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SUFFICIENT AND NECESSARY CONDITIONS FOR THE GENERALIZED DISTRIBUTION SERIES TO BE IN SUBCLASSES OF UNIVALENT FUNCTIONS

ДОСТАТНІ ТА НЕОБХІДНІ УМОВИ ДЛЯ ТОГО, ЩОБ РