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EXISTENCE AND REGULARITY RESULTS FOR DEGENERATE PARABOLIC PROBLEMS IN THE PRESENCE OF STRONGLY INCREASING REGULARIZING LOWER-ORDER TERMS AND L^m-DATA/DIRAC MASS

РЕЗУЛЬТАТИ ЩОДО ІСНУВАННЯ ТА РЕГУЛЯРНОСТІ ДЛЯ ВИРОДЖЕНИХ ПАРАБОЛІЧНИХ ЗАДАЧ ЗА НАЯВНОСТІ СИЛЬНО ЗРОСТАЮЧИХ РЕГУЛЯРИЗУЮЧИХ ЧЛЕНІВ НИЖЧОГО ПОРЯДКУ ТА L^m-ДАНИХ/МАСИ ДІРАКА

We study the existence and regularity results for degenerate parabolic problems in the presence of strongly increasing regularizing lower-order terms and L^m-data/Dirac mass.

Досліджено результати щодо існування та регулярності для вироджених параболічних задач за наявності сильно зростаючих регуляризуючих членів нижчого порядку та L^m-даних/маси Дірака.

1. Introduction. This paper deals with a class of degenerate parabolic problems whose simplest model is

$$\begin{aligned} &u_t + Au + g(t,x,u) = f \quad \text{in} \quad Q := (0,T)\times\Omega,\\ &u(0,x) = u_0 \quad \text{in} \quad \Omega, \qquad u(t,x) = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with lateral boundary $\partial\Omega$, T is a positive constant, $u_0 \in L^1(\Omega)$ and $f \in L^m(Q)$ with $m \geq 1$, in presence of a lower-order term of asymptote type $g : (0,T)\times\Omega\times(0,\sigma) \to \mathbb{R}^+$ which is a Carathéodory[2] function satisfying

$$h(s) \leq g(t,x,s) \leq \rho(t,x)\gamma(s) \quad \text{a.e.} \quad (t,x)\in Q \quad \forall s\in[0,\sigma) \quad \forall t\in[0,T], \tag{1.2}$$

where $0 \leq \rho \in L^1(Q)$ and $\gamma(s), h(s) : [0,\sigma) \to \mathbb{R}^+$ are continuous and increasing real functions such that $\gamma(0) = h(0) = 0$ and $\lim_{s\to\sigma^-} h(s) = +\infty$. We explicitly notice that, due to the structure of h in (1.2), the function $\gamma(s)$ goes to infinity as s approaches σ, let us also stress that assumption $\gamma(0) = 0$ is only technical and it can be removed with the use of a slightly different approximation procedure in the existence result (for the sake of simplicity, we do not treat this case here). Observe that the nonlinear term g has an asymptote in σ, and due to this structure on g, it is natural to consider initial datum u_0 which are measurable and strictly less than σ a.e. on Ω. The differential operator

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[2] I.e., $g(\cdot,\cdot,s)$ is measurable on Q for every $s\in(0,\sigma)$ and continuous on $\mathbb{R}$ for a.e. $(t,x)\in Q$.

A is defined as $A\colon u \to -\operatorname{div}[a((t,x,u)\nabla u]$, where $a\colon (0,T)\times\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory[3] function satisfying for a.e.[4] $(t,x)\in Q$ and every $s\in\mathbb{R}$ the assumption

$$\frac{\alpha}{(1+|s|)^{\gamma}} \le a(t,x,s) \le \beta \quad \text{with} \quad 0\le\gamma<1+\frac{2}{N}, \tag{1.3}$$

where $\alpha,\ \beta$ are two positive constants. Assumption (1.3) implies that the differential operator Au is well defined on $L^2\big(0,T;H_0^1(\Omega)\big)$, but it fails to be coercive on the space when u becomes large, see [53] for more details. Due to the lack of coercivity, the classical methods, see [43, 45], cannot be applied to get an existence result even for sufficiently regular data (see [5, 11, 15, 40, 54] and [4, 13, 16, 21, 31, 39] for details on degenerate problems).

A particular motivation for dealing with lower-order terms of asymptote type as in (1.1) comes from the study of semilinear equations of some functionals in the calculus of variation, see [30]. As a simple example the Dirichlet problem

$$-\Delta u + |u|^{p-1}u = f \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \qquad \partial\Omega,$$

which admits, under the assumption that the datum f belongs to $L^1(\Omega)$, a weak solution $u\in L^p(\Omega)$ such that $\nabla u\in L^q(\Omega)$ with $q<\dfrac{2p}{p+1}$. We point out that the considered lower-order term $|u|^{p-1}u$ has a regularity effect on the solution (see [20] for details). On the other hand, in [14], the authors consider a semilinear Dirichlet problem with an asymptote different from zero in the lower-order term whose model is

$$-\Delta u + \frac{u}{1-u} = f \quad \text{in} \qquad \Omega,$$

$$u = 0 \quad \text{on} \qquad \partial\Omega,$$

and they prove an existence result of a weak solution for any nonnegative L^1-data. A stronger effect can be observed if we consider a lower-order term $h(u)$ where $h\colon [0,\sigma)\to\mathbb{R}^+$ is a continuous and increasing function with vertical asymptote in σ $(\sigma>0)$, more precisely, for the semilinear elliptic problems

$$-\Delta u + h(u) = f \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega.$$

In [38], the authors studied the existence of solutions of the nonlinear problem

[3] I.e., $a(\cdot,\cdot,s)$ is measurable on Q for every $s\in\mathbb{R}$ and continuous on $\mathbb{R}$ for a.e. $(t,x)\in Q$.

[4] Almost every.

$$\begin{aligned} -\Delta u + g(u) &= \mu \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.4}$$

where μ is a bounded measure and $g\colon (-\infty, 1) \to \mathbb{R}$ is a continuous nondecreasing function such that $g(0) = 0$. In this paper, they assume that the nonlinearity g satisfies

$$\lim_{s\uparrow 1} g(s) = +\infty.$$

Recall that, a solution[5] of (1.4) exists and is unique (see [29]). It has been proved by Boccardo [11] (in the spirit of Brezis–Strauss, see [30]), that, for every $\mu \in L^1(\Omega)$, problem (1.4) has a solution. Moreover, Boccardo shows that (1.4) has no solution if μ is a Dirac mass δ_a with $a \in \Omega$. Consequently, in [38], the authors introduced the notion of good measure, i.e., μ is a good measure (relative to g) if problem (1.4) has a solution u. They investigate under what conditions on g and μ problem (1.4) admits a solution (they point out to what extent assumption (1.4) makes problem (1.4) different compared to the case where g is a continuous function defined for every $s \in \mathbb{R}$ studied by authors in [29]), and they characterize the set of good measures associated to g (sufficient condition for a measure to be good) by using the dimensional Hausdorff measure of a set (see also [42]). The method in the study of problem (1.4) uses a standard procedure which consists in approximating g with bounded continuous functions defined the whole $\mathbb{R}$, i.e., $g_n : \mathbb{R} \to \mathbb{R}$ are nondecreasing functions satisfying $0 \le g_1(s) \le g_2(s) \le \ldots$ for every $s \in \mathbb{R}$, $g_n(s) \to g(s)$ $\forall s < 1$, and $g_n(s) \to +\infty$ $\forall s \ge 1$. Even though the existence of solutions of problem (1.4) may fail for some diffuse measures, $L^1(\Omega)$ is not the largest set where (1.4) has a solution for any g, the characterization of the set of all measures, possibly singular, in $\mathcal{M}(\Omega)$ which are good for every g are also given.

Let us recall that this type of questions has been discussed in the case of nonlinear coercive operators, in [9, 28, 30]; more especially for Dirichlet problems of the form

$$\begin{aligned} &-\operatorname{div}(a(x,u,\nabla u)) + g(x,u) = F \in W^{-1,p'}(\Omega) \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

and an existence result of a weak solution u, such that $g(x,u) \in L^1(\Omega)$ and $g(x,u)u \in L^1(\Omega)$, was proved (recall that if u belongs to $W_0^{1,p}(\Omega)$ then $g(x,u)$, in general, does not belongs to $L^1(\Omega)$). We just refer the reader to the case of lower-order terms of power type $g(x,u) = |u|^{r-2}u$, with $r > 1$, considered in [22] for $W^{-1,p'}$-data. Another asymptotic behavior result has been proved for problems[6]

[5] By a solution of (1.4) we mean that $u \in L^1(\Omega)$, $u \le 1$ a.e., $g(u) \in L^1(\Omega)$ and

$$-\int\limits_{\Omega} u\Delta\varphi + \int\limits_{\Omega} g(u)\varphi = \int\limits_{\Omega} \varphi d\mu \quad \forall \varphi \in C^2(\overline{\Omega}), \quad \varphi = 0 \text{ on } \partial\Omega.$$

[6] Observe that $\dfrac{ps}{s+1} < p$ and $\dfrac{ps}{s+1} \to p$ as $s \to +\infty$.

$$-\mathrm{div}(a(x,u,\nabla u)) + |s|^{s-1}u = f \quad \text{in} \quad \Omega,$$

$$u \in W_0^{1,q}(\Omega) \cap L^s(\Omega), \quad q < \frac{ps}{s+1},$$

where f is a nonnegative function in $L^1(\Omega)$. Finally, in [48], the authors provide an existence result of a positive solution $u \in L^2\big(0,T; H_0^1(\Omega)\big)$ for nonlinear parabolic problems with singular lower-order terms; more precisely they consider, the semilinear problems of the type

$$u_t - \mathrm{div}(M(t,x,u)\nabla u) + \frac{u}{1-u} = f(t,x) \quad \text{in} \quad Q := (0,T)\times\Omega,$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \Omega, \qquad u(t,x) = 0 \quad \text{on} \quad (0,T)\times\partial\Omega,$$

where $M(t,x,s) := (m_{i,j}(t,x,s))_{i,j\,:\,1,\ldots,N}$ is a symmetric matrix whose coefficients $m_{i,j} : (0,T)\times\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory[7] abstract functions such that there exist $0 < \alpha \le \beta$ satisfying

$$\alpha|\zeta|^2 < M(t,x,s)\zeta\cdot\zeta, \qquad |M(t,x,s)| \le \beta \quad \text{a.e.} \quad x\in\Omega \quad \forall (s,\zeta)\in\mathbb{R}\times\mathbb{R}^N \quad \forall t\in(0,T),$$

under the assumptions that $u_0 \in L^1(\Omega)$ and f is a nonnegative function in $L^1(Q)$, and in [1], the author provide a complete picture of the situation in the case of nonlinear parabolic operators with monotone operators and general measure data.

