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POINTWISE HEMI-SLANT RIEMANNIAN SUBMERSIONS

ТОЧКОВІ НАПІВПОХИЛІ РІМАНОВІ ЗАНУРЕННЯ

We introduce a new type of submersions, which is called *pointwise hemi-slant Riemannian submersions,* as a generalization of slant Riemannian submersions, hemi-slant submersions, and pointwise slant submersions from Kaehler manifolds onto Riemannian manifolds. We obtain some geometric interpretations of this kind of submersions with respect to the total manifold, the base manifold, and the fibers. Moreover, we present non-trivial illustrative examples in order to demonstrate the existence of submersions of this kind. Finally, we obtain some curvature equalities and inequalities with respect to a certain basis.

Запропоновано новий тип занурень, які називаються *точковими напівпохилими рімановими зануреннями,* як узагальнення похилих ріманових занурень, напівпохилих занурень і поточкових похилих занурень з келерових многовидів на ріманові многовиди. Отримано деякі геометричні інтерпретації цього виду занурень щодо загального многовиду, базового многовиду та волокон. Крім того, наведено нетривіальні наочні приклади, що демонструють наявність таких занурень. Насамкінець отримано кілька рівностей та нерівностей для кривини щодо деякого базису.

1. Introduction. In the 1998's, pointwise slant submanifolds as a generalization of slant submanifolds were first defined by Etayo [9] under the name of quasi-slant submanifolds and such submanifolds have been recently studied deeply by Chen and Garay [8]. They obtain simple characterizations, give a method how to construct such submanifolds in Euclidean space and investigate geometric and topological properties of pointwise slant submanifolds. Since slant submanifolds include holomorphic submanifolds and totally real submanifolds, the class of pointwise slant submanifolds is a general notion for the theory of submanifolds of almost Hermitian manifolds. After that many geometers study this area and there are a lot of results on this topic (see [3, 5, 17, 18, 30, 31, 33]).

On the other hand, Riemannian submersions between Riemannian manifolds were studied by O'Neill [15] and Gray [10]. It is well-known that Riemannian submersions have several applications both in physics and in mathematical physics: the Yang – Mills theory [6], Kaluza – Klein theory [7, 13], supergravity and superstring theories [14, 35], etc. Later such submersions were considered between manifolds with differentiable structures. As an analogue of holomorphic submanifolds, Watson defined almost Hermitian submersions between almost Hermitian manifolds and he showed that the base manifold and each fiber have the same kind of structure as the total space in most cases [34]. For more information about Riemannian submersions, there are books which cover recent results on this topic [11, 29].

In the 2010's, B. Şahin introduced the anti-invariant Riemannian submersions [26], semi-invariant Riemannian submersions [27] and slant submersions [28] from almost Hermitian manifolds onto

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Riemannian manifolds as an analogue of anti-invariant submanifolds, semi-invariant submanifolds and slant submanifolds, respectively. Afterwards, as a natural generalization of slant submersions, the notion of hemi-slant submersions has defined by Taştan et al. in [32]. Recently, there are many research papers on the geometry of hemi-slant submersions between various Riemannian manifolds [1, 2, 16, 19, 20].

In the 2014's, J. W. Lee and B. Şahin defined the notion of pointwise slant submersions, as a generalization of slant submersions which can be seen analogue of pointwise slant submanifolds and obtained several basic results in this setting in [12]. More precisely, let Ψ be a Riemannian submersion from an almost Hermitian manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). If at each given point $p \in M_1$ the Wirtinger angle $\theta(X)$ between J_1X and the space $\left(\ker \Psi_*\right)_p$ is independent of the choice of the nonzero vector $X \in \Gamma(\ker \Psi_*)$, then we say that Ψ is a pointwise slant submersion. In this case, the angle θ can be regarded as a function on M_1, which is called the slant function of the pointwise slant submersion. One can find many papers related to this notion [21–25].

Motivated by the above papers, as a generalization of pointwise slant submersion, we define the notion of pointwise hemi-slant submersions from almost Hermitian manifolds onto Riemannian manifolds.

This paper is organized as follows. In Section 2, we remind some notions, which are used later. In Section 3, we first recall almost Hermitian manifolds and define the notion of pointwise hemi-slant Riemannian submersions from almost Hermitian manifolds to Riemannian manifolds, giving a figure which shows the subclasses of the map and non-trivial (proper) examples and investigate integrability, totally geodesic foliations, J-pluriharmonicity, J-invariant and the totally geodesic map, respectively. Last subsection is devoted to study curvature relations between the total space, the base space, and the fibers.

2. Preliminaries. In this section, we give necessary background for Riemannian submersions.

Let (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, where $\dim(M_1)$ is greater than $\dim(M_2)$. A surjective mapping $\Psi\colon (M_1, g_1) \to (M_2, g_2)$ is called a *Riemannian submersion* [15] if

(S1) Ψ has maximal rank

and

(S2) Ψ_*, restricted to $\ker \Psi_*^{\perp}$, is a linear isometry.

In this case, for each $q \in M_2$, $\Psi^{-1}(q)$ is a k-dimensional submanifold of M_1 and called a *fiber*, where $k = \dim(M_1) - \dim(M_2)$. A vector field on M_1 is called *vertical* (resp., *horizontal*) if it is always tangent (resp., orthogonal) to fibers. A vector field X on M_1 is called *basic* if X is horizontal and Ψ-related to a vector field X_* on M_2, i.e., $\Psi_* X_p = X_{*\Psi(p)}$ for all $p \in M_1$. We will denote by $\mathcal{V}$ and $\mathcal{H}$ the projections on the vertical distribution $\ker \Psi_*$, and the horizontal distribution $(\ker \Psi_*)^{\perp}$, respectively. As usual, the manifold (M_1, g_1) is called *total manifold* and the manifold (M_2, g_2) is called *base manifold* of the submersion $\Psi\colon (M_1, g_1) \to (M_2, g_2)$. The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$, defined as follows:

$$\mathcal{T}_U V = \mathcal{V}\nabla_{\mathcal{V}U}\mathcal{H}V + \mathcal{H}\nabla_{\mathcal{V}U}\mathcal{V}V, \tag{2.1}$$

$$\mathcal{A}_U V = \mathcal{V}\nabla_{\mathcal{H}U}\mathcal{H}V + \mathcal{H}\nabla_{\mathcal{H}U}\mathcal{V}V \tag{2.2}$$

for any vector fields U and V on M_1, where ∇ is the Levi–Civita connection of g_1. It is easy to see that $\mathcal{T}_U$ and $\mathcal{A}_U$ are skew-symmetric operators on the tangent bundle of M_1 reversing the vertical and the horizontal distributions. We now summarize the properties of the tensor fields $\mathcal{T}$ and $\mathcal{A}$. Let $V,\ W$ be vertical and $X,\ Y$ be horizontal vector fields on M_1. Then we have

$$\mathcal{T}_V W = \mathcal{T}_W V, \tag{2.3}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\,\mathcal{V}[X,Y]. \tag{2.4}$$

On the other hand, from (2.1) and (2.2), we obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{2.5}$$

$$\nabla_V X = \mathcal{T}_V X + \mathcal{H}\nabla_V X, \tag{2.6}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{2.7}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{2.8}$$

where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W.$ If X is basic, then

$$\mathcal{H}\nabla_V X = \mathcal{A}_X V.$$

***Remark* 2.1.** In this paper, we will assume all horizontal vector fields as basic vector fields.

It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form while $\mathcal{A}$ acts on the horizontal distribution and measures of the obstruction to the integrability of this distribution. For details on Riemannian submersions, we refer to O'Neill's paper [15] and to the book [11].

Let Ψ be a C^{∞}-map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2). The second fundamental form of Ψ is given by

$$(\nabla\Psi_*)(X,Y) = \nabla^{\Psi}_X \Psi_* Y - \Psi_*(\nabla_X Y) \quad \text{for} \quad X,Y \in \Gamma(TM_1), \tag{2.9}$$

where ∇^{Ψ} is the pullback connection and we denote conveniently by ∇ the Levi–Civita connections of the metrics g_1 and g_2 [4].

