DOI: 10.3842/umzh.v75i10.7293

UDC 517.9

Luis Alejandro Molano Molano[1] (Escuela de Matemáticas y Estadística Universidad Pedagógica y Tecnológica de Colombia, Duitama)

INVERSE PROBLEMS, SOBOLEV – CHEBYSHEV POLYNOMIALS AND ASYMPTOTICS

ОБЕРНЕНІ ЗАДАЧІ, ПОЛІНОМИ СОБОЛЄВА – ЧЕБИШОВА ТА АСИМПТОТИКА

Let (u, v) be a pair of quasidefinite and symmetric linear functionals with $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ as respective sequences of monic orthogonal polynomial (SMOP). We define a sequence of monic polynomials $\{R_n\}_{n\geq 0}$ as follows:

$$\frac{P'_{n+2}(x)}{n+2} + b_n \frac{P'_n(x)}{n} - Q_{n+1}(x) = d_n R_{n-1}(x), \quad n \geq 1.$$

We give necessary and sufficient conditions for $\{R_n\}_{n\geq 0}$ to be orthogonal with respect to a quasidefinite linear functional w. In addition, we consider the case where $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are monic Chebyshev polynomials of the first and second kinds, respectively, and study the relative outer asymptotics of Sobolev polynomials orthogonal with respect to the Sobolev inner product

$$\langle p, q\rangle_S = \int\limits_{-1}^{1} pq(1-x^2)^{-1/2}dx + \lambda_1 \int\limits_{-1}^{1} p'q'(1-x^2)^{1/2}dx + \lambda_2 \int\limits_{-1}^{1} p''q''d\mu(x),$$

where μ is a positive Borel measure associated with w and $\lambda_1, \lambda_2 > 0,$ λ_2 is a linear polynomial of λ_1.

Нехай (u, v) — пара квазівизначених симетричних лінійних функціоналів, в яких $\{P_n\}_{n\geq 0}$ і $\{Q_n\}_{n\geq 0}$ є відповідними послідовностями монічних ортогональних поліномів (ПМОП). Послідовність монічних поліномів $\{R_n\}_{n\geq 0}$ визначено таким чином:

$$\frac{P'_{n+2}(x)}{n+2} + b_n \frac{P'_n(x)}{n} - Q_{n+1}(x) = d_n R_{n-1}(x), \quad n \geq 1.$$

Наведено необхідні та достатні умови для того, щоб послідовність $\{R_n\}_{n\geq 0}$ була ортогональною до квазівизначеного лінійного функціонала w. Крім того, розглянуто випадок, коли $\{P_n\}_{n\geq 0}$ і $\{Q_n\}_{n\geq 0}$ — монічні поліноми Чебишова першого і другого роду відповідно, та вивчено відносну зовнішню асимптотику поліномів Соболєва, ортогональних щодо соболєвського скалярного добутку

$$\langle p, q\rangle_S = \int\limits_{-1}^{1} pq(1-x^2)^{-1/2}dx + \lambda_1 \int\limits_{-1}^{1} p'q'(1-x^2)^{1/2}dx + \lambda_2 \int\limits_{-1}^{1} p''q''d\mu(x),$$

де μ — додатна борелівська міра, пов'язана з w і $\lambda_1, \lambda_2 > 0,$ λ_2 — лінійний поліном від λ_1.

1. Introduction. In the constructive theory of orthogonal polynomials on the real line, there are two fundamental problems. If u is a quasidefinite linear functional and ϕ is a function in the dual space of polynomials with complex coefficients, to seek conditions in order to the functional $v := \phi(u)$ is also quasidefinite, is said to be a *direct problem*. For instance, the so-called canonical spectral transformations (Christoffel, Uvarov and Geronimus), of linear functionals have been extensively analyzed in this direction (see [7, 12, 13, 26 – 28]).

On the other hand, relations such as

[1] E-mail: luis.molano01@uptc.edu.co.

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$$\sum_{k=0}^{m_1} r_{k,n} P_{n+p-k}^{(p)}(x) = \sum_{k=0}^{m_2} s_{k,n} Q_{n+q-k}^{(q)}(x), \tag{1.1}$$

where $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are sequences of monic polynomials with real coefficients, at least one of them orthogonal with respect to a quasidefinite linear functional, $m_1,\ m_2,\ p$ and q non-negative integers such that $p \geq m_1,\ q \geq m_2,$ and the sequences $\{r_{k,n}\}_{n\geq 0}$ and $\{s_{k,n}\}_{n\geq 0}$ satisfy certain regularity conditions, have been widely studied since the early 90s. Relations like (1.1) are related with the so-called *inverse problems*. Namely, given the relation (1.1), if the sequence $\{P_n\}_{n\geq 0}$ is orthogonal with respect to a quasidefinite linear functional $u,$ look for necessary and sufficient conditions such that $\{Q_n\}_{n\geq 0}$ is also orthogonal with respect to some quasidefinite linear functional, is known in literature as an *inverse problem*. A subsequent issue is to search for an algebraic relation between the linear functionals. Inverse problems are addressed in the papers [3] and [21] for $m_1,\ m_2$ and $q = 0.$ In [24] relation (1.1) is studied for $p = q = 0,$ and in [16] and [17] an inverse problem is studied in a general way. Recently, in [4], the case $p = q = 0,\ m_1 = 3,\ m_2 = 1$ is studied. Algebraic relations like (1.1) also have a close connection to the so-called *Sobolev orthogonality*. If $\mu_i,\ i = 0, \ldots, k,$ are positive Borel measures supported on infinite subsets of the real line with finite moments, the inner product on the linear space of polynomials

$$\langle p, q\rangle_S = \sum_{i=0}^{k} \int_{\mathbb{R}} p^{(i)}(x) q^{(i)}(x) d\mu_i = \sum_{i=0}^{k} \left\langle p^{(i)}, q^{(i)} \right\rangle_{\mu_i} \tag{1.2}$$

is said to be a *Sobolev inner product*. This kind of inner products appears for the first time in the pioneer work [18] on an extremal problem related to smooth polynomial approximation. Applications of Sobolev orthogonality include spectral methods in numerical analysis for ODE and PDE. By means of the well-known concept of *coherent pair*, a significant connection between an algebraic relation like (1.1) and a nonstandard inner product like (1.2) can be established. This concept is introduced in the seminal work [15] where (1.2) is studied for $k = 1,\ \{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are associated with μ_0 and μ_1, respectively, and satisfying (1.1) with $p = 1,\ m_1 = 1,\ m_2 = q = 0.$ Numerous and relevant contributions have been published since then. The concept of coherent pair is extended in several ways and, in particular, asymptotic properties of Sobolev polynomials orthogonal with respect to $\langle p, q\rangle_S = \int_{\mathbb{R}} p(x)q(x)d\mu_0 + \lambda \int_{\mathbb{R}} p^{(m)}(x)q^{(m)}(x)d\mu_1,\ \lambda > 0,\ m \in \mathbb{N},$ are studied. The survey [20] is highly recommended as well as the contributions therein. Maybe, the importance of asymptotics for orthogonal polynomials lies in its applications: linear predictors in the theory of stochastic processes, random matrix theory, Fisher–Hartwig conjectures and Ising models, study of algorithms, entropy, among others. Regarding all those matters, the surveys [2, 19, 22, 23] are especially recommended. On the other hand, in [10] the relation

$$P_{n+1}^{[i]} + a_{n,1}^{[i]} P_n^{[i]} + a_{n,2}^{[i]} P_{n-1}^{[i]} + b_n(Q_{n+1} + c_n Q_n) = (1 + b_n) R_{n+1} + d_n R_n \tag{1.3}$$

is studied, where $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are sequences of monic polynomials orthogonal with respect to the quasidefinite linear functionals u, v and w, respectively, with $P_k^{[i]} := P_{k+i}^{(i)}/(k+1)_i$, $i = 0, 1$, and $a_n^{[i]} b_n c_n d_n \neq 0$, $n \geq 0$. Moreover, the functionals u and v are related through the rational relation $\rho(x)u = v$, where ρ is a monic polynomial. Among others, a rational relation between u and w is obtained, assuming u is semiclassical in the case $i = 1$. For the cases $i = 0, 1$, (1.3) can be seen as an extended coherence relation like (1.1) since hypothesis of the problem lead to both structure relations and three terms recurrence relations (in short TTRR), satisfied by the sequences. However, in the context of inverse problems, when a relation like (1.3) is considered, the degrees of polynomials in the desired rational relation are improved compared with those obtained through the results in [24] and starting from an extended coherence relation. In order to study some asymptotic properties for polynomials orthogonal with respect to (1.2) with $k > 1$, the aim of this work is to consider

$$\frac{P'_{n+2}(x)}{n+2} + b_n \frac{P'_n(x)}{n} - Q_{n+1}(x) = d_n R_{n-1}(x), \quad n \geq 1, \tag{1.4}$$

involving three sequences of monic orthogonal polynomials.