The purpose of the present paper is to extend the results, obtained in the elliptic case in [11, 41], to the evolution framework motivated by their applications in a variety of contexts; we cite for example: stochastic control problems [10, 11], growth paterns in clusters and fronts of solidification (growth of tumors [12], flame propagation [17] and growth water flow in a water-absorbing fissurized porous rock [23]), by proving a new regularizing effect of strongly increasing lower-order terms on *entropy* solutions for degenerate parabolic problems with summable data. More precisely, we prove that if the lower-order term is defined through the composition with a continuous, but unbounded, function on some real interval $|0,\sigma)$, and f belongs to $L^m(Q)$, $m \ge 1$, the solutions are bounded. In fact, we study, in the first part of the paper, the existence of a distributional solution of problem

$$\begin{aligned} &u_t - \mathrm{div}(a(t,x,u)\nabla u) + |u|^{p-1}u = f \quad \text{in} \quad Q := (0,T)\times\Omega,\\ &u(0,x) = u_0 \quad \text{in} \quad \Omega, \qquad u(t,x) = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \end{aligned} \tag{1.5}$$

where f belongs to $L^m(Q)$ (observe that the presence of the lower term $|u|^{p-1}u$ guarantees the existence of a distributional solution if f is an L^1-function). On the contrary, problem (1.5) without lower-order term may have no solution because the summability of the gradient of the solutions may be lower than 1. Let us specify that a distributional solution of problem (1.5) is a function $u \in L^2\big(0,T; H_0^1(\Omega)\big) \cap C([0,T]; L^1(\Omega))$ such that $|\nabla u|$ belongs to $L^1(Q)$, which satisfies

[7] I.e., $m_{i,j}(\cdot,\cdot,s)$ is measurable on Q for every $s\in\mathbb{R}$ and $m_{i,j}(t,x,\cdot)$ is continuous on $\mathbb{R}$ for a.e. $(t,x)\in Q$.

$$-\int\limits_{\Omega} u_0\varphi(0)dx - \int\limits_0^T \langle \varphi_t, u\rangle dt + \int\limits_Q a(t,x,u)\nabla u\cdot\nabla\varphi dxdt + \int\limits_Q |u|^{p-1}u\varphi dxdt = \int\limits_Q f(t,x)\varphi dxdt$$

for any $\varphi \in L^2\big(0,T;H_0^1(\Omega)\big)\cap L^\infty(Q)$ with $\varphi_t \in L^2(0,T;H^{-1}(\Omega))$ and $\varphi(T)=0$. Note that the notion of entropy solution, introduced in [8, 55], is useful in the case where the data is not sufficiently regular and the solution of problem (1.5) does not necessary belong to a Sobolev space (see [3]), while the second part of the paper will be devoted mainly to the study of degenerate parabolic problems having a singular lower-order term of asymptote-type

$$\begin{aligned} &u_t - \operatorname{div}(a(t,x,u)\nabla u) + h(u) = f \quad \text{in} \quad Q := (0,T)\times\Omega, \\ &u(0,x) = u_0 \quad \text{in} \quad \Omega, \qquad u(t,x) = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \end{aligned} \tag{1.6}$$

where $h\colon [0,\sigma)\to\mathbb{R}$ is a continuous and increasing function such that $h(0)=0$ and $\lim_{s\to\sigma^-} h(s) = +\infty$ under the condition that f is an L^1-function or a Dirac mass. In this case, existence/nonexistence and regularity of solutions depending on both the data and on the assumptions of the lower-order term need a completely different approach. Namely, we argue by localizing the problem on sets of zero capacity, and then we look for the asymptotic behavior of the lower-order term with respect to the singular datum. The proof of the results will be based on approximation methods and compactness arguments where the key role is played by a specific choice of test functions depending on the function h. In some particular cases, we shall prove some a priori estimates, inspired by [47], that will be essential to get some convergence results, and finally, we shall use some techniques, introduced in [33, 50], to prove the strong convergence of truncates.

This paper is organized as follows. In Section 2, we give an account on some regularity results concerning problems without lower-order terms and we define a notion of entropy solution needed to give sense to the problem. In Section 3, we prove our first main result for problem (1.5) with L^m-data, while Subsection 4.1 is devoted to the proof of the second main result for problem (1.6) under the assumption that f belongs to $L^1(Q)$. Finally, we establish a nonexistence result for problem (1.6) with Dirac mass as data in Subsection 4.2.

2. Some preliminary results and a priori estimates. Let us consider the following class of parabolic problems with degenerate coercivity:

$$\begin{aligned} &u_t - \operatorname{div}(a(t,x,u)\nabla u) + |u|^{p-1}u = f \quad \text{in} \quad Q := (0,T)\times\Omega, \\ &u(0,x) = u_0 \quad \text{in} \quad \Omega,\ u(t,x) = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \end{aligned} \tag{2.1}$$

where u_0 and f belong, respectively, to $L^1(\Omega)$ and $L^m(Q)$ with $m\ge 1$. First of all, observe that if the summability conditions on f will be weaken, the gradient of u may no longer be in $L^1(Q)$. To overcome this difficulty, we may give the meaning of solutions for problem (2.1) by using the concept of entropy solutions (on a complete account in this topic, see [8] for elliptic equations and [6, 37, 46, 52, 55, 56] for parabolic equations). To this aim, let us denote by T_k, for every $k>0$, the usual truncation function, $S_k(s)$ its primitive function and $G_k(s)$ an auxiliary function defined by

$$T_k(s) = \min\{k, \max\{-k, s\}\}, \qquad S_k(s) = \int_0^s T_k(\tau)d\tau \qquad \text{and} \qquad G_k(s) = s - T_k(s) \quad \forall s \in \mathbb{R}. \tag{2.2}$$

In order to define the notion of entropy solution, we need the following lemma.

Lemma 2.1. *If $T_k(u) \in L^2\big(0, T; H_0^1(\Omega)\big)$ for every $k > 0$, then there exists a unique measurable function $v: Q \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|\le k\}} \quad \textit{a.e. in } Q,$$

where $\chi_{\{|u|<k\}}$ denotes the characteristic function over the set $\{|u| < k\}$, and ∇u the derivative of u as the unique function v which satisfies the above equality. Furthermore, $u \in L^2\big(0, T; H_0^1(\Omega)\big)$ if and only if $v \in L^2(Q)$, and then $v = \nabla u$ in the usual weak sense.

Proof. Up to minor changes, the proof is the same as [8, Lemma 2.1].

Definition 2.1. *A measurable function $u \in L^\infty(0, T; L^1)(\Omega)$ is an entropy solution of problem* (2.1) *if $|\nabla u|^p \in L^1(Q)$, $T_k(u) \in L^2\big(0, T; H_0^1(\Omega)\big)$ for every $k > 0$,*

$$\int_\Omega S_k(u(t) - \varphi(t))dx \in C([0, T]),$$

and

$$\int_\Omega S_k(u(T) - \varphi(T))dx - \int_\Omega S_k(u_0 - \varphi(0))dx + \int_0^T \langle \varphi_t, T_k(u - \varphi)\rangle dt$$

$$+ \int_Q a(t, x, u)\nabla u \cdot \nabla T_k(u - \varphi)dxdt + \int_Q |u|^{p-1} u T_k(u - \varphi)dxdt$$

$$\le \int_Q f T_k(u - \varphi)dxdt$$

for every $k > 0$ and all $\varphi \in L^2\big(0, T; H_0^1(\Omega)\big) \cap L^\infty(Q)$ such that $\varphi_t \in L^2(0, T; H^{-1}(\Omega)) + L^1(Q)$.

***Remark* 2.1.** This definition is useful in the case where the solution of problem (2.1) does not necessary belong to a Sobolev space. Indeed, about the gradient of the solution, it has a sense under the weak hypotheses that $\nabla T_k(u) \in L^2(Q)$, we don't need that $\nabla u \in L^1(Q)$, as for distributional solutions.

For any $0 < q < +\infty$, we introduce the Marcinkiewicz space[8] $\mathcal{M}^q(Q)$ as follows (see [5, 8, 9] for details).

Definition 2.2. *The set of measurable functions $u: Q \to \mathbb{R}$ such that the functional*

$$[u]_q = \sup_{k>0} \operatorname{meas}\big\{(t, x) \in Q : |u(t, x)| > k\big\}^{\frac{1}{q}}$$

is finite is called a Marcinkiewicz space and is denoted by $\mathcal{M}^q(Q)$, $0 < q < +\infty$.

[8] Also known as weak-Lebesgue space.

***Remark* 2.2.** Recall that:

(i) The Marcinkiewicz space $\mathcal{M}^q(Q)$ is a Banach space endowed with the norm

$$\|u\|_q := \sup_{s>0} s^{\frac{1-q}{q}} \int\limits_0^s u^\star(\tau)d\tau,$$

where $u^\star = \inf\{k > 0 \colon \mathrm{meas}\{|u| > k \le \tau\}\}$ defines the nonincreasing rearrangement of u, see [18].

(ii) Since Ω is bounded, then, for $q > 1$, we have the continuous embedding

$$L^q(Q) \hookrightarrow \mathcal{M}^q(Q) \hookrightarrow L^{p-\epsilon}(Q) \quad \forall \epsilon \in (0, p-1].$$

(iii) For $r < q$, we have $\mathcal{M}^q(Q) \hookrightarrow \mathcal{M}^r(Q)$ (see also [16, 34]).

Now we state three embedding theorems that will play a central role in our paper. The first one is an Aubin – Simon type result that we state in a form general enough to our purpose, while the second one is the well-known Gagliardo – Nirenberg embedding theorem followed by an important consequence of it for the evolution case.

Theorem 2.1 (Aubin – Simon result)**.** *Let u_n be a bounded sequence in $L^q(0,T;W_0^{1,q}(\Omega))$ such that $(u_n)_t$ is bounded in $L^1(Q) + L^{s'}(0,T;W^{-1,s'}(\Omega))$ with $q, s > 1$. Then u_n is relatively strongly compact in $L^1(Q)$, that is, up to subsequences, u_n strongly converges in $L^1(Q)$ to some function $u \in L^1(Q)$.*

Proof. See [57, Corollary 4].

Let us define, for every $p > 1$, the functional space S^p defined by

$$S^p = \left\{u \in L^p(0,T,W_0^{1,p}(\Omega)),\ \ u_t \in L^1(Q) + L^{p'}(0,T;W^{-1,p'}(\Omega)\right\}$$

and endowed with its natural norm $\|u\|_{S^p} = \|u\|_{L^p\left(0,T;W_0^{1,p}(\Omega)\right)} + \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))+L^1Q)}$.

Theorem 2.2 (trace result)**.** *Let $p > 1$, then we have the continuous injection*

$$S^p \underset{\text{cont}}{\hookrightarrow} C(0,T;L^1(\Omega)).$$

Proof. See [52, Theorem 1.1].

Theorem 2.3 (Gagliardo – Nirenberg)**.** *Let v be a function in $W_0^{1,q}(\Omega) \cap L^\rho(\Omega)$ with $q \ge 1$ and $\rho \ge 1$. Then there exists a positive constant C, depending on N, q and ρ, such that*

$$\|v\|_{L^\gamma(\Omega)} \le C\|\nabla v\|^\theta_{(L^q(\Omega))^N}\|v\|^{1-\theta}_{L^\rho(\Omega)}$$

for every θ and γ satisfying

$$0 \le \theta \le 1, \quad 1 \le \gamma \le +\infty, \qquad \frac{1}{\gamma} = \theta\left(\frac{1}{q} - \frac{1}{N}\right) + \frac{1-\theta}{\rho}.$$

Proof. See [49, Lecture II].

An immediate consequence of the previous result is the following embedding result.