The map Ψ is called a totally geodesic map if $(\nabla\Psi_*)(X,Y) = 0$ for $X,Y \in \Gamma(TM_1)$ [4].

3. Pointwise hemi-slant Riemannian submersions. In this section, we first recall almost Hermitian manifolds and define and study pointwise hemi-slant Riemannian submersions in the complex context.

A manifold M_1 is called an *almost Hermitian manifold* [36] if it admits a tensor field J of type $(1,1)$ on itself such that, for any $X,Y \in \Gamma(TM_1)$,

$$J^2 = -I, \quad g_1(X,Y) = g_1(JX,JY). \tag{3.1}$$

An almost Hermitian manifold M_1 is called *Kaehler manifold* [36] if, for all $X,Y \in \Gamma(TM_1)$,

$$(\nabla_X J)Y = 0, \tag{3.2}$$

where ∇ is the Levi–Civita connection with respect to the Riemannian metric g_1 and I is the identity operator on the tangent bundle TM_1.

Now, we give the definition of the pointwise hemi-slant Riemannian submersion.

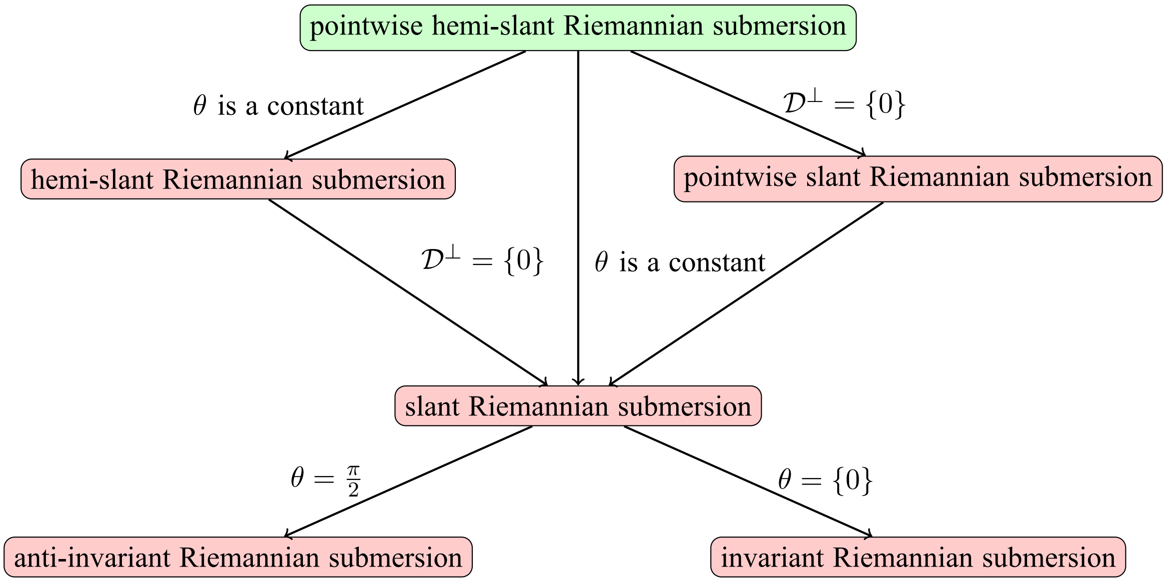

Fig. 1. The progress of the map.

Definition 3.1. *Let (M_1, g_1, J) be an almost Hermitian manifold and (M_2, g_2) be a Riemannian manifold. A Riemannian submersion $\Psi\colon (M_1, g_1, J) \to (M_2, g_2)$ is called a pointwise hemi-slant Riemannian submersion if the vertical distribution $\ker \Psi_*$ of Ψ decomposes into two orthogonal complementary distributions $\mathcal{D}^\perp$ (anti-invariant) and $\mathcal{D}^\theta$ (pointwise slant).*

In this case, we have the decomposition

$$\ker \Psi_* = \mathcal{D}^\perp \oplus \mathcal{D}^\theta, \tag{3.3}$$

where $\mathcal{D}^\perp$ is anti-invariant, $\mathcal{D}^\theta$ is pointwise slant distributions and the angle $\theta = \theta(U)$ between JU and the space $\left(\mathcal{D}^\theta\right)_q$ $(\forall q \in M_1)$ is independent of the choice of nonzero vector $U \in \Gamma\left(\mathcal{D}^\theta\right)_q$, which is called slant function of the pointwise hemi-slant Riemannian submersion (see Fig. 1).

We call the pointwise hemi-slant Riemannian submersion $\Psi\colon (M_1, g_1, J) \to (M_2, g_2)$ proper if $\mathcal{D}^\perp \neq \{0\}$ and $\theta \neq 0, \dfrac{\pi}{2}$.

***Example* 3.1.** Let $\mathbb{R}^8$ be the standard Euclidean space with the standard metric g_1. Consider $\{J_1, J_2\}$ a pair of almost complex structures on $\mathbb{R}^8$ satisfying $J_1 J_2 = -J_2 J_1$, where

$$J_1(x_1, \ldots, x_8) = (-x_3, -x_4, x_1, x_2, -x_7, -x_8, x_5, x_6),$$

$$J_2(x_1, \ldots, x_8) = (-x_2, x_1, x_4, -x_3, -x_6, x_5, x_8, -x_7).$$

For any real-valued function $f\colon \mathbb{R}^8 \to \mathbb{R} - \left\{\dfrac{\pi}{2}\right\}$, we define a new almost complex structure J_f on $\mathbb{R}^8$ by $J_f = (\cos f) J_1 + (\sin f) J_2$. Then $\mathbb{R}^8_f = \left(\mathbb{R}^8, J_f, g_1\right)$ is an almost Hermitian manifold.

Define a map $\Psi\colon \mathbb{R}^8_f \to \mathbb{R}^4$,

$$\Psi(x_1, \ldots, x_8) = (x_2, x_3, x_6, x_8).$$

Then the differential Ψ_* of the map Ψ is

$$\Psi_* := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is obvious that $\operatorname{rank} \Psi_* = 4$.

Consider the following decomposition, for the local coordinates on $\mathbb{R}^8_f$, $\partial x_i = \dfrac{\partial}{\partial x_i}$:

$$\ker \Psi_* = \mathcal{D}^\perp \oplus \mathcal{D}^\theta,$$

where

$$\mathcal{D}^\perp = \{a\partial x_1 + b\partial x_4,\ a, b \in \mathbb{R}\}$$

and

$$\mathcal{D}^\theta = \{c\partial x_5 + d\partial x_7,\ c, d \in \mathbb{R}\}.$$

It can be seen that $\mathcal{D}^\theta$ has the slant function $\theta = f$.

Then the set $\left(\ker \Psi_*\right)^\perp$ can be expressed as

$$(\ker \Psi_*)^\perp = \Big\{ e((\sin f)\partial x_2 + (\cos f)\partial x_3) + m((-\cos f)\partial x_2 + (\sin f)\partial x_3)\, g\partial x_6$$
$$+ h\partial x_8,\ e, m, g, h \in \mathbb{R} \Big\}.$$

In this case, the following calculation can be obtained for each horizontal vector field in $\left(\ker \Psi_*\right)^\perp$:

$$g_1((\sin f)\partial x_2 + (\cos f)\partial x_3, (\sin f)\partial x_2 + (\cos f)\partial x_3) = 1,$$

$$g_2(\phi_*((\sin f)\partial x_2 + (\cos f)\partial x_3), \phi_*((\sin f)\partial x_2 + (\cos f)\partial x_3))$$
$$= g_2((\sin f)\partial y_2 + (\cos f)\partial y_3, (\sin f)\partial y_2 + (\cos f)\partial y_3) = 1,$$

where g_2 is the Euclidean metric on $\mathbb{R}^4$ and $\partial y_i = \dfrac{\partial}{\partial y_i}$ are local coordinates on $\mathbb{R}^4$. Therefore, the restriction of Ψ_* onto $\left(\ker \Psi_*\right)^\perp$ preserves the lenghts of the horizontal vectors. It follows that Ψ is a pointwise hemi-slant Riemannian submersion with the slant function $\theta = f$.