As far as we known, in the literature there are no neither studies of inverse problems suppose the use of more than two sequences of orthogonal polynomials nor asymptotic results involving at least three different nondiscrete measures in the respective Sobolev inner product. The structure of this manuscript is the following. In Section 2, we present the basic background. Section 3 deals with an inverse problem associated with (1.4) when $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are orthogonal. In Section 4, we study analytic properties of Sobolev polynomials orthogonal with respect to a Sobolev inner product associated to (1.4) and, finally, we address asymptotic properties of these Sobolev polynomials with respect to the Chebyshev polynomials of the first kind.

2. Preliminaries. Let $\mathbf{P}$ be the linear space of polynomials with real coefficients and $\mathbf{P}'$ be the algebraic dual of $\mathbf{P}$. As it is usual, $\langle u, p\rangle$ is the action of u on $p \in \mathbf{P}$. If $u \in \mathbf{P}'$ and $u_n := \langle u, x^n\rangle$, then u is said to be a *moment functional* associated with the *moment sequence* $\{u_n\}_{n\geq 0}$. In addition, u is **quasidefinite** if leading principal submatrices of the *Hankel matrix* $(u_{i+j})_{i,j=0}^{\infty}$ are nonsingular and it is **positive-definite** if $\langle u, \pi(x)\rangle > 0$ for every nonnegative and nonzero $\pi \in \mathbf{P}$. Finally, u is called **symmetric** if $u_{2n+1} = 0$ for $n \geq 0$. If u is positive-definite, then there exists a positive Borel measure μ supported on an infinite set $E \subseteq \mathbb{R}$ such that u has the integral representation $\langle u, p\rangle = \displaystyle\int_E p(x)d\mu(x)$, $p \in \mathbf{P}$. Given a quasidefinite linear functional $u \in \mathbf{P}'$, a bilinear form $\langle , \rangle_u : \mathbf{P} \times \mathbf{P} \to \mathbb{R}$ is defined in a natural way as $\langle p, q\rangle_u := \langle u, pq\rangle$. If u is positive definite then the bilinear form is an inner product on $\mathbf{P}$ and the induced norm will be represented as $\|p\|_\mu = \left\langle u, p^2\right\rangle^{1/2}$.

Theorem 2.1 (Favard's theorem, [6, Theorem 4.4]). *$\{P_n\}_{n\geq 0}$ is a SMOP with respect to a quasidefinite linear functional u if and only if there exist sequences $\{\beta_n\}_{n\geq 1}$ and $\{\gamma_n\}_{n\geq 1}$, with $\gamma_n \neq 0$ for $n \geq 1$, such that $P_0(x) = 1$, $P_1(x) = x - \beta_0$, and*

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \geq 1.$$

Moreover, for $n \geq 1$, $\beta_n = \dfrac{\left\langle u, xP_n^2\right\rangle}{\left\langle u, P_n^2\right\rangle}$ and $\gamma_n = \dfrac{\left\langle u, P_n^2\right\rangle}{\left\langle u, P_{n-1}^2\right\rangle}$.

The monic *Gegenbauer polynomials* $\left\{C_n^{(\alpha)}\right\}_{n\geq 0}$ are orthogonal with respect to

$$\langle p,q\rangle_{\alpha-1/2} := \int\limits_{-1}^{1} p(x)q(x)(1-x^2)^{\alpha-1/2}dx, \quad \alpha > -1/2. \tag{2.1}$$

The norm induced by (2.1) is defined as (see [25])

$$\left\|C_n^{(\alpha)}\right\|_{\alpha-1/2} = \frac{\pi}{2^{2\alpha+2n-1}}\frac{n!\Gamma(n+2\alpha)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha)}.$$

In particular, Gegenbauer polynomials satisfy the structure relations

$$C_n^{(\alpha)}(x) = C_n^{(\alpha+1)}(x) - \frac{n(n-1)}{4(n+\alpha)(n+\alpha-1)}C_{n-2}^{(\alpha+1)}(x), \quad n > 1,$$

$$C_n^{(\alpha)}(x) = \frac{1}{n+1}\left(C_{n+1}^{(\alpha)}\right)'(x) - \frac{n}{4(n+\alpha)(n+\alpha-1)}\left(C_{n-1}^{(\alpha)}\right)'(x), \quad n > 1. \tag{2.2}$$

If $\alpha = 0, 1$ we get the *Chebyshev polynomials* of the first and second kind, respectively. In the sequel we will write $C_n^{(0)} := T_n$ and $C_n^{(1)} := U_n$ for every $n \geq 0$. In particular, on asymptotics for polynomials of first kind the next results are well-known.

Proposition 2.1 (see [25, Chapter 8]). *$T_n(z^*) \approx z^n/2^n$ for $z^* = (z+z^{-1})/2$ with $z \in \mathbb{C}\backslash\mathbb{T}$ and $\mathbb{T} = \{z, |z| \leq 1\}$.*

***Corollary* 2.1.** $\lim_{n\to\infty} \dfrac{T_{n-1}(z^*)}{T_{n+1}(z^*)} = \dfrac{4}{z^2}$ *uniformly on compact subsets of* $\mathbb{C}\backslash\mathbb{T}$.

Let $\mu_0, \ldots, \mu_k$ be positive Borel measures supported on $\mathbb{R}$ such that $\operatorname{supp}(\mu_0)$ is infinite, μ_k is not trivial, and, for every i, $\langle,\rangle_{\mu_i}$ and $\|.\|_{\mu_i}$, will denote the inner product and induced norm in $L^2(\mu_i)$, respectively. Then (1.2) is called a *Sobolev inner product*, and $\|.\|_S$ will denote the induced norm. A sequence of polynomials $\{S_n\}_{n\geq 0}$, with $\deg S_n = n$ for $n \geq 0$, orthogonal with respect to (1.2) is called a sequence of *Sobolev orthogonal polynomials*. Sobolev orthogonality is said to be *nonstandard* since the *multiplication operator* $\mathcal{M}_x : \mathbf{P} \to \mathbf{P}$, $M_x(f) = xf$, is not symmetric with respect to $\langle,\rangle_S$, and, as a consequence, usual properties of the standard orthogonality such as the existence of a TTRR is no longer valid. Next we provide information on algebraic connection between Sobolev polynomials and certain extensions of coherent pairs known in the literature (see [9]), as *symmetric* (1,1)-*coherent pairs*.