***Corollary* 2.1.** *Let $v \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^\infty(0,T;L^\rho(\Omega))$ with $q \ge 1$ and $\rho \ge 1$. Then $v \in L^\sigma(Q)$ with $\sigma = q\dfrac{N+\rho}{N}$ and*

$$\int\limits_Q |s|^\sigma dxdt \le C\|v\|^{\frac{\rho q}{N}}_{L^\infty(0,T;L^\rho(\Omega))} \int\limits_Q |\nabla v|^q dxdt.$$

Proof. See [35, Proposition 3.1].

Finally, in order to use some intermediary results, let us denote by

$$W = \Big\{u \in L^p\big(0,T;W_0^{1,p}(\Omega)\big),\;\; u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))\Big\}$$

endowed with its natural norm $\|u\|_W = \|u\|_{L^p\left(0,T;W_0^{1,p}(\Omega)\right)} + \|u_t\|_{L^{p'}(0,T;W^{-1,p'}(\Omega))}$.

Theorem 2.4. *Let $1 < p < \infty$, then $C_0^\infty([0,T]\times\Omega)$ is dense in W.*

Proof. See [36, Theorem 2.11].

Let us emphasize that, if $u \in W \cap L^\infty(Q)$, then the approximating sequence of functions in $C_0^\infty([0,T]\times\Omega)$ that exists thanks to Theorem 2.4, can be chosen to be bounded. In the following, when u_t is said to belong to a space $L^q(a,b,\tilde{V})$ ($\tilde{V}$ being a Banach space) this means that there exists a function $z \in L^q(a,b;\tilde{V}) \cap D'(a,b;V)$ such that[9]

$$\langle u_t, \psi\rangle = -\int_a^b u\psi_t dt = \langle z,\psi\rangle \quad \forall \psi \in C_0^\infty(a,b).$$

We recall the following classical embedding result.

Theorem 2.5. *Let H be a Hilbert space such that $V \underset{\text{dense}}{\hookrightarrow} H \hookrightarrow V'$ and let $u \in L^p(a,b;V)$ be such that u_t, defined in the distributional sense, belongs to $L^{p'}(a,b;V')$. Then u belongs to $C([a,b];H)$.*

Proof. See [32, Chapter XVIII, Section 2, Theorem 1].

Here we give a further result that will be very useful in what follows, it is a generalization of the *integration by parts* formula

$$\int_a^b \langle v, u_t\rangle dt + \int_a^b \langle u, v_t\rangle dt = \big(u(a), v(b)) - (u(a), v(a)\big),$$

where $\langle\cdot,\cdot\rangle$ is the duality between V and V' and $(\cdot,\cdot)$ is the scalar product in H.

Lemma 2.2 (integration by parts formula)**.** *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous piecewise C^1-function such that $f(0)=0$ and f' is zero away from a compact set of $\mathbb{R}$. Let us denote $F(s) = \int_0^s f(\sigma)d\sigma$. If $u \in L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ is such that $u_t \in L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big) + L^1(Q)$ and if $\psi \in C^\infty(\bar{Q})$, then we have (here we have chosen the continuous representative of u)*

$$\int_0^T \langle u_t, f(u)\psi\rangle dt = \int_\Omega F(u(T))\psi(T)dx - \int_\Omega F(u(0))\psi(0)dx - \int_Q \psi_t F(u)dxdt.$$

Proof. See [37, Lemma 7.1].

Finally, we mention that if $\gamma > 1 + \dfrac{2}{N}$ the effect of the degenerate coercivity is even worst, that is, problem (2.1) has no solution even if the datum f is constant (see [4, 16] for details).

The following intermediary lemma gives some a priori estimates satisfied by gradients of solutions.

[9] Here $D'(a,b;V)$ denotes the space of vector valued distributions which is the space of linear continuous functions from $C_0^\infty(a,b)$ into V.

Lemma 2.3. *Let u be a measurable function in $\mathcal{M}^{\mu}(Q)$ with $\mu > 0$, and assume that there exist two nonnegative constants $\nu > \gamma$ such that*

$$\int_Q |\nabla T_k(u)|^2 dxdt \le M(1+k)^{\gamma} k^{\nu-\gamma} \quad \forall k > 0,$$

where M is a positive constant (independent of k). Then $|\nabla u| \in \mathcal{M}^{\delta}(Q)$ with $\delta = \dfrac{2\mu}{\mu+\nu}$.

Proof. See [47, Lemma 2.3].

***Remark* 2.3.** Lemma 2.3 is true for sequences two, that is, u is a measurable function such that

$$\operatorname{meas}\big\{|\{|u_n| \ge k\}|\big\} \le \frac{M_1}{k^{\mu}}, \quad \mu > 0,$$

where M_1 is a positive constant (independent of k), and there exist two positive constants $\nu > \gamma$ such that

$$\int_Q |\nabla T_k(u)|^2 dxdt \le M(1+k)^{\gamma} k^{\nu-\gamma} \quad \forall k > 0.$$

Then

$$\operatorname{meas}\big\{|\{|\nabla u| \ge k\}|\big\} \le M_2\left(\frac{k^{\nu}}{l^2} + \frac{1}{k^{\mu}}\right),$$

where $M_2 = \max\{2^{\gamma}M, M_1\}$. By minimizing with respect to k, we easily get

$$\operatorname{meas}\big\{|\nabla u| > l\big\} \le \frac{M_3}{l^{\delta}}, \quad \text{that is,} \quad k = \left(\frac{\mu}{\nu}\right)^{\frac{1}{\mu+\nu}} l^{\frac{2}{\mu+\nu}},$$

where M_3 is a positive constant independent of l.

In [2] (see also [8, 19]), the existence of a weak solution of problem (2.1) is solved by the following tool, which we recall here being the key result for the whole theory.

Lemma 2.4. *Let $C(k) > 0$ (dependent of k) and $(u_n)_{n\in\mathbb{N}} \subset \mathcal{T}_0^{1,p}(Q)$ such that $T_k(u_n) \in L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ and*

$$\int_Q |\nabla T_k(u_n)|^p dxdt \le C(k) \quad \forall k > 0.$$

Then there exists a measurable function u such that $T_k(u) \in L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ and a subsequence, not relabeled, satisfying

$$u_n \to u \quad \text{a.e. in } Q,$$

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{wekaly in} \quad L^p\big(0,T;W_0^{1,p}(\Omega)\big) \quad \text{and a.e. in } Q \quad \text{for every} \quad k > 0.$$

Proof. See [2, Proposition 3.12].

3. Regularizing effect of the lower-order term $|u|^{p-1}u$. In order to discuss the regularizing effect of the lower-order term $|u|^{p-1}u$ on the entropy solution of problem (2.1), we need to consider the approximate problem

$$
\begin{aligned}
&(u_n)_t - \operatorname{div}(a(t,x,T_n(u_n))\nabla u_n) + |u_n|^{p-1}u_n = f_n \quad \text{in} \quad Q := (0,T)\times\Omega,\\
&u_n(0,x) = u_0^n(x) \quad \text{in} \quad \Omega, \qquad u_n(t,x) = 0 \quad \text{on} \quad (0,T)\times\partial\Omega,
\end{aligned} \tag{3.1}
$$

where T_n is defined in (2.2), u_0^n approaches u_0 in $L^1(\Omega)$ and $f_n \in \mathcal{D}(Q)$ such that

$$
\|f_n\|_{L^m(Q)} \le \|f\|_{L^m(Q)}, \qquad f_n \to f \quad \text{strongly in} \quad L^m(Q) \quad \forall n\in\mathbb{N} \quad \forall m \ge 1. \tag{3.2}
$$

Thus, from the well-known results of [43, 45] we have the following lemma.

Lemma 3.1. *Let $f \in L^m(Q)$ with $m \ge 1$. Then there exists a solution $u_n \in C([0,T]; L^2(\Omega)) \cap L^2\big(0,T; H_0^1(\Omega)\big)$ with $(u_n)_t \in L^2(0,T;H^{-1}(\Omega))$ of problem* (3.1) *satisfying*

$$
\int\limits_Q |u_n|^{pm}dxdt \le \int\limits_Q |f|^m dxdt + C \tag{3.3}
$$

and

$$
\int\limits_0^T \langle (u_n)_t, \varphi\rangle dt + \int\limits_Q a(t,x,T_n(u_n))\nabla u_n\cdot\nabla\varphi dxdt + \int\limits_Q |u_n|^{p-1}u_n\varphi dxdt = \int\limits_Q f_n\varphi dxdt \tag{3.4}
$$

for every $\varphi \in L^2\big(0,T;H_0^1(\Omega)\big)$ with $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Proof. Since the functional

$$
u \mapsto -\operatorname{div}(a(t,x,T_n(u_n))\nabla u_n) + |u_n|^{p-1}u_n
$$

is well defined and satisfies the standard assumptions of parabolic operators (see [43, 45]), there exists a solution u_n of problem (3.1) satisfying the weak formulation (3.4). Moreover, to check the inequality (3.3) we need to deal separately with the cases $m=1$ and $m>1$.

Case 1: $m=1$. For all $\tau\in(0,T]$ and all $k>0$, using $\dfrac{T_k(u_n)}{k}\chi_{(0,\tau)}(t)$ with $\chi_{(0,\tau)}$, denotes the characteristic function in $(0,\tau]$ as test function in the weak formulation (3.4) of problem (3.1), we get

$$
\begin{aligned}
&\int\limits_0^\tau\int\limits_\Omega (u_n)_t \frac{T_k(u_n)}{k}dxdt\\
&\qquad + \int\limits_0^\tau\int\limits_\Omega \frac{1}{k}a(t,x,T_n(u_n))\nabla u_n\cdot\nabla T_k(u_n)dxdt + \int\limits_0^\tau\int\limits_\Omega |u_n|^{p-1}u_n\frac{T_k(u_n)}{k}dxdt\\
&= \int\limits_0^\tau\int\limits_\Omega f_n\frac{T_k(u_n)}{k}dxdt.
\end{aligned}
$$

By using assumption (1.3) and Hölder's inequality, we obtain

$$\frac{1}{k}\int\limits_{\Omega}\Theta_k(u_n(\tau,x))dx+\frac{\alpha}{k}\int\limits_0^{\tau}\int\limits_{\Omega}\frac{|\nabla T_k(u_n)|^2}{(1+|u_n|)^{\gamma}}dxdt$$

$$+\int\limits_0^{\tau}\int\limits_{\Omega}|u_n|^{p-1}\frac{T_k(u_n)}{k}dxdt$$

$$\le\int\limits_0^{\tau}\int\limits_{\Omega}f_n\frac{T_k(u_n)}{k}dxdt\le\int\limits_Q|f|dxdt+\frac{1}{k}\int\limits_{\Omega}\Theta_k(u_0^n(x))dx,$$

where $\Theta_k(s)=\int_0^s T_k(\tau)d\tau$ (the primitive function of $T_k(s)$). Since $\Theta_k(s)\ge 0,\ |\Theta_1(s)|\ge|s|-1$, by virtue of $\Theta_k(u_n(\tau,x))\ge\dfrac{|T_k(u_n(\tau,x))|^2}{2}$, we have

$$\operatorname*{ess\,sup}_{0\le t\le T}\int\limits_{\Omega}\frac{|T_k(u_n(t,x))|^2}{k}dx$$

$$+\frac{\alpha}{k}\int\limits_Q\frac{|\nabla T_k(u)|^2}{(1+|u_n|)^{\gamma}}dxdt+\int\limits_Q|u_n|^{p-1}u_n\frac{T_k(u_n)}{k}dxdt$$

$$\le\int\limits_Q|f|dxdt+\int\limits_{\Omega}\frac{|T_k(u_0^n)|^2}{2}dx.$$

Then, by letting k tends to infinity, dropping positive terms and using Fatou's lemma, we obtain estimate (3.3) for the case $m=1$.