***Example* 3.2.** Let $J_{\mathbb{R}^6}$ be an almost complex structure on $\mathbb{R}^6$ as follows:

$$J_{\mathbb{R}^6}(x_1, \ldots, x_6) = (x_2, -x_1, \ldots, x_6, -x_5).$$

Consider a map $\Psi : \mathbb{R}^6 \to \mathbb{R}^3$ by

$$\Psi(x_1, \ldots, x_6) = (x_1 \cos\alpha - x_3 \sin\alpha, x_2 \sin\beta - x_4 \cos\beta, x_5),$$

where $\alpha, \beta : \mathbb{R}^6 \to \mathbb{R}$ are real valued functions. Then the map Ψ is a pointwise hemi-slant Riemannian submersion such that

$$\mathcal{D}^\perp = \operatorname{span}\left\{\frac{\partial}{\partial x_6}\right\} \quad \text{and} \quad \mathcal{D}^\theta = \operatorname{span}\left\{\sin\alpha \frac{\partial}{\partial x_1} + \cos\alpha \frac{\partial}{\partial x_3}, \cos\beta \frac{\partial}{\partial x_2} + \sin\beta \frac{\partial}{\partial x_4}\right\}$$

with the hemi-slant function θ with $\cos\theta = \sin(\alpha + \beta)$.

Let Ψ be a pointwise hemi-slant Riemannian submersion from an almost Hermitian manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then, for any $U \in \Gamma(\ker \Psi_*)$ and $\xi \in \Gamma(\ker \Psi_*)^{\perp}$, while we can consider

$$U = FU + QU,$$

where $U|_{\mathcal{D}^{\perp}} = FU$ and $U|_{\mathcal{D}^{\theta}} = QU$, we may also write

$$JU = PU + NU, \tag{3.4}$$

$$J\xi = \mathcal{B}\xi + \mathcal{C}\xi, \tag{3.5}$$

where $PU, \mathcal{B}\xi \in \Gamma(\ker \Psi_*)$ and $NU, \mathcal{C}\xi \in \Gamma(\ker \Psi_*)^{\perp}$. It is known that each distribution ($\mathcal{D}^{\perp}$ and $\mathcal{D}^{\theta}$) which are mentioned in Definition 3.1 are P-invariant. In this case, $(\ker \Psi_*)^{\perp}$ can be decomposed as

$$(\ker \Psi_*)^{\perp} = J\mathcal{D}^{\perp} \oplus N\mathcal{D}^{\theta} \oplus \mu,$$

where μ is the orthogonal distribution of $J\mathcal{D}^{\perp} \oplus N\mathcal{D}^{\theta}$ in $(\ker \Psi_*)^{\perp}$.

One can obtain the following fundamental equalities for a pointwise hemi-slant Riemannian submersion:

$$\text{(a)}\ \ P^2 + \mathcal{B}N = -I, \qquad \text{(b)}\ \ NP + \mathcal{C}N = 0,$$

$$\text{(c)}\ \ P\mathcal{B} + \mathcal{B}\mathcal{C} = 0, \qquad \text{(d)}\ \ N\mathcal{B} + \mathcal{C}^2 = -I,$$

where I is the identity operator on TM_1. Moreover, one can obtain the following lemma.

Lemma 3.1. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then we have*

$$\text{(a)}\ \ P^2U = (-\cos^2\theta)U, \qquad \text{(b)}\ \ \mathcal{B}NU = (-\sin^2\theta)U,$$

$$\text{(c)}\ \ \mathcal{B}NX = -X, \qquad \text{(d)}\ \ NPU + \mathcal{C}NU = 0,$$

$$\text{(e)}\ \ P^2X = 0, \qquad \text{(f)}\ \ P^2U + \mathcal{B}NU = -U,$$

where $X \in \Gamma(\mathcal{D}^{\perp})$, $U \in \Gamma(\mathcal{D}^{\theta})$.

We give the following lemma which is usual for the slant distribution $\mathcal{D}^{\theta}$.

Lemma 3.2. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then, for all $U \in \Gamma(\mathcal{D}^{\theta})$, we have*

$$g_1(PU, PU) = \cos^2\theta\, g_1(U, U),$$

$$g_1(NU, NU) = \sin^2\theta\, g_1(U, U).$$

3.1. Integrability. In this subsection, we investigate the integrability for pointwise hemi-slant Riemannian submersions. First, we study the integrability of the distributions which are mentioned in Definition 3.1. In this direction, we give the following lemma.

Lemma 3.3. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then we have the following equality:*

$$g_1(\mathcal{A}_{NY}NU, X) = g_1(\mathcal{A}_{NX}NU, Y), \tag{3.6}$$

$$NX \perp NU, \tag{3.7}$$

where $X, Y \in \Gamma(\mathcal{D}^{\perp})$ and $U \in \Gamma(\mathcal{D}^{\theta})$.

Proof. The skew-symmetry of the O'Neill tensor $\mathcal{A}$, the parallelism of the complex structure J, and the equation (2.8) yield us (3.6).

On the other hand, (3.4) and Lemma 3.1(b) give (3.7), which complete the proof.

Theorem 3.1. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then the totally real distribution $\mathcal{D}^{\perp}$ is always integrable.*

Proof. For $X, Y \in \Gamma(\mathcal{D}^{\perp})$ and $U \in \Gamma(\mathcal{D}^{\theta})$, by using (2.5), (2.8), (3.1), (3.2), (3.4), (3.5), the skew-symmetry of $\mathcal{T}$ and $\mathcal{A}$ and Lemma 3.1, it follows that

$$\begin{aligned} g_1(\nabla_X Y, U) &= g_1(\nabla_X JY, PU) + g_1(\nabla_X JY, NU) \\ &= -g_1(\nabla_X Y, P^2 U) - g_1(\nabla_X Y, NPU) + g_1(\mathcal{A}_{NY}X, NU) \\ &= \cos^2\theta g_1(\nabla_X Y, U) - g_1(\mathcal{T}_X Y, NPU) + g_1(\mathcal{A}_{NY}X, NU) \\ &\Rightarrow \sin^2\theta g_1(\nabla_X Y, U) = g_1(\mathcal{T}_Y NPU - \mathcal{A}_{NY}NU, X) \\ &\Rightarrow g_1(\nabla_X Y, U) = \csc^2\theta g_1(\mathcal{T}_Y NPU - \mathcal{A}_{NY}NU, X). \end{aligned} \tag{3.8}$$

Interchanging X and Y in (3.8) gives us

$$g_1(\nabla_Y X, U) = \csc^2\theta g_1(\mathcal{T}_X NPU - \mathcal{A}_{NX}NU, Y). \tag{3.9}$$

Therefore, the skew-symmetry of $\mathcal{T}$, (2.3), Lemma 3.3, (3.8), and (3.9) yield us

$$g_1([X, Y], U) = 0, \quad \text{i.e.,} \quad [X, Y] \in \mathcal{D}^{\perp},$$

which implies that $\mathcal{D}^{\perp}$ is integrable.

Theorem 3.2. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then the pointwise slant distribution $\mathcal{D}^{\theta}$ is integrable if and only if*

$$g_1(\mathcal{T}_X NPV + \mathcal{A}_{NV}NX, U) = g_1(\mathcal{T}_X NPU + \mathcal{A}_{NU}NX, V), \tag{3.10}$$

where $X \in \Gamma(\mathcal{D}^{\perp})$ and $U, V \in \Gamma(\mathcal{D}^{\theta})$.

