text{for all} \quad \lambda > 0.$$

Lemma 3.2. *Let $x(t)$ be a solution of the r-homogeneous system* (3.3) *with the degree k for an initial condition $x_0 \in \mathbb{R}^n$ and κ verifies Assumption* 3.1. *Then $y(t) = \Delta_\lambda\Big(x(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)\Big)$ for all $t \ge t_0$ and $1 + \iota(\alpha - 1) \neq 0$ with the initial condition $y_0 = \Delta_\lambda(x_0)$ is a solution of the modification system*

$$y^\alpha(t) = f\Big(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t, y(t)\Big), \quad t \ge t_0. \tag{3.4}$$

Proof. For $i = 1, \ldots, n,\ y_i(t) = \lambda^{r_i} x_i(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)$ for all $t \geq t_0$. We have

$$\begin{aligned}
y_i^{\alpha}(t) &= \left[\lambda^{r_i} x_i(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)\right]^{\alpha} = \lambda^{r_i}\left[x_i(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)\right]^{\alpha} \\
&= \lambda^{r_i} \lim_{\varepsilon\to 0} \frac{x\Big(\lambda^{\frac{k}{1+\iota(\alpha-1)}}(t + \varepsilon\kappa^{1-\alpha}(t))\Big) - x(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)}{\varepsilon} \\
&= \lambda^{r_i} \lim_{\varepsilon\to 0} \frac{x\Big(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t + \varepsilon\lambda^{\frac{k}{1+\iota(\alpha-1)}}\kappa^{1-\alpha}(t)\Big) - x(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)}{\varepsilon} \\
&= \lambda^{r_i} \lim_{\varepsilon\to 0} \frac{x\Big(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t + \varepsilon\lambda^{k}(\lambda^{\frac{k\iota}{1+\iota(\alpha-1)}}\kappa(t))^{1-\alpha}\Big) - x(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)}{\varepsilon} \\
&= \lambda^{r_i+k} \lim_{\eta\to 0} \frac{x\Big(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t + \eta\kappa^{1-\alpha}(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)\Big) - x(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)}{\eta} \\
&= \lambda^{r_i+k} x_i^{\alpha}(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t), \quad \eta = \varepsilon\lambda^k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
y^{\alpha}(t) &= \big(y_1^{\alpha}(t), \ldots, y_n^{\alpha}(t)\big)^{\mathrm{T}} \\
&= \Big(\lambda^{r_1+k}\, x_1^{\alpha}\big(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t\big), \ldots, \lambda^{r_n+k}\, x_n^{\alpha}\big(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t\big)\Big)^{\mathrm{T}} \\
&= \Big(\lambda^{r_1+k} f_1\Big(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t, x(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)\Big), \ldots, \lambda^{r_n+k} f_n\Big(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t, x(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)\Big)\Big)^{\mathrm{T}} \\
&= \lambda^k\Big(\lambda^{r_1} f_1\Big(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t, x(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)\Big), \ldots, \lambda^{r_n} f_n\Big(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t, x(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)\Big)\Big)^{\mathrm{T}} \\
&= \lambda^k \Delta_\lambda\Big(f\big(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t, x(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)\big)\Big) \\
&= f\Big(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t, \Delta_\lambda\big(x(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)\big)\Big) = f(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t, y(t)).
\end{aligned}$$

Then $y(t) = \Delta_\lambda\big(x(\lambda^{\frac{k}{1+\iota(\alpha-1)}} t)\big)$ is a solution of system (3.4).

4. Homogeneous Lyapunov function. Let us consider the following conformable fractional derivative system:

$$T_\alpha x(t) = f(x(t)), \quad x(t_0) = x_0. \tag{4.1}$$

In the case of the classical derivative, if the origin is asymptotically stable for a r-homogeneous system of class C^1, then there exists a positive definite r-homogeneous Lyapunov function V which satisfies $fV(x) < 0\ \forall x \neq 0$. In the case where f is homogeneous and uniquely continuous, Rosier [23] showed the following results.