Case 2: $m>1$. Taking $\varphi=|u_n|^{p(m-1)}\mathrm{sign}(u_n)$ as test function in the weak formulation (3.4) of problem (3.1), it is easy to prove estimate (3.3). In fact, we have

$$\int\limits_0^T\langle(u_n)_t,|u_n|^{p(m-1)}\mathrm{sign}(u_n)\rangle dt$$

$$+\int\limits_Q a(t,x,u_n)\nabla u_n\cdot\nabla(|u_n|^{p(m-1)}\mathrm{sign}(u_n))dxdt$$

$$+\int\limits_Q|u_n|^{p-1}u_n|u_n|^{p(m-1)}\mathrm{sign}(u_n)dxdt$$

$$=\int\limits_Q f_n|u_n|^{p(m-1)}\mathrm{sign}(u_n)dxdt.$$

The second term of the above inequality is nonnegative, then, by the integration by parts formula and Hölder's inequality of exponent $m,$ we get

$$\int\limits_{\Omega} \frac{|u_n(\tau)|^{p(m-1)+1}}{p(m-1)+1}dx + \int\limits_{Q} |u_n|^{pm}dxdt \leq \int\limits_{Q} \|f_n\|_m \left[\int\limits_{Q} |u_n|^{pm}\right]^{\frac{m-1}{m}} + C.$$

Hence, since $m > 1$, by using (3.2), we obtain

$$\int\limits_{Q} |u_n|^{pm}dxdt \leq \|f\|_{L^m(Q)} \left(\int\limits_{Q} |u_n|^{pm}\right)^{q-\frac{1}{m}} + C.$$

Inequality (3.3) is proved.

Now, we state our first main result.

Theorem 3.1. *Under assumption* (1.3) *and* $f \in L^1(Q)$:

(i) *If* $p > \gamma + 1$, *there exists a distributional solution* u *of problem* (1.5) *such that*

$$u \in L^s(0, T; W_0^{1,s}(\Omega)) \cap L^p(Q) \quad \textit{with} \quad s < \frac{2p}{\gamma+1}.$$

(ii) *If* $0 < p \leq \gamma + 1$, *there exists an entropy solution of problem* (1.5) *such that*

$$|u|^p \in L^1(Q) \qquad \textit{and} \qquad |\nabla u| \in \mathcal{M}^{\frac{2p}{\gamma+1}(Q)}.$$

Proof. The proof is divided in two steps.

Step 1: $p > \gamma + 1$. Let $\psi(s) = [(1 + |s|)^{1-\lambda} - 1]\mathrm{sign}(u_n)$ for all $s \in \mathbb{R}$ with $\lambda > 1$ is a positive constant, which will be determined lately. For every $\tau \in (0, T]$, using $\psi(u_n(t, x))\chi_{(0,\tau)}(t)$ as test function in the weak formulation (3.4) of problem (3.1) and using assumption (1.3), we get

$$\int\limits_{\Omega} \Psi(u_n(\tau, x))dx + (1-\lambda)\alpha \int\limits_{0}^{\tau}\int\limits_{\Omega} \frac{|\nabla u_n|^2}{(1+|u_n|)^\gamma}(1+|u_n|)^{-\lambda}dxdt$$

$$+ \int\limits_{0}^{\tau}\int\limits_{\Omega} |u_n|^{p-1}u_n\Big[(1+|u_n|)^{1-\lambda} - 1\Big]\mathrm{sign}(u_n)dxdt$$

$$\leq \int\limits_{0}^{\tau}\int\limits_{\Omega} |f_n|\Big[(1+|u_n|)^{1-\lambda} - 1\Big]dxdt + \int\limits_{\Omega} \Psi(u_n(0, x))dx,$$

where $\Psi(s) = \int_0^s \psi(\tau)d\tau$. Recalling the definition of $\psi(s)$, we have

$$\Psi(s) \geq \frac{1}{2-\lambda}|s|^{2-\lambda} \quad \forall s \in \mathbb{R}.$$

Then, by dropping the third term in the left-hand side, since it is nonnegative and using Hölder's inequality, we obtain

$$\frac{1}{2-\lambda}\int\limits_{\Omega}|u_n(\tau,x)|^{2-\lambda}dx+(1-\lambda)\alpha\int\limits_0^{\tau}\int\limits_{\Omega}|\nabla u_n|(1+|u_n|)^{-\gamma-\lambda}dxdt$$

$$\leq \|f_n\|_{L^1(Q)}\left(\int\limits_{\Omega}|(1+|u_n|)^{1-\lambda}-1|dxdt\right)+\frac{1}{2-\lambda}(1+\|u_0^n\|_{L^1(\Omega)})^{2-\lambda}+C,$$

the above estimate and (3.2) yield

$$\int\limits_{\Omega}\Big[|u_n(t,x)|^{\frac{2-\lambda-\gamma}{2}}\Big]^{\frac{2(2-\lambda)}{2-\lambda-\gamma}}dx+C_{\lambda,\alpha}\int\limits_{Q}|\nabla|u_n|^{\frac{2-\lambda-\gamma}{2}}|^2$$

$$\leq C\big(\|f\|_{L^1(Q)}\big)\int\limits_{Q}|u_n|^{1-\lambda}dxdt+C(\|u_0\|_{L^1(\Omega)}),$$

where $C\big(\|f\|_{L^1(Q)}\big)$ and $C(\|u_0\|_{L^1(\Omega)})$ are two constants independent of n. Thus,

$$C_\lambda\operatorname*{ess\,sup}_{0\leq t\leq T}\int\limits_{\Omega}(1+|u_n|)^{3-\lambda}dx+C_{\lambda,\alpha}\int\limits_0^{\tau}\int\limits_{\Omega}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\gamma+\lambda}}dxdt$$

$$\leq C\big(\|f\|_{L^1(Q)}\big)\left(\int\limits_{Q}|u_n|^{1-\lambda}dxdt\right)+C(\|u_0\|_{L^1(\Omega)}),$$

where C_λ and $C_{\lambda,\alpha}$ are two positive constants (independent of n). Now, for every $1\leq q<2$ and $\lambda>1$, we have by Hölder's inequality with exponent $\dfrac{2}{q}$ such that

$$\int\limits_{Q}|\nabla u_n|^q dxdt=\int\limits_{Q}\frac{|\nabla u_n|^q}{(1+|u_n|)^{\frac{q}{2}(\gamma+\lambda)}}(1+|u_n|)^{\frac{q}{2}(\gamma+\lambda)}dxdt$$

$$\leq C\int\limits_{Q}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\gamma+\lambda}}dxdt.$$

Thanks to Lemma 3.1, the right-hand side is uniformly bounded if $\dfrac{q(\gamma+1-p)}{1-q}=p$, i.e., $q=\dfrac{2p}{\gamma+1}$. Since $\lambda>1$, then $q>1$ and so we get that u_n is uniformly bounded in $L^\beta\big(0,T,W_0^{1,\beta}(\Omega)\big)$ for $\beta=\dfrac{2p}{\gamma+1}$. As a consequence there exists a function $u\in L^\beta\big(0,T;W_0^{1,\beta}(\Omega)\big)$ with $\beta<\dfrac{2p}{\gamma+1}$ such that, up to subsequences, u_n weakly converges to $L^\beta\big(0,T;W_0^{1,\beta}(\Omega)\big)$. Moreover, u_n converges to u a.e. in Q which implies that $u\in L^p(Q)$.

Now, we are able to prove that u is a *distributional* solution of problem (1.5) by passing to the limit, as n tends to infinity, in the approximate problem (3.1). To this aim, let $\varphi \in C_0^\infty(Q)$. Then the approximating sequences satisfy

$$\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } L^\beta(Q), \qquad \beta = \frac{2p}{\gamma+1},$$

$$a(t,x,T_n(u_n)) \cdot \nabla\varphi \to a(t,x,u)\cdot\nabla\varphi \quad \text{in } L^m(Q) \quad \forall m \geq 1.$$

Thanks to the convergence results, all but the lower-order term pass to the limit on n. Actually, the only term that give some difficulties is the term with $|u_n|^{p-1}u_n$. We can write by choosing $\psi_i(u_n)$ as test function in (3.1) where ψ_i is a positive sequence of increasing and uniformly $C^\infty(Q)$-functions satisfying

$$\psi_i(s) = \begin{cases} 1, & \text{if } \ s \geq h, \\ 0, & \text{if } \ |s| < h, \\ -1, & \text{if } \ s \leq -h. \end{cases}$$

Thus, we obtain, by taking the limit on i, that

$$\int\limits_{\{|u_n|>h\}} |u_n|^p dxdt \leq \int\limits_{\{|u_n|>h\}} |f| dxdt. \tag{3.5}$$

In order to prove the equiintegrability of the lower-order term, let E be any measurable subset of Q. Then we have, for any $h > 0$ and using inequality (3.5), that

$$\int\limits_E |u_n|^p dxdt \leq h^p(E) + \int\limits_{E\cap\{|u_n|>h\}} |u_n|^p dxdt \leq h^p|E| + \int\limits_{\{|u_n|>h\}} |f| dxdt.$$

Since f belongs to $L^1(Q)$, there exists h_ϵ for every $\epsilon > 0$ such that

$$\int\limits_{\{|u_n|>h_\epsilon\}} |f| dxdt \leq \epsilon \Rightarrow \int\limits_E |u_n|^p dxdt \leq h_\epsilon^p |E| + \epsilon,$$

and so

$$\lim_{|E|\to 0} \int\limits_E |u_n|^p dxdt \leq \epsilon \quad \forall \epsilon > 0,$$

and, finally,

$$\lim_{|E|\to 0} \int\limits_E |u_n|^p dxdt = 0 \quad \text{uniformly with respect to } n.$$

Hence, using Vitali's theorem, we get

$$|u_n|^{p-1}u_n \to |u|^{p-1}u \quad \text{in } L^1(Q),$$

which concludes that u is a *distributional* solution of problem (1.5).