Proof. Let $X \in \Gamma(\mathcal{D}^{\perp})$ and $U, V \in \Gamma(\mathcal{D}^{\theta})$. With the help of (2.5), (2.7), the skew-symmetry of $\mathcal{T}$ and $\mathcal{A}$, (3.1), and Lemma 3.3, we have

$$\begin{aligned} g_1(\nabla_U V, X) &= g_1(\nabla_U PV, JX) + g_1(\nabla_U NV, JX) \\ &= -g_1(\nabla_U P^2 V, X) - g_1(\nabla_U NPV, X) + g_1(\mathcal{A}_{NV}U, NX) \\ &= \cos^2\theta g_1(\nabla_U V, X) - g_1(\mathcal{T}_X NPV, U) - g_1(\mathcal{A}_{NV}NX, U) \end{aligned}$$

$$\Rightarrow \sin^2\theta g_1(\nabla_U V, X) = -g_1(\mathcal{T}_X NPV + \mathcal{A}_{NV} NX, U)$$

$$\Rightarrow g_1(\nabla_U V, X) = -\csc^2\theta g_1(\mathcal{T}_X NPV + \mathcal{A}_{NV} NX, U). \tag{3.11}$$

By interchanging U and V in (3.11), it follows that

$$g_1(\nabla_V U, X) = -\csc^2\theta g_1(\mathcal{T}_X NPU + \mathcal{A}_{NU} NX, V). \tag{3.12}$$

Thus, (3.11) and (3.12) show that $\mathcal{D}^\theta$ is integrable if and only if (3.10) holds.

3.2. Totally geodesic foliations. In this subsection, we investigate the totally geodesic foliations for pointwise hemi-slant Riemannian submersions.

First, we prove the following theorem for $\mathcal{D}^\perp$.

Theorem 3.3. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then any of the following first two conditions below implies the third condition:*

(a) *the anti-invariant distribution $\mathcal{D}^\perp$ defines totally geodesic foliations on M_1,*

(b) $g_2((\nabla\Psi_*)(U, JV), \Psi_*(\mathcal{N}Z)) = g_1(\mathcal{T}_U JV, PZ)$,

(c) $\mathcal{C}(\mathcal{T}_U JV + H\nabla_U JV)$ *has no component in* $\Gamma\big(\ker\Psi_*\big)^\perp$,

(d) $g_1(\mathcal{T}_V NPZ - \mathcal{A}_{NV} NZ, U) = 0$

for $U, V \in \Gamma\big(\mathcal{D}^\perp\big)$ *and* $Z \in \Gamma\big(\mathcal{D}^\theta\big)$.

Proof. Let $U, V \in \Gamma\big(\mathcal{D}^\perp\big)$ and $Z \in \Gamma\big(\mathcal{D}^\theta\big)$. With the help of (3.2), (3.4), we get

$$g_1(\nabla_U V, Z) = g_1(\nabla_U JV, PZ) + g_1(\nabla_U JV, NZ).$$

Now, by using (2.6), we obtain

$$g_1(\nabla_U V, Z) = g_1(\mathcal{T}_U JV, PZ) + g_2(\Psi_*(\mathcal{H}\nabla_U JV), \Psi_*(NZ))$$

$$= g_1(\mathcal{T}_U JV, PZ) - g_2((\nabla\Psi_*)(U, JV), \Psi_*(NZ)).$$

On the other hand, for $U, V \in \Gamma\big(\mathcal{D}^\perp\big)$ and $Z \in \Gamma\big(\mathcal{D}^\theta\big)$, from (3.2), we have

$$g_1(\nabla_U V, Z) = g_1(-J(\nabla_U JV), Z).$$

Taking into account (2.6) and (3.4), we get

$$g_1(\nabla_U V, Z) = -g_1(J(\mathcal{T}_U JV + \mathcal{H}\nabla_U JV), Z) = -g_1(\mathcal{C}(\mathcal{T}_U JV + \mathcal{H}\nabla_U JV), Z).$$

Further, (2.5) and (3.8) yield us

$$g_1\Big(\hat{\nabla}_U V, Z\Big) = \csc^2\theta g_1(\mathcal{T}_V NPZ - \mathcal{A}_{NV} NZ, U),$$

which says the anti-invariant distribution $\mathcal{D}^\perp$ defines totally geodesic foliations on M if and only if condition (d) holds.

It is known that covariant derivative of P is defined by

$$(\nabla_X P)Y = \hat{\nabla}_X PY - P\hat{\nabla}_X Y, \tag{3.13}$$

for all X, Y tangent vector fields. If $\nabla P \equiv 0$, then the canonical structure P is called *parallel*.

Under the assumption of the parallelism of P, we may give interesting result for the totally geodesicity of the anti-invariant distribution $\mathcal{D}^\perp$ as in the following theorem.

Theorem 3.4. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2) with the parallel canonical structure P. Then $\mathcal{D}^\perp$ defines totally geodesic foliations on the fibers.*

Proof. To prove the totally geodesicity of $\mathcal{D}^\perp$, we will use the following fact: $\mathcal{D}^\perp$ defines totally geodesic foliations on the fibers if and only if, for all $X, Y \in \Gamma(\mathcal{D}^\perp)$, $\hat{\nabla}_X Y \in \mathcal{D}^\perp$. By (2.5) and (3.8), we have

$$\sin^2\theta g_1\Big(\hat{\nabla}_X Y, U\Big) = g_1(\mathcal{T}_Y NPU - \mathcal{A}_{NY}NU, X),$$

where $X, Y \in \Gamma(\mathcal{D}^\perp)$, $U \in \Gamma(\mathcal{D}^\theta)$. Therefore, (3.1), Lemma 3.1 and (3.13) yield us

$$\sin^2\theta g_1\Big(\hat{\nabla}_X Y, U\Big) = -\sin^2\theta g_1\Big(\hat{\nabla}_X Y, \mathcal{B}NU\Big) = \sin^4\theta g_1\Big(\hat{\nabla}_X Y, U\Big)$$

$$\Rightarrow \big(\sin^2\theta - \sin^4\theta\big) g_1\Big(\hat{\nabla}_X Y, U\Big) = 0$$

$$\Rightarrow \sin^2\theta\cos^2\theta g_1\Big(\hat{\nabla}_X Y, U\Big) = 0,$$

which says that $\hat{\nabla}_X Y \in \mathcal{D}^\perp$ and completes the proof.

Theorem 3.5. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then the following assertions are equivalent to each other:*

(a) *the distribution* $\ker\Psi_*$ *defines totally geodesic foliations on* M_1,

(b) $\sin^2\theta g_1(\nabla_U V, X) = \sin 2\theta X(\theta) g_1(QU, QV) + g_1(\sin^2\theta[X, U] + \cos^2\theta \mathcal{V}\nabla_X PU + B\mathcal{A}_X\mathcal{T}PU + \mathcal{T}\mathcal{V}\nabla_X\mathcal{T}PU + \mathcal{T}\mathcal{A}_X FU + \mathcal{B}\mathcal{H}\nabla_X FU + \mathcal{A}_X F\mathcal{T}QU, V)$ *for* $U, V \in \Gamma\big(\ker\Psi_*\big)$ *and* $X \in \Gamma\big(\ker\Psi_*\big)^\perp$.

***3.3. J-pluriharmonicity of* Ψ.** In this subsection, we are going to investigate the J-pluriharmonicity of the Ψ with respect to the distibutions on the total space. First, we give the following definition.

Definition 3.2. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Ψ is called J-pluriharmonic, $\big(\ker\Psi_*\big)^\perp$-$J$-pluriharmonic,* $\ker\Psi_*$*-J-pluriharmonic, $\mathcal{D}^\perp$-J-pluriharmonic, $\mathcal{D}^\theta$-J-pluriharmonic, $\big(\mathcal{D}^\perp - \mathcal{D}^\theta\big)$-$J$-pluriharmonic and $\Big(\big(\ker\Psi_*\big)^\perp - \ker\Psi_*\Big)$-$J$-pluriharmonic if*

$$(\nabla\Psi_*)(X, Y) + (\nabla\Psi_*)(JX, JY) = 0$$

for any $X, Y \in \Gamma(TM_1)$, for any $X, Y \in \Gamma\big(\ker\Psi_*\big)^\perp$, for any $X, Y \in \Gamma\big(\ker\Psi_*\big)$, for any $X, Y \in \Gamma\big(\mathcal{D}^\perp\big)$, for any $X, Y \in \Gamma\big(\mathcal{D}^\theta\big)$, for any $X \in \Gamma\big(\ker\Psi_*\big)^\perp$, $Y \in \Gamma\big(\ker\Psi_*\big)$.