Theorem 2.2 [8, Section 3.1]. *Consider symmetric and positive Borel measures μ_1, μ_2 supported on infinite subsets of $\mathbb{R}$ and the Sobolev inner product on $\mathbf{P}$*

$$\langle p,q\rangle_\lambda = \int\limits_{\mathbb{R}} p(x)q(x)d\mu_1(x) + \lambda\int\limits_{\mathbb{R}} p'(x)q'(x)d\mu_2(x) = \langle p,q\rangle_{\mu_1} + \lambda\langle p',q'\rangle_{\mu_2}, \quad \lambda > 0.$$

Let $\{S_n\}_{n\geq 0}$, $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are the SMOP associated with $\langle,\rangle_\lambda$, μ_1 and μ_2, respectively. Suppose that there exist $\{a_n\}_{n\geq 0}$ and $\{\eta_n(\lambda)\}_{n\geq 0}$ such that

$$S_{n+3}(x) + \eta_n(\lambda)S_{n+1}(x) = P_{n+3}(x) + \frac{n+3}{n+1}a_nP_{n+1}(x) \tag{2.3}$$

holds. Then there exists a sequence $\{r_n\}_{n\geq 0}$ *such that*

$$\frac{P'_{n+3}(x)}{n+3} + a_n \frac{P'_{n+1}(x)}{n+1} = R_{n+2}(x) + r_n R_n(x).$$

Proposition 2.2 (see [11, Lemma 3.5, Theorem 3.8]). *The Sobolev coefficients* $\{\eta_n(\lambda)\}_{n\geq 0}$ *in* (2.3) *depend on* λ *and satisfy*

$$\eta_n(\lambda) = \frac{r_n(n+1)(n+3)\|R_n\|^2_{\mu_2}\lambda + \dfrac{n+3}{n+1} a_n \|P_{n+1}\|^2_{\mu_1}}{\|S_{n+1}\|^2_\lambda},$$

where $\|.\|_\lambda$ *denotes the induced norm by* $\langle , \rangle_\lambda$. *Moreover,*

$$\eta_{2n}(\lambda) = (A_{2n}\lambda + B_{2n})\frac{Q_n(\lambda)}{Q_{n+1}(\lambda)}, \qquad \eta_{2n+1}(\lambda) = (A_{2n+1}\lambda + B_{2n+1})\frac{\widetilde{Q}_n(\lambda)}{\widetilde{Q}_{n+1}(\lambda)},$$

where $\{Q_n\}$ *and* $\{\widetilde{Q}_n\}$ *are sequences of polynomials satisfying*

$$Q_{n+1}(\lambda) = (C_{2n}\lambda + D_{2n})Q_n(\lambda) - (A_{2n-2}\lambda + B_{2n-2})^2 Q_{n-1}(\lambda), \tag{2.4}$$

$$\widetilde{Q}_{n+1}(\lambda) = (C_{2n+1}\lambda + D_{2n+1})\widetilde{Q}_n(\lambda) - (A_{2n-1}\lambda + B_{2n-1})^2 \widetilde{Q}_{n-1}(\lambda), \tag{2.5}$$

$$A_n = r_n(n+1)(n+3)\|R_n\|^2_{\mu_2}, \qquad B_n = \frac{n+3}{n+1} a_n \|P_{n+1}\|_{\mu_1}, \tag{2.6}$$

$$C_n = (n+1)^2\|R_n\|^2_{\mu_2} + \left(\frac{n+1}{n-1}a_{n-2}\right)^2 \|P_{n-1}\|_{\mu_1}, \quad D_n = \|P_{n+1}\|_{\mu_1}, \tag{2.7}$$

with the initial conditions $C_1 = 4\|R_1\|^2_{\mu_2}$, $\widetilde{Q}_0 = Q_0 = 1$, $Q_1(\lambda) = \lambda + \|P_1\|_{\mu_1}$ *and* $\widetilde{Q}_1(\lambda) = 4\|R_1\|^2_{\mu_2}\lambda + \|P_2\|^2_{\mu_1}$.

The next theorem describes asymptotics for the ratio of solutions of nonstandard TTRR whose coefficients are analytical in certain region of the complex plane. Such TTRR are known as R_{II} type recurrence relations, studied for the first time in [14].

Theorem 2.3 [5, Theorem 2]. *Consider the sequence of functions* $(w_n)_{n\geq 0}$ *satisfying*

$$w_{n+1}(z) = p_n(z)w_n(z) - q_n^2(z)w_{n-1}(z)$$

with $p_n(z) \to p(z)$ *and* $q_n(z) \to q(z)$ *locally uniformly on a domain* G, *and* $p(z) \neq 0$, $z \in G$. *If we define* $\rho_\pm(z) = p(z) \pm \sqrt{p^2(z) - 4q^2(z)}$, $\Gamma = \big\{z \in G \,|\, |\rho_+(z)| = |\rho_-(z)|\big\}$, *and* $E = \{z \in G | \rho_+(z) = 0\}$, *then* $\dfrac{w_{n+1}}{w_n}$ *converges locally uniformly on* $G \backslash (\Gamma \cup E)$ *to the zero of greatest absolute value of the equation* $x^2 - p(z)x + q^2(z) = 0$.

3. An inverse problem. In this section we pose an inverse problem associated with the three monic sequences of polynomials $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ satisfying (1.4). Let (u, v) be a pair of symmetric quasidefinite linear functionals and the respective SMOP's, $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$, satisfy the TTRR

$$P_{n+1}(x) = xP_n(x) - \beta_n^u P_{n-1}(x), \quad n \geq 0, \tag{3.1}$$

and

$$Q_{n+1}(x) = xQ_n(x) - \beta_n^v Q_{n-1}(x), \quad n \geq 0, \tag{3.2}$$

respectively, with the initial conditions $P_{-1} = Q_{-1} := 0,\ P_0 = Q_0 = 1$. Then, given a sequence of numbers $\{b_n\}_{n\geq 1}$, we define $\{R_n\}_{n\geq 0}$ through the algebraic relation

$$\frac{P'_{n+2}(x)}{n+2} + b_n \frac{P'_n(x)}{n} - Q_{n+1}(x) = d_n R_{n-1}(x), \quad n \geq 1, \tag{3.3}$$

where $d_n = \dfrac{n}{n+2} k_{n+2}^{u,n} - k_{n+1}^{v,n-1} + b_n$ and $k_n^{w,n-2} := -\sum_{j=1}^{n-1} \beta_j^w,\ w \in \{u, v\}$ for $n \geq 2$. As a consequence $d_n = \sum_{j=1}^{n} \beta_j^v - \dfrac{n}{n+2} \sum_{j=1}^{n+1} \beta_j^u + b_n$. We will assume that $\{b_n\}_{n\geq 1}$ is chosen in such a way that $d_n \neq 0$ for every n. Taking derivative in (3.1) and replacing in (3.3), we get

$$d_n R_{n-1}(x) = \frac{xP'_{n+1}(x)}{n+2} + \left(\frac{b_n}{n} - \frac{\beta_{n+1}^u}{n+2}\right) P'_n(x) + \frac{P_{n+1}(x)}{n+2} - Q_{n+1}(x). \tag{3.4}$$

Analogously, we can obtain the relation

$$\frac{d_n}{d_{n-1}} x \frac{P'_{n+1}(x)}{n+1} + \frac{d_n}{d_{n-1}} \frac{b_{n-1}}{(n-1)} x P'_{n-1}(x) - \frac{d_n}{d_{n-1}} x Q_n(x) = d_n x R_{n-2}(x). \tag{3.5}$$

Introducing the term $d_n x R_{n-2}(x)$ in (3.4) and by using (3.5), we have

$$\begin{aligned} d_n R_{n-1}(x) = d_n x R_{n-2}(x) &+ \left(\frac{1}{n+2} - \frac{d_n}{d_{n-1}(n+1)}\right) x P'_{n+1}(x) \\ &+ \left(\frac{b_n}{n} - \frac{\beta_{n+1}^u}{n+2}\right) P'_n(x) - \frac{d_n}{d_{n-1}} \frac{b_{n-1}}{(n-1)} x P'_{n-1}(x) \\ &+ \frac{d_n}{d_{n-1}} x Q_n(x) + \frac{P_{n+1}(x)}{n+2} - Q_{n+1}(x). \end{aligned}$$