Step 2: $0 < p \leq \gamma + 1$. For all $\tau \in (0, T]$, choosing $T_k(u_n(t,x))\chi_{(0,\tau)}(t)$ as test function in problem (3.1) and using assumption (1.3) and the integration by parts formula we have

$$\int\limits_{\Omega} \Theta_k(u_n(\tau,x))dx + \alpha \int\limits_0^{\tau}\int\limits_{\Omega} \frac{|\nabla T_k(u_n)|^2}{(1+|u_n|)^{\tau}} dxdt$$

$$+ \int\limits_0^{\tau}\int\limits_{\Omega} |u_n|^{p-1}u_n T_k(u_n) dxdt$$

$$= \int\limits_0^{\tau}\int\limits_{\Omega} f_n T_k(u_n) dxdt,$$

where $\Theta_k(s) = \displaystyle\int_0^s T_k(s)ds$ is the primitive function of $T_k(s)$. By virtue of $\Theta_k(u_n(\tau,x)) \geq \dfrac{|T_k(u_n(\tau,x))|^2}{2}$, we obtain

$$\operatorname*{ess\,sup}_{0\leq t<T} \int\limits_{\Omega} |T_k(u_n(\tau,x))|^2 dx + \int\limits_{Q} \frac{|\nabla T_k(u_n)|^2}{(1+|u_n|)^{\tau}} dxdt + \int\limits_0^{\tau}\int\limits_{\Omega} |u_n|^{p-1}u_n T_k(u_n) dxdt \leq Ck.$$

Hence,

$$\int\limits_{Q} |\nabla T_k(u_n)|^2 dxdt = \int\limits_{Q} \frac{|\nabla T_k(u_n)|^2}{(1+|T_k(u_n)|)^{\tau}}(1+|T_k(u_n)|)^{\tau} dxdt \leq Ck(1+k)^{\gamma}.$$

Then, by Lemma 2.4 and up to a subsequence, there exists a function $u \in L^{\infty}(0,T;L^1(\Omega))$ such that $T_k(u) \in L^2\big(0,T;H_0^1(\Omega)\big)$, and

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } \; L^2\big(0,T;H_0^1(\Omega)\big),$$

$$u_n \to u \quad \text{a.e. in } \; Q,$$

and, by Fatou's lemma, $|u|^p \in L^1(Q)$. However, by passing to the limit as n tends to infinity, we get

$$\int\limits_{Q} |\nabla T_k(u)|^2 dxdt \leq Ck(1+k)^{\gamma},$$

which implies, by Lemma 2.3, that $|\nabla u|$ belongs to $\mathcal{M}^{\frac{2p}{\gamma+1}}(Q)$.

We have to check that u is an *entropy* solution of problem (1.5). To do that let us choose $T_k(u_n-\varphi)$ with $\varphi \in L^2\big(0,T;H_0^1(\Omega)\big) \cap L^{\infty}(Q)$ such that $\varphi_t \in L^2(0,T;H^{-1}(\Omega))$ and $\varphi(T,x) = 0$, as test function in the weak formulation (3.4), to get

$$\overbrace{\int_0^T \Theta_k(u_n-\varphi)(T,x)dx}^{(\mathcal{A}_1)} - \overbrace{\int_\Omega \Theta_k(u_n-\varphi)(0,x)dx}^{(\mathcal{A}_2)} + \overbrace{\int_0^T \langle \varphi_t, u_n-\varphi\rangle dt}^{(\mathcal{A}_3)}$$

$$+ \overbrace{\int_Q a(t,x,T_k(u_n))\nabla u_n \cdot \nabla T_k(u_n-\varphi)dxdt}^{(\mathcal{A}_4)} + \overbrace{\int_Q |u_n|^{p-1}u_nT_k(u_n-\varphi)dxdt}^{(\mathcal{A}_5)}$$

$$= \overbrace{\int_Q f_nT_k(u_n-\varphi)dxdt}^{(\mathcal{A}_6)}.$$

Let us analyze this equality term by term, we can write

$$(\mathcal{A}_4) := \int_Q a(t,x,T_k(u_n))\nabla u_n \cdot \nabla T_k(u_n-\varphi)dxdt$$

$$= \int_Q a(t,x,T_k(u_n))|\nabla T_k(u_n-\varphi)|^2 dxdt \qquad (\mathcal{A}_{4.1})$$

$$+ \int_Q a(t,x,T_n(u_n))\nabla\varphi \cdot \nabla T_k(u_n-\varphi)dxdt. \qquad (\mathcal{A}_{4.2})$$

Since $T_k(u_n-\varphi)$ converges to $T_k(u-\varphi)$ *weakly in $L^\infty(Q)$ and weakly in $L^2\big(0,T;H_0^1(\Omega)\big)$ and u_n converges to u a.e. in Q, we get

$$(\mathcal{A}_{4.1}) = \liminf_{n\to\infty}\int_Q a(t,x,T_k(u_n))|\nabla T_k(u_n-\varphi)|^2dxdt \geq \int_Q a(t,x,u)|\nabla T_k(u-\varphi)|^2dxdt,$$

while

$$(\mathcal{A}_{4.2}) = \lim_{n\to\infty}\int_Q a(t,x,T_k(u_n))\nabla u_n \cdot \nabla T_k(u_n-\varphi)dxdt \geq \int_Q a(t,x,u)\nabla u \cdot \nabla T_k(u-\varphi)dxdt.$$

Now, using the monotone convergence theorem, we get

$$\lim_{n\to\infty}(\mathcal{A}_1)+(\mathcal{A}_2) = \int_\Omega \Theta_k(u-\varphi)dx - \int_\Omega \Theta_k(u(0)-\varphi(0))dx$$

$$= \int_0^\tau\int_\Omega \Theta_k(u-\varphi)_t dxdt = \int_0^t \langle (u-\varphi)_t, T_k(u-\varphi)\rangle_{W^{-1,p'}(\Omega),W_0^{1,p}(\Omega)}dt.$$

Since $T_k(u_n-\varphi)$ converges to $T_k(u-\varphi)$ weakly in $L^2\big(0,T;H_0^1(\Omega)\big)$, we have

$$(\mathcal{A}_3)=\int_0^T\langle\varphi_t,u_n-\varphi\rangle dt\underset{n\to\infty}{\to}\int_0^T\langle\varphi_t,T_k(u-\varphi)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}dt.$$

Finally, we sum all terms to find

$$\lim_{n\to\infty}(\mathcal{A}_1)+(\mathcal{A}_2)+(\mathcal{A}_3)=\int_0^T\langle u_t,T_k(u-\varphi)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}dt,$$

and, as we mentioned above, this is enough to prove that u is an *entropy* solution of problem (1.5).

Theorem 3.1 is proved.

Our second main result concerns the case where $f\in L^m(Q)$ with $m>1$.

Theorem 3.2. *Under assumption* (1.3) *and* $f\in L^m(Q)$ *with* $m>1$:

(i) *If* $p\ge\dfrac{\gamma+1}{m-1}$, *there exists a distributional solution* u *of problem* (1.5) *such that*

$$u\in L^2\big(0,T;H_0^1(\Omega)\big)\cap L^{pm}(Q).$$

(ii) *If* $\dfrac{\gamma}{m-1}<p<\dfrac{\gamma+1}{m-1}$, *there exists a distributional solution* u *of problem* (1.5) *such that*

$$|u|^{pm}\in L^1(Q)\qquad\text{and}\qquad u\in L^{\frac{2m}{\gamma+1+p}}\Big(0,T;W^{1,\frac{2m}{\gamma+1+p}}(\Omega)\Big).$$

(iii) *If* $0<p\le\dfrac{\gamma}{m-1}$, *there exists an entropy solution* u *of problem* (1.5) *such that*

$$|u|^{pm}\in L^1(Q)\qquad\text{and}\qquad|\nabla u|\in\mathcal{M}^{\frac{2pm}{\gamma+1+p}}(Q).$$

(iv) *If* $p<\dfrac{\gamma}{m-1}$, *there exists a distributional solution* u *of problem* (1.5) *for any* $m>1$.

Proof. The proof is divided in three steps.

Step 1: $p\ge\dfrac{\gamma+1}{m-1}$. By the definition of $\psi(s)$ and $\Psi(s)$ in the proof of Theorem 3.1, we also have by choosing test function $\Psi(u_n)=\big[(1+|u_n|)^{\gamma+1}-1\big]\operatorname{sign}(u_n)$ in the weak formulation of problem (3.4) that

$$C_\lambda\int_\Omega|u_n(\tau,x)|^{\gamma+2}dx+(\gamma+1)\alpha\int_0^\tau\int_\Omega\frac{|\nabla u_n|^2(1+|u_n|)^\gamma}{(1+|u_n|)^\gamma}dxdt\le\int_Q f_n\big[(1+|u_n|)^{\gamma+1}-1\big]dxdt,$$

which implies that

$$\operatorname*{ess\,sup}_{0\le t\le T}\int_\Omega|u_n(t,x)|^{\gamma+2}dx+\int_Q|\nabla u_n|^2dxdt\le C_{\lambda,\gamma,\alpha}\int_Q|f||u_n|^{\gamma+1}dxdt.$$

On the other hand, using the Hölder's inequality, Lemma 3.1, and the fact that $pm\ge(\gamma+1)\dfrac{m}{m-1}$, we obtain

$$\int_Q |f||u_n|^{\gamma+1}dxdt \le \|f\|_{L^m(Q)}\left[\int_Q |u_n|^{(\gamma+1)(\frac{m}{m-1})}\right]^{\frac{m-1}{m}} < \infty,$$

which implies that

$$\int_Q |\nabla u_n|^2 dxdt \le C \quad \forall n \in \mathbb{N}.$$

Hence, up to a subsequence, there exists a function $u \in L^2\big(0,T;H_0^1(\Omega)\big)$ such that, up to a subsequence, u_n converges to u weakly in $L^2\big(0,T;H_0^1(\Omega)\big)$ and a.e. in Q. Moreover, $u \in L^{pm}(Q)$.

Now, we shall use the approximate formulation (3.4) of problem (3.1) in order to prove that u is a solution of problem (1.5). In fact, thanks to the convergence of $a(t,x,T_k(u_n))\nabla T_k(u_n)\cdot\nabla\varphi$ to $a(t,x,u)\nabla u\cdot\nabla\varphi$ in $L^r(Q)$, for any $r \ge 1$, and due to assumption (1.3), we get

$$\lim_{n\to\infty}\int_Q a(t,x,T_k(u_n))\nabla u_n\cdot\nabla\varphi dxdt = \int_Q a(t,x,u)\nabla u\cdot\nabla\varphi dxdt.$$

Moreover, since $|u_n|^{p-1}u_n$ is uniformly bounded in $L^m(Q)$ with $m > 1$, and using the a.e. convergence of u_n to u, we conclude that

$$|u_n|^{p-1}u_n \to |u|^{p-1}u \quad \text{in} \quad L^1(Q).$$

Hence, the desired result holds.