We first prove the following theorem.

Theorem 3.6. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Suppose that the map Ψ is a $\mathcal{D}^\perp$- J-pluriharmonic. Then the map Ψ is a* $\ker\Psi_*$*-geodesic map if and only if $\mathcal{T} = \{0\}$, which gives that the fibers are totally geodesic submanifolds.*

Proof. For any $U, V \in \Gamma(\mathcal{D}^{\perp})$, since $\mathcal{D}^{\perp}$-J-pluriharmonic, by virtue of (2.9), we have

$$0 = (\nabla\Psi_*)(U, V) + (\nabla\Psi_*)(JU, JV) = -\Psi_*(\mathcal{T}_U V) + (\nabla\Psi_*)(JU, JV),$$

which gives the proof.

For the slant distribution $\mathcal{D}^{\theta}$, we have the following theorem.

Theorem 3.7. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Suppose that the map Ψ is a $\mathcal{D}^{\theta}$-J-pluriharmonic. Then the map Ψ is a $N\mathcal{D}^{\theta}$-geodesic map if and only if $\mathcal{T}_U V + \mathcal{T}_{PU} PV + \mathcal{H}\nabla_{PV} NW + \mathcal{A}_{NV} PW = 0$.*

Proof. Given $U, V \in \Gamma(\mathcal{D}^{\theta})$, since $\mathcal{D}^{\theta}$-J-pluriharmonic, by virtue of (2.9), we obtain

$$0 = (\nabla\Psi_*)(V, W) + (\nabla\Psi_*)(JV, JW) = -\Psi_*(\mathcal{T}_V W) + (\nabla\Psi_*)(NV, NW) -$$

$$- \Psi_*(\mathcal{T}_{PV} PW + \mathcal{H}\nabla_{PV} NW + \mathcal{A}_{NV} PW),$$

$$(\nabla\Psi_*)(NV, NW) = -\Psi_*(\mathcal{T}_V W + \mathcal{T}_{PV} PW + \mathcal{H}\nabla_{PV} NW + \mathcal{A}_{NV} PW),$$

which completes the proof.

For $(\mathcal{D}^{\perp} - \mathcal{D}^{\theta})$-$J$-pluriharmonicity, we prove the following theorem.

Theorem 3.8. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Suppose that the map Ψ is a $(\mathcal{D}^{\perp} - \mathcal{D}^{\theta})$-$J$-pluriharmonic. Then the following assertions are equivalent:*

(i) *the anti-invariant distribution $\mathcal{D}^{\perp}$ defines a totally geodesic foliations on M_1,*

(ii) $\nabla^{M_2}_{\psi_* JV} \Psi_* NW = \Psi_*(\mathcal{C}\mathcal{A}_{JV} W + N\mathcal{V}\nabla_{JV} W)$.

Proof. For $V \in \Gamma(\mathcal{D}^{\perp})$ and $W \in \Gamma(\mathcal{D}^{\theta})$, since the map Ψ is a $(\mathcal{D}^{\perp} - \mathcal{D}^{\theta})$-$J$-pluriharmonic, by using (2.9), we get

$$0 = (\nabla\Psi_*)(V, W) + (\nabla\Psi_*)(JV, JW)$$

$$= -\Psi_*(\nabla_V W) + \nabla^{M_2}_{\Psi_*(JV)} \Psi_*(NW) - \Psi_*(J\nabla_{JV} W)$$

$$= -\Psi_*(\nabla_V W) + \nabla^{M_2}_{\Psi_*(JV)} \Psi_*(NW) - \Psi_*(\mathcal{C}\mathcal{A}_{JV} W + N\mathcal{V}\nabla_{JV} W),$$

$$\Psi_*(\nabla_V W) = \nabla^{N}_{\Psi_*(JV)} \Psi_*(NW) - \Psi_*(\mathcal{C}\mathcal{A}_{JV} W + N\mathcal{V}\nabla_{JV} W),$$

which gives the proof.

Finally, for $\Big((\ker\Psi_*)^{\perp} - \ker\Psi_*\Big)$-$J$-pluriharmonicity, we have the following theorem.

Theorem 3.9. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Suppose that the map Ψ is a $((\ker\Psi_*)^{\perp} - \ker\Psi_*)$-$J$-pluriharmonic. Then the following assertions are equivalent*:

(i) *the horizontal distribution $(\ker\Psi_*)^{\perp}$ defines a totally geodesic foliations on M_1,*

(ii) $(\nabla\Psi_*)(\mathcal{C}X, NU) = -\Psi_*(\mathcal{T}_{\mathcal{B}X}\phi U + \mathcal{H}\nabla_{\mathcal{B}X} NU + \mathcal{A}_{\mathcal{C}X} PU)$ *for any $X \in \Gamma(\ker\Psi_*)^{\perp}$ and $U \in \Gamma(\ker\Psi_*)$.*

Proof. For $X \in \Gamma\big(\ker \Psi_*\big)^{\perp}$ and $U \in \Gamma\big(\ker \Psi_*\big)$, since the map Ψ is a $\big((\ker \Psi_*)^{\perp} - \ker \Psi_*\big)$-$J$-pluriharmonic, by using (2.9), we get

$$\begin{aligned} 0 &= (\nabla\Psi_*)(X,U) + (\nabla\Psi_*)(JX,JU) \\ &= -\Psi_*(\nabla_X U) + (\nabla\Psi_*)(\mathcal{B}X, PU) + (\nabla\Psi_*)(\mathcal{B}X, NU) \\ &\quad + (\nabla\Psi_*)(\mathcal{C}X, PU) + (\nabla\Psi_*)(\mathcal{C}X, NU) \\ &= -\Psi_*(\nabla_X U) - \Psi_*(\mathcal{T}\mathcal{B}X, PU) - \Psi_*(\mathcal{H}\nabla_{\mathcal{B}X} NU) \\ &\quad - \Psi_*(\mathcal{A}_{\mathcal{C}X} PU) + (\nabla\Psi_*)(\mathcal{C}X, NU), \\ (\nabla\Psi_*)(\mathcal{C}X, NU) &= -\Psi_*(\nabla_X U) - \Psi_*(\mathcal{T}\mathcal{B}X, PU + \mathcal{H}\nabla_{\mathcal{B}X} NU + \mathcal{A}_{\mathcal{C}X} PU), \end{aligned}$$

which completes the proof.

3.4. J-invariant and totally geodesic maps of Ψ. In this subsection, we will find necessary and sufficient conditions for the map to be the J-invariant of the distibutions on the total space. First, we give the following definition.

Definition 3.3. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Ψ is called J-invariant, $\big(\ker \Psi_*\big)^{\perp}$-$J$-invariant, $\ker \Psi_*$-J-invariant, $\mathcal{D}^{\perp}$-J-invariant, $\mathcal{D}^{\theta}$-J-invariant, $\big(\mathcal{D}^{\perp} - \mathcal{D}^{\theta}\big)$-$J$-invariant and $\Big(\big(\ker \Psi_*\big)^{\perp} - -\ker \Psi_*\Big)$-$J$-invariant if*

$$(\nabla\Psi_*)(Z,W) = (\nabla\Psi_*)(JZ,JW)$$

for any $Z, W \in \Gamma(TM_1)$, for any $Z, W \in \Gamma\big(\ker \Psi_*\big)^{\perp}$, for any $Z, W \in \Gamma\big(\ker \Psi_*\big)$, for any $Z, W \in \Gamma\big(\mathcal{D}^{\perp}\big)$, for any $Z, W \in \Gamma\big(\mathcal{D}^{\theta}\big)$, for any $Z \in \Gamma\big(\ker \Psi_*\big)^{\perp}$, $W \in \Gamma\big(\ker \Psi_*\big)$.

We first prove the following theorem.