If we define

$$\begin{aligned} T(x) := &\left(\frac{1}{n+2} - \frac{d_n}{d_{n-1}(n+1)}\right) x P'_{n+1}(x) + \left(\frac{b_n}{n} - \frac{\beta_{n+1}^u}{n+2}\right) P'_n(x) \\ &- \frac{d_n}{d_{n-1}} \frac{b_{n-1}}{(n-1)} x P'_{n-1}(x) + \frac{d_n}{d_{n-1}} x Q_n(x) + \frac{P_{n+1}(x)}{n+2} - Q_{n+1}(x), \end{aligned}$$

after straightforward computations it is possible to show that $\deg(T) \leq n-3$. The relation $k_{n+2}^{u,n} = k_{n+1}^{u,n-1} - \beta_{n+1}^u$ will be useful in such an aim. We will define δ_n in such a way that $-\delta_n d_n R_{n-3} = T$. Thus, we get the TTRR

$$R_{n-1}(x) = xR_{n-2}(x) - \delta_n R_{n-3}(x), \quad n \geq 3, \tag{3.6}$$

with $R_0 = 1,\ R_1 = x$. Then $\{R_n\}_{n\geq 0}$ is a SMOP if and only if there exists a sequence $\{\delta_n\}_{n\geq 1}$, $\delta_n \neq 0$, such that $-\delta_n d_n R_{n-3} = T$, where, if we compare the coefficients of x^{n-3}, we must get

$$-\delta_n d_n = \left(\frac{n-2}{n+2} - \frac{d_n(n-3)}{d_{n-1}(n+1)}\right)k_{n+1}^{u,n-3} + \left(\frac{b_n}{n} - \frac{\beta_{n+1}^u}{n+2}\right)(n-2)k_n^{u,n-2}$$

$$- b_{n-1}\frac{d_n}{d_{n-1}}\frac{n-3}{n-1}k_{n-1}^{u,n-3} + \frac{d_n}{d_{n-1}}k_n^{v,n-4} - k_{n+1}^{v,n-3}, \quad n \geq 4.$$

According to the definition of d_n in terms of recurrence coefficients, we obtain $k_{n+1}^{u,n-3} = \beta_3^u\beta_1^u + \sum_{k=1}^{n-3}\beta_{k+3}^u\left(\sum_{j=1}^{k+1}\beta_j^u\right)$. Then, if we define

$$\xi_n^w = \sum_{k=0}^{n-3}\sum_{j=1}^{k+1}\beta_{k+3}^w\beta_j^w, \quad w \in \{u,v\}, \quad n \geq 3, \tag{3.7}$$

it follows that

$$\delta_n = \frac{\dfrac{n-2}{n+2}\xi_n^u - \xi_n^v + \left(\dfrac{b_n}{n} - \dfrac{\beta_{n+1}^u}{n+2}\right)(n-2)k_n^{u,n-2} + \dfrac{d_n}{d_{n-1}}y(n,u,v)}{-d_n}, \tag{3.8}$$

where

$$y(n,u,v) := \xi_{n-1}^v - (n-3)\left(\frac{b_{n-1}k_{n-1}^{u,n-3}}{n-1} - \frac{\xi_n^u}{n+1}\right). \tag{3.9}$$

As a second assumption, we will assume the coefficients $\{b_n\}$ also are chosen in such a way that $\delta_n \neq 0$. Then we show that $\{R_n\}_{n\geq 0}$ is a SMOP satisfying (3.6) if and only if

$$d_n\delta_n R_{n-3}(x) + \left(\frac{1}{n+2} - \frac{d_n}{d_{n-1}(n+1)}\right)xP'_{n+1}(x) + \left(\frac{b_n}{n} - \frac{\beta_{n+1}^u}{n+2}\right)P'_n(x)$$

$$- \frac{d_n}{d_{n-1}}\frac{b_{n-1}}{n-1}xP'_{n-1}(x) + \frac{d_n}{d_{n-1}}xQ_n(x) + \frac{P_{n+1}(x)}{n+2} - Q_{n+1}(x) = 0 \tag{3.10}$$

holds for $n \geq 3$. Indeed, replacing (3.6) in (4.19), we get (3.10). Reciprocally, replacing Q_n and Q_{n+1} in (3.10), by using (3.3) we obtain

$$d_n x R_{n-2}(x) - d_n\delta_n R_{n-3}(x)$$

$$= d_n R_{n-1}(x) + \frac{1}{n+2}xP'_{n+1}(x) - \frac{\beta_{n+1}^u}{n+2}P'_n(x) + \frac{P_{n+1}(x)}{n+2} - \frac{P'_{n+2}(x)}{n+2}.$$

Taking derivative in (3.1) and replacing $xP'_{n+1}(x)$ in the above formula, we get (3.6). So, we proved the following result.

Proposition 3.1. *Let (u,v) be a pair of symmetric quasidefinite linear functionals, $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are the respective SMOP, and $\{\beta_n^u\}_{n\geq 0}$, $\{\beta_n^v\}_{n\geq 0}$ are the respective recurrence coefficients. We define the sequence $\{R_n\}_{n\geq 0}$ by means of the algebraic relation*

$$\frac{P'_{n+2}(x)}{n+2} + b_n \frac{P'_n(x)}{n} - Q_{n+1}(x) = d_n R_{n-1}(x), \quad n \geq 1, \quad R_0 := 1,$$

where $\{b_n\}_{n\geq 1}$ *satisfies* $d_n = \frac{n}{n+2}\sum_{j=1}^{n+1} \beta_j^u - \sum_{j=1}^{n} \beta_j^v - b_n \neq 0,$ *and*

$$\frac{n-2}{n+2}\xi_n^u - \xi_n^v + \left(\frac{b_n}{n} - \frac{\beta_{n+1}^u}{n+2}\right)(n-2)k_n^{u,n-2} + \frac{d_n}{d_{n-1}} y(n,u,v) \neq 0,$$

with ξ_n^u *and* ξ_n^v *as in* (3.7), *and* $y(n,u,v)$ *as in* (3.9). $\{R_n\}_{n\geq 0}$ *is a SMOP satisfying* (3.6) *with initial conditions* $R_0 = 1,\ R_1 = x,$ *and* $\{\delta_n\}_{n\geq 3},$ *defined in* (3.8), *if and only if*

$$d_n \delta_n R_{n-3}(x) + \left(\frac{1}{n+2} - \frac{d_n}{d_{n-1}(n+1)}\right) x P'_{n+1}(x) + \left(\frac{b_n}{n} - \frac{\beta_{n+1}^u}{n+2}\right) P'_n(x)$$

$$- \frac{d_n}{d_{n-1}} \frac{b_{n-1}}{n-1} x P'_{n-1}(x) + \frac{d_n}{d_{n-1}} x Q_n(x) + \frac{P_{n+1}(x)}{n+2} - Q_{n+1}(x) = 0, \quad n \geq 3.$$

In the sequel, we consider the very particular case $b_n = 0$ for $n \geq 1$. Thus, $d_n = \frac{n}{n+2}k_{n+2}^{u,n} - k_{n+1}^{v,n-1}$ and

$$\delta_n = \frac{\frac{n-2}{n+2}\xi_n^u - \xi_n^v - \frac{\beta_{n+1}^u}{n+2}(n-2)k_n^{u,n-2} + \frac{d_n}{d_{n-1}}\left(\xi_{n-1}^v + \frac{n-3}{n+1}\xi_n^u\right)}{-d_n}.$$

In addition, we consider $\beta_1^v = \frac{1}{2},\ \beta_n^v = \frac{1}{4}$ for $n \geq 2$ and $\beta_n^u = \frac{1}{4}$ for $n \geq 1$. In this way, we get $Q_n = T_n,\ P_n = U_n$. Then (3.3) is written as

$$C_{n+1}^{(2)}(x) - T_{n+1}(x) = d_n R_{n-1}(x) \tag{3.11}$$

with $d_n = \frac{1}{2}\frac{n+1}{n+2}$. After straightforward computations, we obtain

$$\delta_n = \frac{2n^2 + 2n - 3}{8n(n+1)}, \tag{3.12}$$

and, as a consequence, the sequence $\{R_n\}_{n\geq 0}$ satisfies

$$R_{n+1}(x) = xR_n(x) - \frac{2n^2 + 10n + 9}{8(n+2)(n+3)} R_{n-1}(x), \quad n \geq 1. \tag{3.13}$$

Concerning location of the zeros of every R_n, it is possible to show that between two positive zeros of T_n there exists one zero of $C_n^{(2)}$. For such a purpose, the well-known relation

$$(n+1)(1-x^2)C_n^{(2)}(x) = xU_{n+1}(x) - (n+2)T_{n+2}(x)$$

is useful. So, the next result follows from (3.11).

Lemma 3.1. *Let* $\{R_n\}_{n\geq 0}$ *be the SMOP defined by means of the TTRR* (3.13). *For every* $n \geq 1$, *the zeros of* R_n *are real, simple and lie in* $[-1, 1]$.

According to definiton of δ_n, there exists a positive-definite linear functional w whose SMOP is $\{R_n\}_{n\geq 0}$. Let μ denotes the positive Borel measure associated with w. It follows from (3.12) that, for $m \geq 1$,

$$\|R_m\|_\mu^2 = \prod_{n=0}^{m} \frac{2n^2+10n+9}{8(n+2)(n+3)}.$$

Lemma 3.2. *Let* $\{R_n\}_{n\geq 0}$ *be the SMOP defined in* (3.13). *It holds*

$$\lim_{n\to\infty} 4^{n+1}\|R_n\|_\mu^2 = \frac{9}{\Gamma\left(\dfrac{7-\sqrt{7}}{2}\right)\Gamma\left(\dfrac{7+\sqrt{7}}{2}\right)} := R_\mu. \tag{3.14}$$

Proof. After cumbersome computations, from (3.14) we can obtain

$$\|R_m\|_\mu^2 = \frac{12\left(\dfrac{1}{4}\right)^m}{(m+2)!(m+3)!} \frac{\Gamma\left(m+\dfrac{7-\sqrt{7}}{2}\right)\Gamma\left(m+\dfrac{7+\sqrt{7}}{2}\right)}{\Gamma\left(\dfrac{7}{2}-\dfrac{1}{2}\sqrt{7}\right)\Gamma\left(\dfrac{7}{2}+\dfrac{1}{2}\sqrt{7}\right)}.$$

Since for $a, b, x > 0$, it holds $\lim_{x\to\infty} x^{b-a}\dfrac{\Gamma(x+a)}{\Gamma(x+b)} = 1$ (see [1]).

4. Sobolev inner products and asymptotics. Throughout this section we will consider the monic Sobolev polynomials $\left\{S_n^{[\lambda_1,\lambda_2]}\right\}_{n\geq 0}$, orthogonal with respect to the Sobolev inner product

$$\langle p, q\rangle_S = \int_{-1}^{1} p(x)q(x)(1-x^2)^{-1/2}dx$$

$$+ \lambda_1 \int_{-1}^{1} p'(x)q'(x)(1-x^2)^{1/2}dx + \lambda_2 \int_{-1}^{1} p''(x)q''(x)d\mu(x), \tag{4.1}$$

where μ is the positive Borel measure associated with $\{R_n\}_{n\geq 0}$, and we will assume that $\lambda_1, \lambda_2 > 0$. We introduce the auxiliary Sobolev inner products

$$\langle p, q\rangle_{S_1} = \int_{-1}^{1} p(x)q(x)(1-x^2)^{1/2}dx + \eta \int_{-1}^{1} p'(x)q'(x)(1-x^2)^{-1/2}dx, \quad \eta > 0, \tag{4.2}$$

and

$$\langle p, q\rangle_{S_2} = \int_{-1}^{1} p(x)q(x)(1-x^2)^{1/2}dx + \lambda \int_{-1}^{1} p'(x)q'(x)d\mu(x), \quad \lambda > 0. \tag{4.3}$$

Let $\left\{S_n^{(1,\eta)}\right\}_{n\geq 0}$ and $\left\{S_n^{(2,\lambda)}\right\}_{n\geq 0}$ are the respective SMOP, as well as $\|.\|_{(1,\lambda)}$ and $\|.\|_{(2,\lambda)}$ are the respective induced norms.

Lemma 4.1. *Let μ_1 and μ_2 are positive Borel measures supported on infinite subsets of $\mathbb{R}$, $\lambda > 0$, and $\{S_n^\lambda\}_{n\geq 0}$ be the SMOP associated with*

$$\langle p, q\rangle_S = \int_{-1}^{1} p(x)q(x)d\mu_1 + \lambda \int_{-1}^{1} p'(x)q'(x)d\mu_2, \quad \lambda > 0.$$

There exists a sequence of monic polynomials $\{S_n\}_{n\geq 0}$, $\deg S_n = n$, such that

$$S_n(x) = \lim_{\lambda\to\infty} S_n^\lambda(x),$$

satisfying

$$S'_{n+1}(x) = (n+1)P_n^{[2]}(x), \quad n \geq 0, \tag{4.4}$$

where $\left\{P_n^{[2]}\right\}_{n\geq 0}$ is the SMOP associated with μ_2.

Proof. We define $\langle u_i, p\rangle_i := \int_{\mathbb{R}} p(x)d\mu_i$, $i = 1, 2$, and assume that the normalization $\langle u_i, 1\rangle_i := 1$. Notice that the polynomials S_n are well defined, since coefficients in the canonical expansion of every S_n^λ (except the leading coefficient), are proper functions of the parameter λ. Indeed, if Δ_n^λ is the determinant of the leading principal submatrix of order $n+1$ of the Hankel matrix associated with $\langle , \rangle_S$, then it is possible to show (see [22]) through basic determinant properties, that

$$S_n^\lambda(x) = x^n + \sum_{k=1}^{[n/2]} \frac{\Delta_{n-1}^{\lambda,n-2k}}{\Delta_{n-1}^\lambda} x^{n-2k},$$

where $\Delta_{n-1}^{\lambda,n-2k}$ is results from to replace the $(n-2k)$th column of Δ_{n-1}^λ by the vector

$$\left(\langle x^n, 1\rangle_S, \langle x^n, x\rangle_S, \ldots, \left\langle x^n, x^{n-1}\right\rangle_S\right).$$

Also, for $1 \leq m < n$, we get

$$\left\langle u_2, (S_n^\lambda)' x^{m-1}\right\rangle_2 = -\frac{\left\langle u_1, x^m S_n^\lambda\right\rangle_1}{m\lambda}.$$

If $\lambda \to \infty$ then $\left\langle (S_n)', x^{m-1}\right\rangle_{\mu_2} = 0$. Thus, we obtain (4.4).

Lemma 4.2. *For $n \geq 1$,*

$$Q_{n+1}^{[1]}(x) = U_{n+1}(x) - \frac{1}{2}\frac{n}{n-1}U_{n-1}(x) + \frac{1}{16}\frac{n+1}{n-1}U_{n-3}(x) \tag{4.5}$$

and

$$Q_{n+1}^{[1]}(x) = (n+1)\int T_n(x)dx. \tag{4.6}$$

Proof. For $n \geq 1$, $\left\langle S_n^\lambda, 1\right\rangle_S = \left\langle u_1, S_n^\lambda\right\rangle_1$, then, if $\lambda \to \infty$, it holds that

$$\langle u_1, S_n\rangle_1 = 0. \tag{4.7}$$

Now we consider (4.2). It can be written as

$$\langle p,q\rangle_{S_1} = \langle p(x),q(x)\rangle_{1/2} + \eta\langle p'(x),q'(x)\rangle_{-1/2}.$$

From Lemma 4.1, if we define $Q_n^{[1]}(x) = \lim_{\eta\to\infty} S_n^{(1,\eta)}(x)$, we can obtain the relation $\big(Q_{n+1}^{[1]}\big)'(x) = (n+1)T_n(x)$. It now follows from (2.2) with $\alpha = 0$ that

$$Q_{n+1}^{[1]}(x) = U_{n+1}(x) - \frac{1}{2}\frac{n}{n-1}U_{n-1}(x) + \frac{1}{16}\frac{n+1}{n-1}U_{n-3}(x) + K_n.$$

From (2.2) and as a consequence of (4.8), we get $0 = \big\langle S_n^{(1,\eta)},1\big\rangle_{S_1} = \big\langle S_n^{(1,\eta)},1\big\rangle_{1/2}$ for $n \geq 1$. When $n \geq 5$ and $\eta \to \infty$, we have

$$0 = \left\langle Q_n^{[1]},1\right\rangle_{1/2} = \left\langle U_n - \frac{1}{2}\frac{n-1}{n-2}U_{n-2} + \frac{1}{16}\frac{n}{n-2}U_{n-4},1\right\rangle_{1/2} + \langle 1,1\rangle_{1/2}K_n = K_n.$$

Since $U_{-1} := 0$, $K_n = 0$ also for $n = 1,2,3$.

Lemma 4.3. *For $n \geq 2$,*

$$Q_{n+1}^{[2]}(x) = U_{n+1}(x) - \frac{1}{8}\frac{n+3}{n+2}U_{n-1}(x). \tag{4.8}$$

Proof. If we consider (4.3) and Lemma 4.2 with $Q_n^{[2]}(x) := \lim_{\lambda\to\infty} S_n^{(2,\lambda)}(x)$, then

$$\big(Q^{[2]}\big)'_{n+1}(x) = (n+1)R_n(x), \tag{4.9}$$

equivalently, $Q_{n+1}^{[2]}(x) = (n+1)\displaystyle\int R_n(x)dx + k_n$. From (3.11) and (4.6), we get

$$Q_{n+1}^{[2]}(x) = 2\frac{n+1}{n+2}\Big(U_{n+3}(x) - Q_{n+3}^{[1]}\Big) + k_n.$$

Then, as in the above lemma, from (4.7) we can show that $k_n = 0$. Finally, from (4.5), (4.8) is obtained.

Proposition 4.1. *For $n \geq 1$,*

$$S_n^{(2,\lambda)}(x) + c_n(\lambda)S_{n-2}^{(2,\lambda)}(x) = U_n(x) - \frac{1}{8}\frac{n+2}{n+1}U_{n-2}(x) \tag{4.10}$$

with

$$c_n(\lambda) = -\frac{1}{8}\frac{n+2}{n+1}\frac{\|U_{n-2}\|_{1/2}^2}{\Big\|S_{n-2}^{(2,\lambda)}\Big\|_{(2,\lambda)}^2}. \tag{4.11}$$

Proof. By using of $\big\{S_n^{(2,\lambda)}\big\}_{n\geq 0}$ as a basis, we can expand $Q_n^{[2]}$ as follows:

$$Q_n^{[2]}(x) = S_n^{(2,\lambda)}(x) + \sum_{k=0}^{n-2} c_{n,k}S_k^{(2,\lambda)}(x), \qquad c_{n,k} = \frac{\left\langle Q_n^{[2]},S_k^{(2,\lambda)}\right\rangle_{S_2}}{\Big\|S_k^{(2,\lambda)}\Big\|_{(2,\lambda)}^2}.$$

Then, from (4.9) and (4.8) it follows that

$$\left\| S_k^{(2,\lambda)} \right\|_{(2,\lambda)}^2 c_{n,k} = \int_{-1}^{1} \left(U_n(x) - \frac{1}{8}\frac{n+2}{n+1} U_{n-2}(x) \right) S_k^{(2,\lambda)}(x)(1-x^2)^{1/2} dx.$$

Thus, $c_{n,k} = 0$ for $k < n-2$ and $c_{n,n-2} = -\frac{1}{8}\frac{n+2}{n+1}\|U_{n-2}\|_{1/2}^2 / \left\| S_{n-2}^{(2,\lambda)} \right\|_{(2,\lambda)}^2$. If we define $c_n(\lambda) := c_{n,n-2}$, the result follows.

In addition, as a consequence of the extremal property of the norm $\|, \|_{(2,\lambda)}$, from (4.11) we have the next result.

Lemma 4.4. *For $n \geq 1$, $c_n(\lambda)$, defined in* (4.11), *satisfies*

$$\left| c_n(\lambda) \right| \leq \frac{1}{8}\frac{n+2}{n+1}. \tag{4.12}$$

From (4.10) we get

$$c_n(\lambda) = \widetilde{b}_n + \lambda \frac{\displaystyle\int_{-1}^{1} U_n'(x) \left(S_{n-2}^{(2,\lambda)} \right)'(x) d\mu(x)}{\left\| S_{n-2}^{(2,\lambda)} \right\|_{S_2}^2}$$

with $\widetilde{b}_n := -\frac{1}{8}\frac{n+2}{n+1}$, and according to Theorem 2.2 and Proposition 2.2

$$c_{2n+3}(\lambda) = (A_{2n}\lambda + B_{2n}) \frac{Q_n(\lambda)}{Q_{n+1}(\lambda)}, \qquad c_{2n+4}(\lambda) = (A_{2n+1}\lambda + B_{2n+1}) \frac{\widetilde{Q}_n(\lambda)}{\widetilde{Q}_{n+1}(\lambda)},$$

where $\{Q_n\}$ and $\{\widetilde{Q}_n\}$ are sequences of polynomials satisfying the nonstandard TTRR (2.4) and (2.5), respectively, with the coefficients and the initial conditions described there. At the core of our paper, we will study the limit behavior of ratios of polynomials in the such sequences. For the even case, let $\{\widehat{Q}_n\}$ denotes the monic sequence associated to $\{Q_n\}$. Therefore, (2.4) can be written as

$$\widehat{Q}_{n+1}(\lambda) = \left(\lambda + \frac{D_{2n}}{C_{2n}} \right) \widehat{Q}_n(\lambda) - \frac{B_{2n-2}^2}{C_{2n}C_{2n-2}} \widehat{Q}_{n-1}(\lambda) \tag{4.13}$$

with

$$c_{2n+3}(\lambda) = \frac{B_{2n}}{C_{2n}} \frac{\widehat{Q}_n(\lambda)}{\widehat{Q}_{n+1}(\lambda)}.$$

Notice that when $n \to \infty$ we get

$$\frac{D_{2n}}{C_{2n}} = \frac{\pi}{2} \frac{1}{(2n+1)^2 \left(4^{2n+1} \|R_{2n}\|_\mu^2 \right) + \frac{\pi}{32} \left(\frac{2n+3}{n+1} \right)^2} \to 0,$$

and, in the same way, $\frac{B_{2n}}{C_{2n}} \to 0$. On the other hand, after straightforward computations we obtain

$$\frac{B_{2n-2}}{\sqrt{C_{2n}C_{2n-2}}} = -\frac{2n+3}{2n+2} \frac{1}{\sqrt{\left(\frac{(2n+1)^2}{\pi} \left(4^{2n+1} \|R_{2n}\|_\mu^2 \right) + \frac{1}{2} \left(\frac{2n+3}{4n+4} \right)^2 \right)}}$$

$$\times \frac{1}{\sqrt{\left(\frac{(8n-4)^2}{\pi}\left(4^{2n-1}\|R_{2n-2}\|_{\mu}^2\right)+\frac{1}{2}\left(\frac{2n+1}{n}\right)^2\right)}},$$

whence $\frac{B_{2n-2}}{\sqrt{C_{2n}C_{2n-2}}} \to 0$ as $n \to \infty$. This completes the proof of the following lemma.