Step 2: $\dfrac{\gamma}{m-1} < p < \dfrac{\gamma+1}{m-1}$. By choosing $\Psi = \big[(1+|u_n|)^{p(m-1)} - 1\big]\operatorname{sign}(u_n)$ as test function in the weak formulation (3.4) of problem (3.1) and using assumption (1.3) we easily obtain[10]

$$C_p\int_\Omega |u_n(\tau,x)|^{p(m-1)+1}dx + \int_Q \frac{|\nabla u_n|^2}{(1+|u_n|)^{\gamma-p(m-1)+1}}dxdt \le C\int_Q |f||u_n|^{p(m-1)}dxdt. \tag{3.6}$$

On the other hand, using the Hölder's inequality in the right-hand side of the previous inequality and Lemma 3.1, we get

$$\int_Q \frac{|\nabla u_n|^2}{(1+|u_n|)^{\gamma-p(m-1)+1}}dxdt \le C\left[\int_Q |u_n|^{pm}\right]^{1-\frac{1}{m}} \le C \quad \forall n\in\mathbb{N}. \tag{3.7}$$

Moreover, for any[11] $q < 2$, we have by Hölder's inequality with exponent $\dfrac{2}{q}$ such that

$$\int_Q |\nabla u_n|^q dxdt = \int_Q \frac{|\nabla u_n|^q}{(1+|u_n|)^{\frac{q}{2}(\gamma-p(m-1)+1)}}(1+|u_n|)^{\frac{q}{2}(\gamma-p(m-1)+1)}dxdt$$

$$\le C\left(\int_Q (1+|u_n|)^{\frac{q}{2-q}[\gamma-p(m-1)+1]}\right)^{1-\frac{q}{2}}.$$

[10] Observe that $\Psi(s) \ge C_p|s|^{p(m-1)+1} - \tilde{C}_p$.

[11] Note that $q < 2$ since we are assuming that $p < \dfrac{\gamma+1}{m-1}$.

By virtue of Lemma 3.1, the fact that $\frac{q}{2-q}[\gamma - p(m-1)+1] = pm$, i.e., $q = \frac{2pm}{\gamma+p+1}$, the last quantity is bounded. Hence, if $1 < \frac{2pm}{\gamma+p+1}$, we finally have that

$$\int_Q |\nabla u_n|^{\frac{2pm}{\gamma+p+1}} dxdt \le C \quad \forall n \in \mathbb{N},$$

which implies that there exists a function $u \in L^{\frac{2pm}{\gamma+p+1}}(0,T; W^{1,\frac{2pm}{\gamma+p+1}}(\Omega))$ such that, up to a subsequence, u_n converges to u weakly in $L^{\frac{2pm}{\gamma+p+1}}(\Omega)$ and a.e. in Q. Moreover, $|u|^{pm} \in L^1(Q)$. This concludes, by following step 1, that u is a *distributional* solution of problem (1.5).

Step 3: $p \le \frac{\gamma}{m-1}$. We shall study the existence of *entropy* solution of problem (1.5). Replacing, respectively, (3.6) and (3.7), which are independent of the choice of p, by the inequalities

$$\int_Q \frac{|\nabla u_n|^2}{(1+|u_n|)^{\gamma-p(m-1)+1}} dxdt \le C$$

and

$$\int_{\{|u_n|<k\}} |\nabla T_k(u_n)|^2 dxdt \le C(1+k)^{\gamma-p(m-1)+1},$$

where C is a positive constant independent of n. Lemma 2.4 imply that there exists a function $u \in L^\infty(0,T; L^1(\Omega))$ such that $T_k(u) \in L^2\big(0,T; H_0^1(\Omega)\big)$, and, up to a subsequence,

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } L^2\big(0,T; H_0^1(\Omega)\big),$$

$$u_n \to u \quad \text{a.e. in } Q.$$

Hence, by tending n to infinity, we obtain that

$$\int_Q |\nabla T_k(u)|^2 dxdt \le C(1+k)^{\gamma-p(m-1)+1},$$

which implies by Lemma 2.3 that if $p < \frac{\gamma+1}{m-1}$, we have $|\nabla u| \in \mathcal{M}^{\frac{2pm}{\gamma+1+p}}(Q)$. It follows, by using the fact that $|u_n|^{pm} \in L^1(Q)$, that $|u|^{pm} \in L^1(Q)$. Thus, by following step 1, we obtain the desired result.

Theorem 3.2 is proved.

4. Degenerate parabolic problem with asymptote. ***4.1. L^1-data.*** In this subsection, we prove the existence of a solution for nonlinear parabolic problem (1.6) in presence of a singular lower-order term of asymptote type without any dependence on the gradient. The proof will be based on a double approximation argument. If $\|u_0\|_{L^\infty(\Omega)} < \sigma$, then we readapt the argument of [11] in order to pass to the limit in the approximate problem. Then, to handle the general case of the initial data possibly touching the singular value σ, we perform a truncation argument[12]. To this aim, let us define $h_n(s)$ as

[12] Using the strong compactness in $L^1(Q)$ of the approximating lower-order term.

$$h_n(s) = \begin{cases} h(s), & \text{if} \quad h(s) < n \quad \text{and} \quad s < \sigma, \\ n, & \text{if} \quad h(s) \geq n \quad \text{and} \quad s < \sigma, \\ n, & \text{if} \quad s \geq \sigma, \end{cases}$$

and let us consider the approximate problem

$$\begin{gathered} (u_n)_t - \operatorname{div}(a(t,x,T_n(u))\nabla u_n) + h_n(u) = f \quad \text{in } Q := (0,T) \times \Omega, \\ u_n(0,x) = u_0^n(x) \quad \text{in } \Omega, \qquad u_n(t,x) = 0 \quad \text{on } (0,T) \times \partial\Omega. \end{gathered} \tag{4.1}$$

By the standard argument of parabolic operators (see [43, 45]), there exists a solution $u_n \in L^2\big(0,T;H_0^1(\Omega)\big) \cap C([0,T];L^2(\Omega))$ of problem (4.1) such that $(u_n)_t \in L^2(0,T;H^{-1}(\Omega))$. In addition, thanks to [46], there exists $l > 0$ independent of n such that $\|u_n\|_{L^\infty(Q)} \leq l$ for every $n \in \mathbb{N}$.

Now, we state the first main result of this section.

Theorem 4.1. *Let f be nonnegative function in $L^1(Q)$ and u_0 be a nonnegative measurable function such that $u_0 < \sigma$ a.e. on Ω. Under assumption* (1.3), *there exists a solution $u \in L^2\big(0,T;H_0^1(\Omega)\big) \cap L^\infty(Q)$ of problem* (1.6) *such that $0 \leq u(t,x) \leq \sigma$ a.e. in Q.*

Proof. We divide the proof in three steps.

Step 1: We prove a priori estimates for u_n.

Step 2: We prove Theorem 4.1 for a bounded datum $f \in L^\infty(Q)$ and $\|u_0\|_{L^\infty(Q)} < \sigma$.

Step 3: By using an approximate argument, we use step 1 to prove Theorem 4.1.

Step 1: *A priori estimates*. We are going to prove that u_n are a priori bounded in the space $L^2\big(0,T;H_0^1(\Omega)\big)$. To this aim, we shall use $\varphi = u_n$ as test function in approximate problem (4.1) to get

$$\int_0^T \langle (u_n)_t, u_n \rangle dt + \int_Q a(t,x,u_n)\nabla u_n \cdot \nabla u_n dxdt + \int_Q h_n(u_n)u_n dxdt = \int_Q f_n u_n dxdt.$$

Since

$$\int_0^T \langle (u_n)_t, u_n \rangle dt = \frac{1}{2}\int_Q \frac{d}{dt} u_n^2 dxdt = \frac{1}{2}\int_\Omega u_n^2(T)dx - \frac{1}{2}\int_\Omega u_n^2(0)dx,$$

and using assumption (1.3), the boundedness of u_n, Young's inequality and by dropping positive terms, we obtain

$$\frac{1}{2}\int_\Omega u_n^2(T)dx + \alpha \int_Q \frac{|\nabla u_n|^2}{(1+|u_n|)^\gamma} dxdt + \int_Q h_n(u_n)u_n dxdt \leq l \int_Q f dxdt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

This implies that (u_n) is bounded in $L^2\big(0,T;H_0^1(\Omega)\big)$ and, in particular, u_n converges to u in $L^2\big(0,T;H_0^1(\Omega)\big)$ and $h_n(u_n)u_n$ is bounded in $L^1(Q)$. Notice that, since $(u_n)_t$ is uniformly bounded in $L^2(0,T;H^{-1}(\Omega)) + L^1(Q)$, we can use the classical Aubin – Simon compactness argument (see [44, Corollary 4]), to obtain the a.e. convergence of u_n toward u.

Step 2: *The case* $f \in L^\infty(Q)$ *and* $\|u_0\|_{L^\infty(\Omega)} < \sigma$. Let us define $\eta = \max\big\{h^{-1}(\|f\|_{L^\infty(Q)}), \|u_0\|_{L^\infty(\Omega)}\big\} < \sigma$ and $\Gamma = \int_0^s (\tau - \eta)^+ d\tau$. Then we have

$$\int_0^T \langle (u_n)_t, (u_n - \eta)^+ \rangle dt = \int_\Omega \Gamma(u_n(T))dx - \int_\Omega \Gamma(u_n(0))dx \geq - \int_\Omega \Gamma(u_0(x)) = 0.$$

Choosing $\varphi = (u_n - \eta)^+$ as test function in the approximated problem (4.1) and using assumption (1.3), we deduce that

$$\int_Q T_n(h(u_n) - f)(u_n - \eta)^+ dxdt \leq \int_Q (h_n - f)(u_n - \eta)^+ dxdt \leq 0,$$

i.e.,

$$0 \geq \int_{\{n - h(u_n) \geq h(\eta)\}} [h(u_n) - f](u_n - \eta)^+ dxdt + \int_{\{h(u_n) \geq n \geq h(\eta)\}} (n - f)(u_n - \eta)^+ dxdt.$$

By using the fact that the right-hand side of this inequality is nonnegative, we obtain

$$0 \leq h(u_n) \leq \|f\|_{L^\infty(Q)},$$

and so

$$0 \leq u_n \leq \max\big\{h^{-1}(\|f\|_{L^\infty(Q)}), \|u_0\|_{L^\infty(\Omega)}\big\} \leq \sigma - \epsilon,$$

that is, by passing to the limit, $0 \leq u < \eta$. Indeed, using assumption (1.3) and the fact that $h(u_n) \geq 0$, it yields

$$\alpha \int_Q \frac{|\nabla u_n|^2}{(1+\sigma)^\gamma} dxdt \leq \|f\|\sigma + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)},$$

which implies that there exists a function $u \in L^2\big(0, T; H_0^1(\Omega)\big) \cap L^\infty(Q)$ such that, up to a subsequence, u_n converges to u weakly in $L^2\big(0, T; H_0^1(\Omega)\big)$ and a.e. in Q. We proceed now to pass to the limit in the approximated problem, we follow the ideas of [11], using the integration by parts formula and the weak convergence of u_n to u in $L^2\big(0, T; H_0^1(\Omega)\big)$. We readily have, for any $\varphi \in L^2\big(0, T; H_0^1(\Omega)\big) \cap L^\infty(Q)$ with $\varphi_t \in L^2(0, T; H^{-1}(\Omega))$ and $\varphi(T, x) = 0$, that

$$-\int_0^T \langle (u_n)_t, \varphi \rangle dt = -\int_Q u_0(x)\varphi(0,x)dx - \int_0^T \langle \varphi_t, u_n \rangle dt \underset{n\to\infty}{\longrightarrow} -\int_\Omega u_0(x)\varphi(0,x)dx - \int_0^T \langle \varphi_t, u \rangle dt.$$

On the other hand, the weak convergence of $a(t, x, u_n) \cdot \nabla u_n$ to $a(t, x, u) \cdot \nabla u$ in $L^2(Q)$ imply that

$$\int_Q a(t, x, u_n)\nabla u_n \cdot \nabla\varphi dxdt \underset{n\to\infty}{\longrightarrow} \int_Q a(t, x, u)\nabla u \cdot \nabla\varphi dxdt.$$

Now, we prove the equiintegrability of the sequence $(h_n)_{n\in\mathbb{N}}$: for any measurable subset E of Q, we get

$$\int\limits_E h_n(u_n)dxdt = \int\limits_{E\cap\{u_n\le\eta\}} h_n(u_n)dxdt \le \int\limits_{E\cap\{u_n\le\eta\}} h(u_n)dxdt,$$

and so

$$\lim_{\text{meas}(E)\to 0}\int\limits_E h_n(u_n)dxdt = 0.$$

The above equiintegrability of $h_n(u_n)$ and the a.e. convergence to $h(u)$ imply by Vitali's theorem that

$$h_n(u_n) \to h(u) \quad \text{in } \ L^1(Q).$$

Thus, we can pass to the limit in the sequence of approximating problems to deduce that u is a solution of (1.6) with $f \in L^\infty(Q)$ and $\|u_0\|_{L^\infty(\Omega)} < \sigma$.