Theorem 3.10. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Suppose that the map Ψ is a $\mathcal{D}^{\perp}$-J-invariant. The following assertions are equivalent:*

(i) *the anti-invariant distribution $\mathcal{D}^{\perp}$ defines a totally geodesic foliations on M_1,*

(ii) $\nabla^{M_2}_{\Psi_* JX} \Psi_* JZ = \Psi_*(\mathcal{C}\mathcal{A}_{JX} Z + N\mathcal{V}\nabla_{JX} Z)$ *for any $X, Z \in \Gamma\big(\mathcal{D}^{\perp}\big)$.*

Proof. Given $X, Z \in \Gamma\big(\mathcal{D}^{\perp}\big)$, since the map is $\mathcal{D}^{\perp}$-J-invariant, by using (2.9), we get the proof.

For the slant distribution $\mathcal{D}^{\theta}$, we have the following theorem.

Theorem 3.11. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Suppose that the map Ψ is a $\mathcal{D}^{\theta}$-J-invariant. The following assertions are equivalent:*

(i) *the fibers are totally geodesic submanifolds in M_1,*

(ii) $\nabla\Psi_*(NU, NV) = \Psi_*(\mathcal{T}_{PU} PU + \mathcal{H}\nabla_{PU} NV - \mathcal{A}_{NU} PU)$ *for any $U, V \in \Gamma\big(\mathcal{D}^{\theta}\big)$.*

Proof. Given $U, V \in \Gamma(\mathcal{D}^{\theta})$, since $\mathcal{D}^{\theta}$-J-invariant, by virtue of (2.9), we obtain

$$\begin{aligned} &(\nabla\Psi_*)(U,V) = (\nabla\Psi_*)(JU,JV), \\ &-\Psi_*(\nabla_U V) = -\Psi_*(\nabla_{PU} PV) - \Psi_*(\nabla_{PU} NV) - \Psi_*(\nabla_{NU} PV) - \Psi_*(\nabla_{NU} NV), \end{aligned}$$

$$-\Psi_*(\nabla_U V) = -\Psi_*(\mathcal{T}_{PU}PV + \mathcal{H}\nabla_{PU}NV - \mathcal{A}_{NU}PV) - \Psi_*(\nabla_{NU}NV),$$

which completes the proof.

For $(\mathcal{D}^{\perp} - \mathcal{D}^{\theta})$-$J$-invariant, we have the following theorem.

Theorem 3.12. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). The map Ψ is a $(\mathcal{D}^{\perp} - \mathcal{D}^{\theta})$-$J$-invariant if and only if $\nabla^{M_2}_{\Psi_*(JX)}\Psi_*(NU) = \Psi_*(\mathcal{A}_{JX}PU + \mathcal{H}\nabla_{JX}NU - \mathcal{A}_X U)$ for any $X \in \Gamma(\mathcal{D}^{\perp})$ and $U \in \Gamma(\mathcal{D}^{\theta})$.*

Proof. Given $X \in \Gamma(\mathcal{D}^{\perp})$ and $U \in \Gamma(\mathcal{D}^{\theta})$, since $(\mathcal{D}^{\perp} - \mathcal{D}^{\theta})$-$J$-invariant, by virtue of (2.9), we get

$$(\nabla\Psi_*)(X, U) = (\nabla\Psi_*)(JX, JU),$$

$$-\Psi_*(\nabla_X U) = -\Psi_*(\nabla_{JX}PU) - \nabla^{M_2}_{\Psi_*(JX)}\Psi_*(NU) - \Psi_*(\nabla_{JX}NU),$$

$$-\Psi_*(\nabla_U V) = -\Psi_*(\mathcal{A}_{JX}PU + \mathcal{H}\nabla_{JX}NU - \Psi_*(\mathcal{H}\nabla_{JX}NU),$$

which gives the proof.

Finally, for $\left((\ker\Psi_*)^{\perp} - \ker\Psi_*\right)$-$J$-invariant, we have the following theorem.

Theorem 3.13. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). The map Ψ is a $\left((\ker\Psi_*)^{\perp} - \ker\Psi_*\right)$-$J$-invariant if and only if $\mathcal{C}(\mathcal{T}_{\mathcal{B}X}U + \mathcal{A}_{\mathcal{C}X}U) + N(\hat{\nabla}_{\mathcal{B}X}U + \mathcal{V}\nabla_{\mathcal{C}X}U + \mathcal{A}_X U) = 0$ for any $X \in \Gamma(\ker\Psi_*)^{\perp}$ and $U \in \Gamma(\ker\Psi_*)$.*

Proof. Given $X \in \Gamma(\ker\Psi_*)^{\perp}$ and $U \in \Gamma(\ker\Psi_*)$. We assume that the map is invariant. In this case, by virtue of (2.9), we get

$$(\nabla\Psi_*)(X, U) = (\nabla\Psi_*)(JX, JU),$$

$$-\Psi_*(\nabla_X U) = -\Psi_*(\nabla_{\mathcal{B}X}JU) - \Psi_*(\nabla_{\mathcal{C}X}JU),$$

$$-\Psi_*(\nabla_U V) = -\Psi_*(J(\mathcal{T}_{\mathcal{B}X}U + \hat{\nabla}_{\mathcal{B}X}U) + J(\mathcal{A}_{\mathcal{C}X}U + \mathcal{V}\nabla_{\mathcal{C}X}U),$$

$$0 = \Psi_*(\mathcal{C}(\mathcal{T}_{\mathcal{B}X}U + \mathcal{A}_{\mathcal{C}X}U) + N(\hat{\nabla}_{\mathcal{B}X}U + \mathcal{V}\nabla_{\mathcal{C}X}U + \mathcal{A}_X U)),$$

which completes the proof.

Recall that a map Ψ is called totally geodesic if $(\nabla\Psi_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM_1)$. Geometrically the notion implies that for each geodesic β in M_1 the image $\Psi(\beta)$ is a geodesic in M_2.

Theorem 3.14. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then Ψ is totally geodesic if and only if*

$$\omega\mathcal{T}_U JV + \mathcal{C}\mathcal{H}\nabla_U JV = 0,$$

$$\sin 2\theta U(\theta)Z + \mathcal{H}\nabla_U NPZ + \mathcal{C}\mathcal{H}\nabla_U NZ + N\mathcal{T}_U NZ = 0,$$

$$\sin 2\theta X(\theta)Z + \mathcal{H}\nabla_X NPZ + \mathcal{C}\mathcal{H}\nabla_X NZ + N\mathcal{A}_X NZ = 0,$$

and

$$\nabla_X^{\Psi}\Psi_*(Y) = -\Psi_*\big(\mathcal{A}_X P\mathcal{B}Y + \mathcal{H}\nabla_X N\mathcal{B}Y\big) + \mathcal{C}\mathcal{H}\nabla_X \mathcal{C}Y + N\mathcal{A}_X\mathcal{C}Y\big)$$

for $\xi \in \Gamma\big(\ker\Psi_*\big)$, $U, V \in \Gamma\big(\mathcal{D}^{\perp}\big)$, $Z \in \Gamma\big(\mathcal{D}^{\theta}\big)$ *and* $X, Y \in \Gamma\big(\ker\Psi_*\big)^{\perp}$.