Lemma 4.5. *For B_n, C_n and D_n defined in* (2.6) *and* (2.7), *it holds that*

$$\lim_{n\to\infty}\frac{D_{2n}}{C_{2n}} = \lim_{n\to\infty}\frac{B_{2n}}{C_{2n}} = \lim_{n\to\infty}\frac{B_{2n-2}}{\sqrt{C_{2n}C_{2n-2}}} = 0. \tag{4.14}$$

Lemma 4.6. *For $\lambda > 0$,*

$$\lim_{n\to\infty} c_n(\lambda) = 0. \tag{4.15}$$

Proof. Tacking into account Theorem 2.3, if we consider (4.13) and (4.14), we obtain $p(\lambda) = \lambda$, $q(\lambda) = 0$ and $G = \mathbb{C}\backslash\{0\}$. In addition, since $\rho_{\pm}(\lambda) = \lambda \pm \lambda$, we get $\Gamma = E = \{0\}$, and then $\frac{\widehat{Q}_{n+1}(\lambda)}{\widehat{Q}_n(\lambda)} \to \lambda$ locally uniformly on $\mathbb{C}\backslash\{0\}$. We can also deduce an analogous result in the odd case.

Finally, we go back to (4.1). In the sequel, we will also suppose that λ_2 is a linear function of λ_1, i.e., $\lambda_2 = \eta\lambda_1$ for $\eta > 0$. Such an inner product can be expressed as $\langle p, q\rangle_S = \langle p, q\rangle_{-1/2} + \lambda_1\langle p', q'\rangle_{S_2}$. We define, for every n,

$$Q_n^{[\lambda_2]}(x) := \lim_{\lambda_1\to\infty} S_n^{[\lambda_1,\lambda_2]}(x). \tag{4.16}$$

Then

$$\left\langle S_n^{[\lambda_1,\lambda_2]}, x^m\right\rangle_S = \frac{1}{\lambda_1}\left\langle S_n^{[\lambda_1,\lambda_2]}, x^m\right\rangle_{-1/2} + m\left\langle \left(S_n^{[\lambda_1,\lambda_2]}\right)', x^{m-1}\right\rangle_{1/2}$$

$$+ m(m-1)\eta\int_{-1}^{1}\left(S_n^{[\lambda_1,\lambda_2]}\right)''(x)x^{m-2}d\mu(x) = 0,$$

and when $\lambda_1 \to \infty$ necessarily $\left\langle\left(Q_n^{[\lambda_2]}\right)', x^{m-1}\right\rangle_{S_2} = 0$. As a consequence, $\left(Q_{n+1}^{[\lambda_2]}\right)'(x) = (n+1)S_n^{(2,\eta)}(x)$. If we suppose that $\frac{Q_{n+1}^{[\lambda_2]}(x)}{n+1} = \int S_n^{(2,\eta)}(x)dx + k_n$, then, for $n \geq 0$, we have $\left\langle \frac{S_{n+1}^{[\lambda_1,\lambda_2]}}{n+1}, 1\right\rangle_S = 0$. When $\lambda_1 \to \infty$, we obtain

$$\left\langle \frac{Q_{n+1}^{(\lambda_2)}}{n+1}, 1\right\rangle_{-\frac{1}{2}} = \left\langle \int S_n^{(2,\eta)}(x)dx, 1\right\rangle_{-\frac{1}{2}} + k_n\langle 1, 1\rangle_{-\frac{1}{2}} = 0.$$

From (4.10) we get recurrently

$$S_n^{(2,\eta)}(x) = U_n(x) + \sum_{k=1}^{[n/2]} b_k(n,\eta)U_{n-2k}(x), \quad n \geq 2,$$

where the coefficients $b_k(n,\eta)$ depend on n and η. In this way, it is clear that $\int_{-1}^{1}\left(\int S_n^{(2,\eta)}(x)dx\right)\times (1-x^2)^{-1/2}dx=0$ and, as a consequence $k_n=0,$ for every $n.$ Thus,

$$\frac{Q_{n+1}^{[\lambda_2]}(x)}{n+1}=\int S_n^{(2,\eta)}(x)dx. \tag{4.17}$$

From (4.10), (4.17), and $\left\{S_{n+1}^{[\lambda_1,\lambda_2]}\right\}_{n\geq 0}$, as a basis, we get

$$Q_{n+1}^{[\lambda_2]}(x)+c_n(\eta)\frac{n+1}{n-1}Q_{n-1}^{[\lambda_2]}(x)=S_{n+1}^{[\lambda_1,\lambda_2]}(x)+\sum_{k=0}^{n-1}\zeta_{n,k}S_k^{[\lambda_1,\lambda_2]}(x),\quad n\geq 2,$$

$$\zeta_{n,k}=\left(-\frac{1}{8}\frac{n+2}{n-1}\int\limits_{-1}^{1}T_{n-1}(x)S_k^{[\lambda_1,\lambda_2]}(x)(1-x^2)^{-1/2}dx\right.$$

$$\left.+(n+1)c_n(\eta)\lambda_1\left\langle S_{n-2}^{(2,\eta)}(x),\left(S_k^{[\lambda_1,\lambda_2]}\right)'(x)\right\rangle_{S_2}\right)\Big/\left\|S_k^{[\lambda_1,\lambda_2]}\right\|_S^2,\quad k\leq n-1.$$

Then $\zeta_{n,k}=0$ for $k<n-1$ and

$$\zeta_n(\lambda_1,\lambda_2):=\zeta_{n,n-1}=\frac{-\frac{1}{8}\frac{n+2}{n-1}\|T_{n-1}\|_{-1/2}^2+(n^2-1)c_n(\eta)\lambda_1\left\|S_{n-2}^{(2,\eta)}\right\|_{S_2}^2}{\left\|S_{n-1}^{[\lambda_1,\lambda_2]}\right\|_S^2}.$$

As a result,

$$T_{n+1}(x)-\frac{1}{8}\frac{n+2}{n-1}T_{n-1}(x)=S_{n+1}^{[\lambda_1,\lambda_2]}(x)+\zeta_n(\lambda_1,\lambda_2)S_{n-1}^{[\lambda_1,\lambda_2]}(x),\quad n\geq 2,$$

with the respective initial conditions. On the other hand, by using the extremal property of the norm, we obtain

$$\left\|S_n^{[\lambda_1,\lambda_2]}\right\|_S^2=\left\langle S_n^{[\lambda_1,\lambda_2]},S_n^{[\lambda_1,\lambda_2]}\right\rangle_{-1/2}+\lambda_1 n^2\left\langle\frac{1}{n}\left(S_n^{[\lambda_1,\lambda_2]}\right)',\frac{1}{n}\left(S_n^{[\lambda_1,\lambda_2]}\right)'\right\rangle_{S_2}$$

$$\geq\|T_n\|_{-1/2}^2+\lambda_1 n^2\left\|S_{n-1}^{(2,\eta)}\right\|_{S_2}^2,$$

and, therefore,

$$\zeta_n(\lambda_1,\lambda_2)\leq\frac{-\frac{1}{8}\frac{n+2}{n-1}\|T_{n-1}\|_{-1/2}^2+(n^2-1)c_n(\eta)\lambda_1\left\|S_{n-2}^{(2,\eta)}\right\|_{S_2}^2}{\|T_{n-1}\|_{-1/2}^2+\lambda_1(n-1)^2\left\|S_{n-2}^{(2,\eta)}\right\|_{S_2}^2}.$$

So, from (4.12), the next result is proved.