Step 3: *The case* $0 \le f \in L^1(Q)$ *and* $u_0 \in L^1(\Omega)$ *s.t.* $u_0 < \sigma$ *a.e. on* Ω. Let us consider the approximate problems

$$(u_n)_t - \operatorname{div}(a(t,x,u_n)\nabla u_n) + h_n(u_n) = T_n(f) \quad \text{in } \ Q := (0,T)\times\Omega,$$

$$u_n(0,x) = T_{\sigma-\frac{1}{n}}(u_0(x)) \quad \text{in } \ \Omega, \qquad u_n(T,x) = 0 \quad \text{on } \ (0,T)\times\partial\Omega,$$

which admits a solution u_n such that

$$0 \le u_n \le \sigma_n - \max\Big\{h^{-1}(\|f_n\|_{L^\infty(Q)}), \ \ \|T_{\sigma-\frac{1}{n}}(u_0(x))\|_{L^\infty(\Omega)}\Big\} < \sigma.$$

According to the previous step, u_n satisfies

$$\int\limits_0^T \langle (u_n)_t, u_n\rangle dt + \alpha\int\limits_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^\gamma}dxdt + \int\limits_Q h_n(u_n)u_n dxdt \le \int\limits_Q T_n(f)u_n dxdt,$$

and, more precisely,

$$\alpha\int\limits_Q \frac{|\nabla u_n|^2}{(1+\sigma)^\gamma}dxdt \le \sigma\|f\|_{L^1(Q)} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

Therefore, there exists a function $u \in L^2\big(0,T;H_0^1(\Omega)\big)$ such that, up to a subsequence, u_n converges to u weakly in $L^2\big(0,T;H_0^1(\Omega)\big)$ and a.e. in Q such that $0 \le u \le \sigma$.

Now, let us take $\dfrac{1}{\epsilon}T_\epsilon(G_s(u_n))$, where $s,\epsilon > 0$ be such that $s+\epsilon < \sigma$ and G_s is defined in (2.2), as test function in (4.1). Then, using the fact that $0 \le u_n \le \eta$ and dropping positive terms, we get

$$\frac{1}{\epsilon}\int\limits_Q h_n(u_n)T_\epsilon(G_s(u_n))dxdt \le \int\limits_{\{s\le u_n\le\sigma\}} T_n(f)dxdt + \int\limits_{\{s\le u_0<\sigma\}} u_0^n dx$$

for every $s < \sigma$. Indeed, by virtue of Fatou's lemma and tending ϵ to zero, we obtain

$$\int\limits_{\{s\le u_n\}} h(u_n)dxdt \le \int\limits_{\{s\le u_n\}} T_n(f)dxdt + \int\limits_{\{s<u_0\}} u_0 dxdt \le \int\limits_{\{s\le u_n\}} f dxdt + \int\limits_{\{s\le u_0\}} u_0 dx$$

for every $s < \sigma$. On the other hand, for any measurable subset E of Q, we have

$$\int_E h(u_n)dxdt = \int_{E\cap\{0\le u_n<s\}} h(u_n)dxdt + \int_{E\cap\{s\le u_n\}} h(u_n)dxdt$$

$$\le \int_{\{s\le u_n\}} fdxdt + \int_{\{s\le u_0\}} u_0 dx + \int_{\{E\cap\{s<u_n\}\}} h(u_n)dxdt \tag{4.2}$$

for any $s < \sigma$. Since $h(u_n)u_n$ is bounded in $L^1(Q)$, we obtain

$$h(s)s \int_{\{s\le u_n\}} dxdt \le \int_{\{s\le u_n\}} h(u_n)u_n dxdt,$$

so that, by the limit property on h, we get

$$\lim_{s\to\sigma^-} \sup_{n\in\mathbb{N}} \operatorname{meas}\{(t,x)\in Q : s\le u_n\} = 0.$$

Observe that $f \in L^1(Q)$. Then, for any fixed $\epsilon > 0$, there exists $0 < \epsilon_0 < \sigma$ such that

$$\int_{\{s_0\le u_n\}} f < \epsilon, \qquad \int_{\{s_0<u_0<\sigma\}} u_0 < \epsilon.$$

Thus, integral (4.2) implies that

$$\lim_{\operatorname{meas}(E)} \int_E h(u_n)dxdt \le \epsilon,$$

obtaining the equiintegrability of the sequence $(h(u_n))_{n\in\mathbb{N}}$. Hence, the Vitali theorem gives

$$h(u_n) \to h(u) \quad \text{strongly in } L^1(Q).$$

Now, we can pass to the limit as n tends to infinity, in the weak formulation of (4.1), obtaining

$$-\int_\Omega u_0\varphi(0)dx - \int_0^T \langle\varphi_t, u\rangle dt + \int_Q a((t,x,u)\nabla u)\cdot\nabla\varphi dxdt + \int_Q h(u)\varphi dxdt = \int_Q f\varphi dxdt$$

for every $\varphi \in L^2\big(0,T; H_0^1(\Omega)\big) \cap L^\infty(Q)$ with $\varphi_t \in L^2(0,T;H^{-1}(\Omega))$, which concludes the proof of Theorem 4.1.

4.2. Measure data (Dirac mass). In this subsection, we shall study what happens if we try, as in the proof of Theorem 4.1, to deal with irregular data? Both in Section 3 and Subsection 4.1 the assumptions on the data are rather technical since they allow us to get existence of a solution. Obviously the same results of both Theorems 3.1 and 4.1 can be obtained for Radon measures not charging sets of zero capacity (*diffuse* measures), and one would like to prove then for *signed* measures (charging sets of capacity zero). Actually, let $N \ge 2$, $\Omega = B_1(Q)$ and $f = \delta_0$[13] (the Dirac

[13] Note that this fact is true not only for δ_0, but also for any other datum of the form δ_a, $a \in Q$, with $a \ne 0$, or more generally for any λ-measure concentrated on a set of zero 2-capacity.

mass concentrated at the origin of $\mathbb{R}^{N+1}$). To this aim, let us consider the problem

$$\begin{aligned} &u_t - \operatorname{div}(a(t,x,u)\nabla u) + h(u) = \delta_0 \quad \text{in } Q := (0,T)\times\Omega, \\ &u(0,x) = u_0(x) \quad \text{in } \Omega, \qquad u(t,x) = 0 \quad \text{on } (0,T)\times\partial\Omega, \end{aligned} \tag{4.3}$$

where Ω is an open bounded subset in $\mathbb{R}^N$, $h(s)$ and u_0 are defined as above and δ_0 is the Dirac mass in $(t,0)$ with $t\in[0,T]$. It is known that problem (4.3) has a unique distributional solution u belonging to $L^1(0,T;W_0^{1,1}(\Omega))$, which can be explicitly calculated. If we restrict ourselves to the case $N\geq 3$ and the Δ-Laplacian for simplicity, the function $u(t,x) = C_N(|u(t,x)|^{2-N}-1)$, where C_N is a positive constant depending only on the dimension N. The idea consists on approximating[14] δ_0 with a sequence of functions $f_n = \chi_{B_{\frac{1}{N}(0)}}/\operatorname{meas}(B_{\frac{1}{n}(0)})$ satisfying

$$f_n \to \delta_0 \quad \text{in the weak* topology of measures.}$$

Thus, one can consider the approximate problems

$$\begin{aligned} &(u_n)_t - \operatorname{div}(a(t,x,u_n)\nabla u_n) + h_n(u_n) = f_n \quad \text{in} \quad Q := (0,T)\times\Omega, \\ &u_n(0,x) = u_0^n \quad \text{in} \quad \Omega, \qquad u(t,x) = 0 \quad \text{on} \quad (0,T)\times\partial\Omega. \end{aligned}$$

The case where f is a δ-measure turns out to be much more suitable one might expect. It was observed by Bénilan and Brézis (see [7, 24 – 27]) and, in the elliptic coercive case [29, Theorems B.5 and B.6], if $N\geq 3$ and $h(s) = |s|^{p-1}s$ with $p\geq \dfrac{N}{N-2}$, then problem (4.3) has no solution when the right-hand side is a Dirac mass δ_a at a point $a\in\Omega$. Our goal is to analyze the nonexistence result and to describe what happens if one choose δ_0 in our parabolic problem. Concerning the function h, we will assume throughout the rest of the paper the problem

$$\begin{aligned} &u_t - \operatorname{div}(a(t,x,u)\nabla u) + g(t,x,u) = \delta_0 \quad \text{in } Q := (0,T)\times\Omega, \\ &u(0,x) = u_0(x) \quad \text{in } \Omega, \qquad u(t,x) = 0 \quad \text{on } (0,T)\times\partial\Omega, \end{aligned} \tag{4.4}$$

that $g(t,x,s)\colon (0,T)\times\Omega\times[0,\sigma)\to\mathbb{R}^+$ is a Carathéodory function such that

$$h(s) \leq g(t,x,s) \leq \rho(t,x)\gamma(s) \quad \forall s\in[0,\sigma), \quad \text{a.e. in } \Omega \quad \forall t\in[0,T), \tag{4.5}$$

where $0\leq\rho\in L^1(Q)$ and $\gamma(s), h(s)\colon [0,\sigma)\to\mathbb{R}^+$ are continuous and increasing real functions such that $\gamma(0) = h(0) = 0$ and $\lim_{s\to\sigma^-} h(s) = +\infty$. The regularizing effect of the lower-order term g bring a stronger nonexistence result under the form of removable singulartities phenomena. Note that for having such strong nonexistence result, we require assumption (4.5) which is stronger than

[14] One can choose the approximation

$$\varrho_n \star \delta_0 \to \delta_0,$$

where ϱ_n is a smooth approximation of the Dirac mas $\delta_{(0,0)}$, defined by

$$\varrho_n(x,y) = \frac{1}{n^2}\gamma\Big(\frac{x}{n}\Big)\eta\Big(\frac{t}{n}\Big) \geq 0 \quad \text{with} \quad \operatorname{Supp}\gamma \subseteq [-1,1], \quad \operatorname{Supp}\eta \subseteq [-1,1].$$

assumption (1.2). The following result has suggested that similar features could also be observed in studying the effect of perturbations of the data which are possibly very singular, i.e., not necessarily bounded in $L^1(Q)$, but localized around sets of null capacity. We have then the following result, which can found in less generality in [2, 3, 51].