Proof. For $U, V \in \Gamma\big(\mathcal{D}^{\perp}\big)$, from (3.2), we have

$$(\nabla\Psi_*)(U, V) = \Psi_*(J\nabla_U JV).$$

By virtue of (2.6), (3.4) and (3.5), we get

$$(\nabla\Psi_*)(U, V) = \Psi_*(\omega\mathcal{T}_U JV + \mathcal{C}\mathcal{H}\nabla_U JV). \tag{3.14}$$

For $U \in \Gamma\big(\ker\Psi_*\big)$ and $Z \in \Gamma\big(\mathcal{D}^{\theta}\big)$, (2.9), (3.2) and (3.4) imply

$$(\nabla\Psi_*)(U, Z) = \Psi_*\big(\nabla_U P^2 Z + \nabla_U NPZ + N\mathcal{T}_U NZ + \mathcal{C}\mathcal{H}\nabla_U NZ\big).$$

Then, by using Lemma 3.1(a), we derive

$$\sin^2\theta(\nabla\Psi_*)(U, Z) = \Psi_*(\sin 2\theta U(\theta)Z + \mathcal{H}\nabla_U NPZ + \mathcal{C}\mathcal{H}\nabla_U NZ + N\mathcal{T}_U NZ). \tag{3.15}$$

In a similar way, for $X \in \Gamma\big(\ker\Psi_*\big)^{\perp}$ and $Z \in \Gamma\big(\mathcal{D}^{\theta}\big)$, we obtain

$$\sin^2\theta(\nabla\Psi_*)(X, Z) = \Psi_*(\sin 2\theta X(\theta)Z + \mathcal{H}\nabla_X NPZ + \mathcal{C}\mathcal{H}\nabla_X NZ + N\mathcal{A}_X NZ). \tag{3.16}$$

For $X, Y \in \Gamma\big(\ker\Psi_*\big)^{\perp}$, from (2.9), (3.2) and (2.7), we have

$$\begin{aligned}(\nabla\Psi_*)(X, Y) &= \nabla_X^{\Psi}\Psi_*(Y) + \Psi_*(\nabla_X J\mathcal{B}Y) + \Psi_*(J\nabla_X \mathcal{C}Y)\\ &= \nabla_X^{\Psi}\Psi_*(Y) + \Psi_*(\mathcal{A}_X P\mathcal{B}Y + \mathcal{H}\nabla_X N\mathcal{B}Y + \mathcal{C}\mathcal{H}\nabla_X\mathcal{C}Y + N\mathcal{A}_X\mathcal{C}Y).\end{aligned} \tag{3.17}$$

Thus, the proof is complete due to (3.14) – (3.17).

3.5. Curvature relations. In this subsection, the curvature relations between total space, base space and fibers are studied.

Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Let R, R^* and $\hat{R}$ denote the Riemannian curvature tensors of M_1, M_2 and any fiber of submersion, respectively. In this case, we have the following equalities:

$$R(U, V, W, Z) = \hat{R}(U, V, W, Z) - g_1(\mathcal{T}_U W, \mathcal{T}_V Z) + g_1(\mathcal{T}_V W, \mathcal{T}_U Z), \tag{3.18}$$

$$R(U, V, W, X) = g_1((\nabla_V\mathcal{T})(U, W), X) - g_1((\nabla_U\mathcal{T})(V, W), X), \tag{3.19}$$

$$\begin{aligned}R(X, Y, K, L) = R^*(X, Y, K, L) &- 2g_1(\mathcal{A}_X Y, \mathcal{A}_K L)\\ &+ g_1(\mathcal{A}_Y K, \mathcal{A}_X L) + g_1(\mathcal{A}_X K, \mathcal{A}_Y L),\end{aligned} \tag{3.20}$$

$$\begin{aligned}R(X, Y, K, U) = g_1((\nabla_K\mathcal{A})(X, Y), U) &+ g_1(\mathcal{A}_X Y, \mathcal{T}_U K)\\ &- g_1(\mathcal{A}_Y K, \mathcal{T}_U X) - g_1(\mathcal{A}_K X, \mathcal{T}_U K),\end{aligned} \tag{3.21}$$

$$\begin{aligned}R(X, Y, U, V) = g_1((\nabla_U\mathcal{A})(X, Y), V) &- g_1((\nabla_V\mathcal{A})(X, Y), U) + g_1(\mathcal{A}_X U, \mathcal{A}_Y V)\\ &- g_1(\mathcal{A}_X V, \mathcal{A}_Y U) - g_1(\mathcal{T}_U X, \mathcal{T}_V Y) + g_1(\mathcal{T}_V X, \mathcal{T}_U Y),\end{aligned} \tag{3.22}$$

$$R(X,U,Y,V) = g_1((\nabla_X \mathcal{T})(U,V),Y) + g_1((\nabla_U \mathcal{A})(X,Y),V) \\ - g_1(\mathcal{T}_U X, \mathcal{T}_V Y) + g_1(\mathcal{A}_X U, \mathcal{A}_Y V), \tag{3.23}$$

where $U,\ V,\ W,\ Z$ are vertical and $X,\ Y,\ K,\ L$ are horizontal vector fields on tangent space.

While the sectional curvature K is defined by [15], for any non-zero vector fields X and Y,

$$K(X,Y) = \frac{R(X,Y,Y,X)}{g_1(X,X)g_1(Y,Y) - g_1^2(X,Y)}. \tag{3.24}$$

The holomorphic sectional curvature h is

$$h(X) = K(X,X). \tag{3.25}$$

For the sectional curvature, with the help of (3.18) $\sim$ (3.23), one can obtain the following equalities [15]:

$$K(X,Y) = \hat{K}(X,Y) - g_1(\mathcal{T}_X X, \mathcal{T}_Y Y) + \|\mathcal{T}_X Y\|^2, \tag{3.26}$$

$$K(\xi,X) = g((\nabla_\xi \mathcal{T})(X,X),\xi) + \|\mathcal{A}_\xi X\|^2 - \|\mathcal{T}_X \xi\|^2, \tag{3.27}$$

$$K(\xi,\eta) = K^*(\xi,\eta) - 3\|\mathcal{A}_\xi \eta\|^2, \tag{3.28}$$

where $X,Y \in \Gamma(\ker \Psi_*),\ \xi,\eta \in \Gamma(\ker \Psi_*)^\perp$ are orthonormal and $K,\ K^*$ and $\hat{K}$ are the sectional curvatures of $M_1,\ M_2$ and the fibers of the submersion, respectively.

Now, we give some results for the sectional curvature by calculating with respect to different 2-planes.

Theorem 3.15. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then we have*

$$K(X,Y) = K^*(NX,NY) - 3\|\mathcal{A}_{NX} NY\|^2, \tag{3.29}$$

where $X,Y \in \Gamma(\mathcal{D}^\perp)$ are the orthonormal vector fields.

Proof. Let $X,Y \in \Gamma(\mathcal{D}^\perp)$ are the orthonormal vector fields. By using the facts that $K(X,Y) = K(JX,JY)$ and $JX = NX,\ JY = NY$ with (3.4) and (3.28), we obtain

$$K(X,Y) = K(NX,NY) = K^*(NX,NY) - 3\|\mathcal{A}_{NX} NY\|^2,$$

which completes the proof.

As a natural result of Theorem 3.15 with (3.26), we obtain an inequality between the sectional curvature of the fibers and the base manifold.

***Corollary* 3.1.** *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then we get the following relation:*

$$K^*(JX,JY) - \hat{K}(X,Y) \leq \|\mathcal{T}_X Y\|^2 + 3\|\mathcal{A}_{JX} JY\|^2,$$

where $X,Y \in \Gamma(\mathcal{D}^\perp)$ are the orthonormal vector fields.

Theorem 3.16. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then we obtain*

$$K(X, (\sec\theta)PU) = -g_1((\nabla_{NX}\mathcal{T})(U,U), NX) - \|\mathcal{A}_{NX}U\|^2 + \|\mathcal{T}_U NX\|^2 + K^*(NX, \alpha) + \|\mathcal{A}_{NX}\alpha\|^2,$$

where $\alpha = (\sec\theta)(\csc\theta)NPU$.

Proof. Let $X \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(\mathcal{D}^\theta)$ be orthonormal vector fields. The equation (3.1), (3.4) and the J-invariance of K yield us

$$\begin{aligned} K(X, (\sec\theta)PU) &= g_1(JX, (\sec\theta)P^2U) + K(JX, (\sec\theta)\, NPV) \\ &= -K(NX, (\cos\theta)U) + K(NX, (\sec\theta)\, NPU), \end{aligned}$$

which let us to have with (3.27) and (3.24)

$$K(X, (\sec\theta)PU) = -g_1((\nabla_{NX}\mathcal{T})(U,U), NX) - \|\mathcal{A}_{NX}U\|^2 + \|\mathcal{T}_U NX\|^2 + \frac{R(NX, (\sec\theta)\, NPU, (\sec\theta)\, NPU, NX)}{g_1(NX, NX) g_1((\sec\theta)\, NPU, (\sec\theta)\, NPU)}. \tag{3.30}$$

If the last term of (3.30) is multiplied and divided by $\csc^2\theta$ to obtain orthonormal vector fields in Riemannian curvature tensor R, with the fact that $g_1(NX, NX) = 1$ and P-invariance of $\mathcal{D}^\theta$ it follows that

$$K(X, (\sec\theta)PU) = -g_1((\nabla_{NX}\mathcal{T})(U,U), NX) - \|\mathcal{A}_{NX}U\|^2 + \|\mathcal{T}_U NX\|^2 + \frac{R(NX, (\sec\theta)(\csc\theta)\, NPU, (\sec\theta)(\csc\theta)\, NPU, NX)}{\|(\sec\theta)(\csc\theta)\, NPU\|^2},$$

which completes the proof with (3.20).