Theorem 4.1 [23]. *Let f be a function satisfying*:

(i) $f \in C(\mathbb{R}^n, \mathbb{R}^n), f(0) = 0,$

(ii) *f is homogeneous: there exist $(r_1,\ldots,r_n)\in((0,+\infty))^n$ and $\tau\in\mathbb{R}$ such that*

$$f_i(t^{r_1}x_1,\ldots,t^{r_n}x_n)=t^{\tau+r_i}f_i(x_1,\ldots,x_n)\quad \forall x=(x_i)_{i=1,n}\in\mathbb{R}^n\backslash\{0\}\quad \forall t>0,$$

(iii) *the trivial solution $x=0$ of system $\dot{x}=f(x)$ is locally asymptotically stable.*

Let p be a positive integer and k be a real number larger than $p\cdot\max_{1\le i\le n} r_i$. Then there exists a function $\bar{V}:\mathbb{R}^n\to\mathbb{R}$ such that:

(1) $\bar{V}\in C^p(\mathbb{R}^n,\mathbb{R})\cap C^\infty(\mathbb{R}^n\backslash\{0\},\mathbb{R})$,

(2) $\bar{V}(0)=0, \bar{V}(x)>0$ *for all* $x\neq 0$ *and* $\bar{V}(x)\to+\infty$ *as* $\|x\|\to+\infty$,

(3) $\bar{V}$ *is homogeneous:* $\bar{V}(t^{r_1}x_1,\ldots,t^{r_n}x_n)=t^k\bar{V}(x)\ \forall x=(x_i)_{i=1,n}\in\mathbb{R}^n\backslash\{0\}\ \forall t>0$,

(4) $\nabla\bar{V}(x)\cdot f(x)<0\ \forall x\neq 0$.

Referring to the above theorem, we make the following lemma.

Lemma 4.1. *Let the system* (4.1) *be a r-homogeneous system with degree τ. If there exists a scalar function $V\in C^\infty(\mathbb{R}^n,\mathbb{R})$ such that*:

(i) $V(0)=0$, $V(x)>0$ *for all* $x\neq 0$ *and* $V(x)\to+\infty$ *as* $\|x\|\to+\infty$,

(ii) $\langle\nabla V(x),f(x)\rangle<0\ \forall x\neq 0$,

then there exists a r-homogeneous Lyapunov function $\overline{V}\in C^p(\mathbb{R}^n,\mathbb{R})$ with degree k, where $p<\dfrac{k}{\max\{r_i\mid 1<i<n\}}$, *such that*:

(1) $\overline{V}(0)=0$, $\overline{V}(x)>0$ *for all* $x\neq 0$ *and* $\overline{V}(x)\to+\infty$ *as* $\|x\|\to+\infty$,

(2) $T_\alpha\overline{V}(x)<0$ *for all* $x\in\mathbb{R}^n\backslash\{0\}$.

Proof. For the proof see [23].

Remark **4.1.** The derivative of the dilation vector is given by

$$T_\alpha\big(\Delta_\lambda(x)\big)=\big(\lambda^{r_1}T_\alpha x_1,\ldots,\lambda^{r_n}T_\alpha x_n\big)^{\mathrm{T}}=\Delta_\lambda(T_\alpha x).$$

Remark **4.2.** The homogeneity of the functions $V:\mathbb{R}^n\to\mathbb{R}$ is verified for the conformable derived $T_\alpha V$. We have

$$V\big(\Delta_\lambda(x)\big)=V\big(\lambda^{r_1}x_1,\ldots,\lambda^{r_n}x_n\big)=\lambda^kV(x)\quad\forall x=(x_i)_{i=1,n}\in\mathbb{R}^n\backslash\{0\}\quad\forall\lambda>0,$$

where $r_1,\ldots,r_n$ are some positive real numbers, and k is a nonnegative real number.