Lemma 4.7. *For* $\lambda_1 > 0$, $\eta > 0$ *and* $n > 1$,

$$\left|\zeta_n(\lambda_1, \lambda_2)\right| \leq \frac{1}{8}\frac{n+2}{n-1}. \tag{4.18}$$

Analogously, and in order to study the asymptotic behavior of the Sobolev coefficients $\left\{\zeta_n(\lambda_1, \lambda_2)\right\}_{n\geq 0}$, by using Proposition 2.2 in the even case, with $\{G_n\}$ being the monic sequence associated to $\{Q_n\}$, we get

$$\zeta_{2n+2}(\lambda_1, \lambda_2) = \frac{A_{2n}\lambda_1 + B_{2n}}{C_{2n}}\frac{G_n(\lambda_1)}{G_{n+1}(\lambda_1)}$$

with

$$G_{n+1}(\lambda_1) = \left(\frac{D_{2n}}{C_{2n}} + \lambda_1\right)G_n(\lambda_1) - \frac{\left(A_{2n-2}\lambda_1 + B_{2n-2}\right)^2}{C_{2n}C_{2n-2}}G_{n-1}(\lambda_1),$$

where

$$A_n = (n+3)(n+1)c_{n+2}(\eta)\left\|S_n^{(2,\eta)}\right\|_{S_2}^2,$$

$$B_n = -\frac{1}{8}\frac{n+4}{n+1}\|T_{n+1}\|_{-1/2}, \quad C_n = (n+1)^2\left\|S_n^{(2,\eta)}\right\|_{S_2}^2 + \frac{1}{64}\left(\frac{n+2}{n-1}\right)^2\|T_{n-1}\|_{-1/2},$$

and $D_n = \|T_{n+1}\|_{-1/2}$. Then, from (4.11), after straightforward computations we obtain

$$\frac{D_{2n}}{C_{2n}} = c_{2n+2}(\lambda)\frac{4}{c_{2n+2}(\lambda)\left(\dfrac{2n+2}{2n-1}\right)^2 - \dfrac{n+2}{2n+3}(2n+1)^2}$$

and

$$\frac{A_{2n-2}\lambda + B_{2n-2}}{\sqrt{C_{2n}C_{2n-2}}}$$

$$= \frac{-4c_{2n}^2(\lambda)c_{2n+2}^2(\lambda)\left(\lambda + \dfrac{1}{(2n-1)^2}\right)}{\sqrt{\left(\dfrac{2n+4}{2n+3}\dfrac{(2n+1)^2}{(2n+2)^2} - 2c_{2n+2}(\lambda)\left(\dfrac{2n+2}{2n-1}\right)^2\right)\left(\dfrac{2n+2}{2n+1} - 2c_{2n}(\lambda)\left(\dfrac{2n}{2n-3}\right)^2\right)}}.$$

By using (4.15), we conclude that $\dfrac{D_{2n}}{C_{2n}} \to 0$ and $\dfrac{A_{2n-2}\lambda + B_{2n-2}}{\sqrt{C_{2n}C_{2n-2}}} \to 0$ as $n \to \infty$. A similar analysis is possible in the odd case. Therefore, the next result is a consequence of Theorem 2.3.

Theorem 4.1. *It holds* $\lim_{n\to\infty}\zeta_n(\lambda_1, \lambda_2) = 0$ *for* $\lambda_1, \eta > 0$ *and* $\lambda_1\eta = \lambda_2$.

Now, we introduce the sequence of polynomials $\{\Phi_n(x)\}$ as follows:

$$\Phi_{n+1}(x) := S_{n+1}^{[\lambda_1,\lambda_2]}(x) + \zeta_n(\lambda_1, \lambda_2)S_{n-1}^{[\lambda_1,\lambda_2]}(x), \quad n \geq 1. \tag{4.19}$$

Lemma 4.8. *Uniformly on compact subsets of the outside of unit circle*

$$\lim_{n\to\infty}\frac{\Phi_{n-1}(z^*)}{\Phi_{n+1}(z^*)}=\frac{4}{z^2}.$$

Proof. Recursively, from (4.19) we can obtain the relation

$$S_{n+1}^{[\lambda_1,\lambda_2]}(x)=\Phi_{n+1}(x)+\sum_{k=1}^{\left[\frac{n+1}{2}\right]}(-1)^k\phi_{n,k}(\lambda_1,\lambda_2)\Phi_{n+1-2k}(x),$$

where $\phi_{n,k}(\lambda_1,\lambda_2)=\prod_{j=1}^{k}\zeta_{n+2-2j}(\lambda_1,\lambda_2)$ for $k\geq 1$. Notice that

$$\zeta_n(\lambda_1,\lambda_2)\frac{S_{n-1}^{[\lambda_1,\lambda_2]}(x)}{\Phi_{n+1}(x)}=\zeta_n(\lambda_1,\lambda_2)\frac{\Phi_{n-1}(x)}{\Phi_{n+1}(x)}+\sum_{k=1}^{\left[\frac{n-1}{2}\right]}(-1)^k\phi_{n+2,k}(\lambda_1,\lambda_2)\frac{\Phi_{n-1-2k}(x)}{\Phi_{n+1}(x)}.$$

For $z^*=\dfrac{z+z^{-1}}{2}$, we get

$$\frac{\Phi_{n-1}(z^*)}{\Phi_{n+1}(z^*)}=\frac{\dfrac{T_{n-1}(z^*)}{T_{n+1}(z^*)}-\dfrac{1}{8}\dfrac{n}{n-3}\dfrac{T_{n-1}(z^*)}{T_{n+1}(z^*)}\dfrac{T_{n-3}(z^*)}{T_{n-1}(z^*)}}{1-\dfrac{1}{8}\dfrac{n+2}{n-1}\dfrac{T_{n-1}(z^*)}{T_{n+1}(z^*)}}.$$

Then, by means of Corollary 2.1 the proof is completed.

Since $\dfrac{\Phi_{n-1-2k}(x)}{\Phi_{n+1}(x)}=\prod_{j=1}^{k+1}\dfrac{\Phi_{n-(2j-1)}(x)}{\Phi_{n-(2j-3)}(x)}$ and

$$(-1)^k\phi_{n+2,k}(\lambda_1,\lambda_2)\frac{\Phi_{n-1-2k}(x)}{\Phi_{n+1}(x)}=(-1)^k\phi_{n+2,k}(\lambda_1,\lambda_2)\prod_{j=1}^{k+1}\frac{\Phi_{n-(2j-1)}(x)}{\Phi_{n-(2j-3)}(x)},$$

it follows from (4.18) and Lemma 4.8 that the sequence $\left\{(-1)^k\phi_{n+2,k}(\lambda_1,\lambda_2)\dfrac{\Phi_{n-1-2k}(z^*)}{\Phi_{n+1}(z^*)}\right\}_{n\geq 0}$ is uniformly bounded and converges uniformly to 0, both on compact subsets of $\mathbb{C}\backslash\mathbb{T}$. As a consequence $\left\{\zeta_n(\lambda_1,\lambda_2)\dfrac{S_{n-1}^{[\lambda_1,\lambda_2]}(z^*)}{\Phi_{n+1}(z^*)}\right\}_{n\geq 1}$ converges to 0 on every compact subset of $\mathbb{C}\backslash\mathbb{T}$. We proved the next consequence.

Theorem 4.2. *Let* $\left\{S_{n+1}^{[\lambda_1,\lambda_2]}\right\}_{n\geq 0}$ *and* $\{\Phi_n(x)\}$ *are the monic sequences defined in* (4.16) *and* (4.19), *respectively. It holds*

$$\lim_{n\to\infty}\frac{S_{n+1}^{[\lambda_1,\lambda_2]}(z^*)}{\Phi_{n+1}(z^*)}=1$$

uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{T}$.

Finally, since

$$\frac{S_{n+1}^{[\lambda_1,\lambda_2]}(z^*)}{T_{n+1}(z^*)}=\frac{S_{n+1}^{[\lambda_1,\lambda_2]}(z^*)}{\Phi_{n+1}(z^*)}\left(1-\frac{1}{8}\frac{n+2}{n-1}\frac{T_{n-1}(z^*)}{T_{n+1}(z^*)}\right),$$

the following result is obtained.

Theorem 4.3. *Consider* $\left\{S_{n+1}^{[\lambda_1,\lambda_2]}\right\}_{n\geq 0}$ *as in* (4.16), *and let* $\{T_n\}_{n\geq 0}$ *be the monic Chebyshev polynomials of the first kind. It holds*

$$\lim_{n\to\infty}\frac{S_{n+1}^{[\lambda_1,\lambda_2]}(z^*)}{T_{n+1}(z^*)}=1-\frac{1}{2z^2}$$

uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{T}$.

Acknowledgements. The author thanks the reviewers for this comments. They have contributed to improve the presentation of the manuscript.

This work was partially supported by Dirección de Investigaciones, Universidad Pedagógica y Tecnológica de Colombia.

The author states that there is no conflict of interest.

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Received 18.08.22