By Theorem 3.16 and (3.26), we have the following result.

***Corollary* 3.2.** *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then we get*

$$\hat{K}(X, (\sec\theta)PU) \geq K^*(NX, \alpha) - \|\mathcal{T}_U NX\|^2 - \|\mathcal{T}_X(\sec\theta)PU\|^2,$$

where $\alpha = (\sec\theta)(\csc\theta)NPU$, $X \in \Gamma(\mathcal{D}^\perp)$ and $U \in \Gamma(\mathcal{D}^\theta)$ are orthonormal vector fields.

Theorem 3.17. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M, g, J) onto a Riemannian manifold (N, g_N). Then we obtain*

$$\begin{aligned} K((\sec\theta)PU, (\sec\theta)PV) = \hat{K}(U,V) &- g_1(\mathcal{T}_U U, \mathcal{T}_V, V) + \|\mathcal{T}_U V\|^2 \\ &+ g_1((\nabla_\beta\mathcal{T})(U,U), \beta) + \|\mathcal{A}_\beta U\|^2 - \|\mathcal{T}_U\beta\|^2 \\ &+ g_1((\nabla_\alpha\mathcal{T})(V,V), \alpha) + \|\mathcal{A}_\alpha V\|^2 - \|\mathcal{T}_V\alpha\|^2 \\ &+ K^*(\alpha, \beta) + 3\,\|\mathcal{A}_\alpha\beta\|^2, \end{aligned} \tag{3.31}$$

where $\alpha = (\sec\theta)(\csc\theta)NPU$, $\beta = (\sec\theta)(\csc\theta)NPV$ and $U, V \in \Gamma(\mathcal{D}^\theta)$ are orthonormal vector fields.

Proof. Let U, V be orthonormal vector fields in $\mathcal{D}^\theta$. In this case, it is known that the two elementer set

$$\{(\sec\,\theta)PU, (\sec\,\theta)PV\}$$

includes an orthonormal couple of vector fields. J-invariance of the sectional curvature K yields us

$$\begin{aligned} K((\sec\,\theta)PU, (\sec\,\theta)PV) &= K\big((\sec\,\theta)P^2U, (\sec\,\theta)P^2V\big) + K\big((\sec\,\theta)P^2U, (\sec\,\theta)NPV\big) \\ &\quad + K\big((\sec\,\theta)NPU, (\sec\,\theta)P^2V\big) + K((\sec\,\theta)NPU, (\sec\,\theta)NPV), \end{aligned}$$

which gives with Lemma 3.1

$$\begin{aligned} K((\sec\,\theta)PU, (\sec\,\theta)PV) &= K((\cos\,\theta)U, (\cos\,\theta)V) - K((\cos\,\theta)U, (\sec\,\theta)NPV) \\ &\quad - K((\sec\,\theta)NPU, (\cos\,\theta)V) + K((\sec\,\theta)NPU, (\sec\,\theta)NPV). \end{aligned} \tag{3.32}$$

Now, we calculate (3.32) term by term such that:
with the help of (3.18) and (3.24):

$$\begin{aligned} K((\cos\,\theta)U, (\cos\,\theta)V) &= \frac{R((\cos\,\theta)U, (\cos\,\theta)V, (\cos\,\theta)V, (\cos\,\theta)U)}{g_1((\cos\,\theta)U, (\cos\,\theta)U)g_1((\cos\,\theta)V, (\cos\,\theta)V)} \\ &= R(U, V, V, U) = \hat{K}(U, V) - g(\mathcal{T}_U U, \mathcal{T}_V V) + \|\mathcal{T}_U V\|^2, \end{aligned} \tag{3.33}$$

by using Lemma 3.2, (3.24), (3.23) and as a result of orthonormalization:

$$\begin{aligned} K((\cos\,\theta)U, (\sec\,\theta)NPV) &= \frac{R((\cos\,\theta)U, (\sec\,\theta)NPV, (\sec\,\theta)NPV, (\cos\,\theta)U)}{g_1((\cos\,\theta)U, (\cos\,\theta)U)g_1((\sec\,\theta)NPV, (\sec\,\theta)NPV)} \\ &= -R(\beta, U, \beta, U) = -g_1((\nabla_\beta \mathcal{T})(U, U), \beta) \\ &\quad + \|\mathcal{T}_U \beta\|^2 - \|\mathcal{A}_\beta U\|^2, \end{aligned} \tag{3.34}$$

since the sectional curvature K is symmetric, interchanging U and V in (3.34);

$$K((\cos\,\theta)V, (\sec\,\theta)NPU) = -g_1((\nabla_\alpha \mathcal{T})(V, V), \alpha) + \|\mathcal{T}_V \alpha\|^2 - \|\mathcal{A}_\alpha V\|^2,$$

(3.24), (3.20) and as a result of orthonormalization:

$$\begin{aligned} &K((\sec\,\theta)NPU, (\sec\,\theta)NPV) \\ &= \frac{R((\sec\,\theta)NPU, (\sec\,\theta)NPV, (\sec\,\theta)NPV, (\sec\,\theta)NPU)}{g_1((\sec\,\theta)NPU, (\sec\,\theta)NPU)g_1((\sec\,\theta)NPV, (\sec\,\theta)NPV)} \\ &= \frac{R(\alpha, \beta, \beta, \alpha)}{\|\alpha\|^2\|\beta\|^2} = K^*(\alpha, \beta) + 3\,\|\mathcal{A}_\alpha \beta\|^2, \end{aligned} \tag{3.35}$$

where $\alpha = (\sec\,\theta)(\csc\,\theta)NPU$ and $\beta = (\sec\,\theta)(\csc\,\theta)NPV$. If (3.33) $\sim$ (3.35) are considered with (3.32), it follows that (3.31), which complete the proof.

As a natural conclusion of Theorem 3.17, with the help of (3.26), we have the following corollary.

***Corollary* 3.3.** *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then we have the following inequality:*

$$\hat{K}((\sec\theta)PU, (\sec\theta)PV) \geq K^*(\alpha, \beta) + \hat{K}(U, V) - g_1(\mathcal{T}_U U, \mathcal{T}_V V)$$
$$- \|\mathcal{T}_U \beta\|^2 - \|\mathcal{T}_V \alpha\|^2 - g_1(\mathcal{T}_{(\sec\theta)PU}(\sec\theta)PU, \mathcal{T}_{(\sec\theta)PV}(\sec\theta)PV),$$

where $\alpha = (\sec\theta)(\csc\theta)NPU$, $\beta = (\sec\theta)(\csc\theta)NPV$ and $U, V \in \Gamma(\mathcal{D}^\theta)$ are orthonormal vector fields.

Theorem 3.18. *Let Ψ be a pointwise hemi-slant Riemannian submersion from a Kaehler manifold (M_1, g_1, J) onto a Riemannian manifold (M_2, g_2). Then we have the following assertions:*

(1) *the holomorphic sectional curvature of the base manifold N coincides with the fibers' provided planes are generated by the distribution $\mathcal{D}^\perp$,*

(2) *the holomorphic sectional curvature of the total manifold M coincides with the fibers' provided planes are generated by* $\ker\Psi_*$.

Proof. (1) Let X be a unit vector field in $\mathcal{D}^\perp$. By (2.4), the J-invariance of K, (3.25) and (3.28), it follows that

$$h(X) = h^*(X).$$

On the other hand, (3.26) yields us with (3.25),

$$h(X) = \hat{h}(X),$$

which gives (1).

(2) It follows from (3.26).

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