Hence, it is clear that

$$T_\alpha V(\Delta_\lambda(x))=\lambda^kT_\alpha V(x)\quad\forall x=(x_i)_{i=1,n}\in\mathbb{R}^n\backslash\{0\}\quad\forall\lambda>0.$$

This property implies that the global behavior of trajectories could be evaluated based on their behavior on S^{n-1}, where $S^{n-1}:=\big\{x\in\mathbb{R}^n\backslash\{0\}\mid\|x\|_r=1\big\}$.

5. Numerical example. Let us consider the system

$$\begin{aligned}
T_\alpha x_1&=x_1^3+x_1x_2^2-\frac{\big(2x_1^4-x_2^4-x_3^8\big)+\big(|2x_1^4-x_2^4-x_3^8|^3+x_1^{12}\big)^{\frac{1}{3}}}{x_1},\\
T_\alpha x_2&=-x_2^3+x_1^2x_2-x_3^4\frac{\big(2x_1^4-x_2^4-x_3^8\big)+\big(|2x_1^4-x_2^4-x_3^8|^3+x_1^{12}\big)^{\frac{1}{3}}}{x_1^3},\\
T_\alpha x_2&=-x_3^5+x_2x_3\frac{\big(2x_1^4-x_2^4-x_3^8\big)+\big(|2x_1^4-x_2^4-x_3^8|^3+x_1^{12}\big)^{\frac{1}{3}}}{x_1^3}.
\end{aligned}\tag{5.1}$$

Suppose that $r = \left(1, 1, \dfrac{1}{2}\right)$, the system (5.1) is r-homogeneous with degree 2.

Let the Lyapunov function $V(t, x_1, x_2)$ by

$$V(x_1, x_2, x_3) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{4}x_3^4.$$

V is r-homogeneous with degree 2. The α-derivative of V along the trajectories of system (5.1) is

$$\begin{aligned} T_\alpha V(x_1, x_2, x_3) = x_1 &\left(x_1^3 + x_1 x_2^2 - \frac{(2x_1^4 - x_2^4 - x_3^8) + \left(|2x_1^4 - x_2^4 - x_3^8|^3 + x_1^{12}\right)^{\frac{1}{3}}}{x_1} \right) \\ &+ x_2 \left(-x_2^3 + x_1^2 x_2 - x_3^4 \frac{(2x_1^4 - x_2^4 - x_3^8) + \left(|2x_1^4 - x_2^4 - x_3^8|^3 + x_1^{12}\right)^{\frac{1}{3}}}{x_1^3} \right) \\ &+ x_2 \left(-x_3^5 + x_2 x_3 \frac{(2x_1^4 - x_2^4 - x_3^8) + \left(|2x_1^4 - x_2^4 - x_3^8|^3 + x_1^{12}\right)^{\frac{1}{3}}}{x_1^3} \right) \\ &\le 2x_1^4 - x_2^4 - x_3^8 - \left(2x_1^4 - x_2^4 - x_3^8 + \left(|2x_1^4 - x_2^4 - x_3^8|^3 + x_1^{12}\right)^{\frac{1}{3}}\right) \\ &\le -\left(|2x_1^4 - x_2^4 - x_3^8|^3 + x_1^{12}\right)^{\frac{1}{3}} < 0. \end{aligned}$$

Hence, the origin is asymptotically stable.

6. Conclusion. In this paper, we study a class of homogeneous fractional systems and proved that the uniform exponential stability of the homogeneous fractional system and the modified system are similar. Then we prove the existence of a Lyapunov homogeneous function of the homogeneous fractional systems. The notion of finite-time stability of homogeneous systems has been introduced in the classical derivative, and there are many interesting results for this type of stability. The finite-time stability of conformable homogeneous systems of fractional order can be considered as a possible direction for future research.

The author states that there is no conflict of interest.

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