Theorem 4.2. *Let δ_0 be the Dirac mass, f_n be a sequence of $L^\infty(Q)$-functions such that*

$$\lim_{n\to+\infty}\int\limits_Q f_n\varphi dxdt = \int\limits_Q \varphi d\delta_0 \quad \forall\varphi\in C^0(\bar{Q}),$$

and u_n be an approximate solution of the differential problem (4.4) *with right-hand side f_n. Then*

$$T_k(u_n)\to 0 \quad \textit{strongly in} \quad L^2\big(0,T;H_0^1(\Omega)\big) \quad \forall k>0.$$

Moreover,

$$\lim_{n\to\infty}\int\limits_Q g_n(t,x,u_n)\varphi dxdt = \int\limits_Q \varphi d\delta_0 \quad \forall\varphi\in C_0^1(Q).$$

***Remark* 4.1.** The result of Theorem 4.2 can be seen as an “exceptional” nonexistence result of problem (4.3). One can perturb the datum f with arbitrary large (concentrating on compact sets K) functions if the datum is considered as $\delta_a,\ a\neq 0$ (Dirac mass concentrated on a point: a set of zero N-capacity), and, in particular, one can take $f_n=\varrho_n\star D^n(\delta_{x_0})$ the convolution of derivatives of the Dirac mass. Indeed, the approximating solutions converges to zero in the whole of Q, so that this very strong perturbation is actually swept away by the regularizing effect of the equation.

Now, we turn to problem (1.1) and we recall that the main tool in the proof of the nonexistence result is the fact that the sequence of approximating solutions (u_n) converges to zero which cannot be a solution for our problem with Dirac mass. Hence, a solution obtained by approximation does not exist. To achieve that, it is of fundamental importance the fact that the datum δ_0 (the Dirac mass concentrated at the origin) is approximated with a sequence of nonnegative $L^\infty(Q)$-functions with support concentrated in $\mathcal{B}_{\frac{1}{n}}(0)$, the unit ball of $\mathbb{R}^N$, and zero elsewhere. Since (u_n) has zero 2-capacity (as every point in $\mathbb{R}^N$), for every $\delta>0$, there exists a function $\psi_\delta\in C_0^\infty(Q)$ (see [50, Lemma 5]), such that

$$0\le\psi_\delta\le 1, \qquad \int\limits_Q |\nabla\psi_\delta|^2 dxdt, \qquad \int\limits_Q f_n(1-\psi_\delta)dxdt = 0.$$

The later being true for every n large enough. As a consequence, we have that ψ_δ converges to zero both strongly in $L^2\big(0,T;H_0^1(\Omega)\big)$ a.e. in Q and in the weak* topology of $L^\infty(Q)$.

In other words, the existence of solution fails for the second member measure of problem (4.4).

Theorem 4.3. *Let δ_0 be the Dirac mass at the origin and (f_n) is a sequence of nonnegative $L^\infty(Q)$-functions with support contained in $\mathcal{B}_{\frac{1}{n}}(0)$ and converging to δ_0, i.e.,*

$$\lim_{n\to+\infty}\int\limits_Q f_n\varphi dxdt = \int\limits_Q \varphi d\delta_0 \quad \forall\varphi\in C^0(\bar{Q}).$$

Suppose that u_n be an approximate solution of problem (4.4) *with datum f_n. Then*

$$T_k(u_n) \to 0 \quad \textit{strongly in } L^2\big(0,T;H_0^1(\Omega)\big) \quad \textit{for every } k<\sigma.$$

Moreover,

$$\lim_{n\to+\infty}\int\limits_Q g_n(t,x,u_n)\varphi dxdt = \int\limits_Q \varphi d\delta_0 \quad \forall\varphi\in C_0^1(\bar{Q}).$$

Proof. The proof is divided in three steps.

Step 1: *Approximate problem and a priori estimates.* Let $\Omega=\mathcal{B}_1(0)$ be the unit ball of $\mathbb{R}^N$ and consider (u_n) as a sequence of solutions of approximate problems

$$\begin{aligned} &(u_n)_t - \operatorname{div}(a(t,x,u_n)\nabla u_n) + g_n(t,x,u_n) = f_n \quad \text{in } Q:=(0,T)\times\Omega,\\ &u_n(0,x)=u_0^n(x) \quad \text{in } \Omega, \qquad u(t,x)=0 \quad \text{on } (0,T)\times\partial\Omega, \end{aligned} \tag{4.6}$$

where u_0^n approaches u_0, $a_n(t,x,s,\zeta)=a(t,x,T_n(s),\zeta)$, $g_n(t,x,s)=T_n(g(t,x,s))$ and (f_n) is a sequence of $L^\infty(Q)$-function that approaches δ_0. Using the positiveness of f_n, u_n is also positive and (f_n) is bounded in $L^1(Q)$, we have

$$0\le u_n\le\sigma \qquad \text{and} \qquad \operatorname{meas}\{(t,x)\in Q: u_n(t,x)=\sigma\}=0.$$

On the other hand, since the support of f_n is disjoint from the ball $\mathcal{B}_{\frac{1}{n}}(0)$ if $n\ge n_0$ with n_0 large enough, the result of Theorem 4.1 implies that $g_n(t,x,u_n)$ is L^1-compact and u_n is bounded in $L^2\big(0,T;H_0^1(\Omega)\big)$. Therefore, up to a subsequence, there exist a subsequence, still denoted by u_n, and a function $u\in L^2\big(0,T;H_0^1(\Omega)\big)$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } L^2\big(0,T;H_0^1(\Omega)\big) \quad \text{and a.e. in } Q,$$

$$a(t,x,u_n,\nabla u_n) \rightharpoonup w \quad \text{weakly in } L^2(Q)^N.$$

Step 2: 1*st asymptotic estimate.* By choosing $(k-T_k(u_n))\psi_\delta$ as test function in the weak formulation of (4.6) satisfied by u_n and integrating by parts, we obtain

$$\int\limits_Q \Theta_k(u_n)(\psi_\delta)_t dxdt - \int\limits_\Omega \Theta_k(u_0^n)\psi_\delta(0)dx$$

$$-\int\limits_Q a(t,x,T_n(u_n))\nabla T_k(u_n)\cdot\nabla T_k(u_n)\psi_\delta dxdt$$

$$+\int\limits_Q (k-T_k(u_n))a(t,x,u_n)\nabla T_k(u_n)\cdot\nabla\psi_\delta dxdt$$

$$+\int\limits_Q g_n(t,x,u_n)(k-T_k(u_n))\psi_\delta dxdt$$

$$= \int\limits_Q f_n(k - T_k(u_n))\psi_\delta dxdt.$$

Since $k - T_k(u_n)$ converges to $k - T_k(u_n)$ both in the weak* topology and a.e. in Q, we have that $\nabla\psi_\delta(k - T_k(u_n))$ converges to $\nabla\psi_\delta(k - T_k(u))$ strongly in $L^p(Q)^N$. Hence,

$$\lim_{\delta\to 0}\lim_{n\to\infty}\int\limits_Q (k - T_k(u_n))a(t,x,T_n(u))\nabla T_k(u_n)\cdot\nabla\psi_\delta dxdt$$

$$= \lim_{\delta\to 0}\int\limits_Q (k - T_k(u))w\cdot\nabla\psi_\delta dxdt = 0.$$

On the other hand, due to the fact that $\Theta_k(u_n)$ converges to Θ_k weakly in $L^2\big(0,T;H_0^1(\Omega)\big)$, we observing that $\Theta_k(u)\in L^2\big(0,T;H_0^1(\Omega)\big)\cap L^\infty(Q)$ and

$$\lim_{n\to\infty}\int\limits_Q \Theta_k(u_n)(\psi_\delta)_t dxdt - \int\limits_\Omega \Theta_k(u_0)\psi_\delta dt = 0.$$

Moreover, we have

$$\lim_{\delta\to 0}\lim_{n\to 0}\int\limits_Q g_n(t,x,T_n(u))(k - T_k(u_n))\psi_\delta dxdt$$

$$= \lim_{\delta\to 0}\lim_{n\to\infty}\int\limits_{\{0\le u_n<k\}} g_n(t,x,u_n)(k-u_n)\psi_\delta dxdt$$

$$= \lim_{\delta\to 0}\int\limits_Q g(t,x,T_k(u))(k - T_k(u))\psi_\delta dxdt = 0,$$

which implies that

$$\lim_{\delta\to 0}\lim_{n\to\infty}\int\limits_Q a(t,x,T_n(u_n))\nabla T_k(u_n)\cdot\nabla T_k(u_n)\psi_\delta dxdt \le 0.$$

Step 3: 2*nd asymptotic estimate.* Now, we choose $T_k(u_n)(1-\psi_\delta)$ as test function in the weak formulation of (4.6), satisfied by u_n, to get

$$\int\limits_Q \Theta_n(u_n)(\psi_\delta)_t dxdt - \int\limits_\Omega \Theta_n(u_0^n)(1-\psi_\delta(0))dx$$

$$+ \int\limits_Q a(t,x,T_k(u_n))\nabla T_k(u_n)\cdot\nabla T_k(u_n)(1-\psi_\delta)dxdt$$

$$-\int\limits_{Q} T_k(u_n)a(t,x,u_n)\nabla u_n\cdot\nabla\psi_\delta dxdt+\int\limits_{Q} g_n(t,x,u_n)T_k(u_n)(1-\psi_\delta)dxdt$$

$$=\int\limits_{Q} f_nT_k(u_n)(1-\psi_\eta)dxdt.$$

Dropping the nonnegative term with g_n, observing that the last term is zero for n large enough and passing to the limit as n tends to infinity, we obtain

$$\lim_{\delta\to 0}\lim_{n\to\infty}\int\limits_{Q} a(t,x,u_n)\nabla T_k(u_n)\cdot\nabla T_k(u_n)(1-\psi_\delta)dxdt\leq 0.$$

Collecting the last inequalities, we easily have

$$0\leq\frac{\alpha(k)}{(1+k)^\gamma}\int\limits_{Q}|\nabla T_k(u_n)|^p dxdt\leq\int\limits_{Q} a(t,x,T_k(u_n))\nabla T_k(u)\cdot\nabla T_k(u)dxdt\leq 0,$$

which implies that, for every $k<\sigma$,

$$T_k(u_n)\to 0\quad\text{in}\ \ L^2\big(0,T;H_0^1(\Omega)\big).$$

Hence, $u=0$. However, $u=0$ is not a solution of equation (4.4).

Theorem 4.3 is proved.

***Remark* 4.2.** The conclusion of the previous example remains true every time that f_n converges to a Dirac mass concentrated at a point $x_0\neq 0$ or a singular measure concentrated on a set of zero capacity.

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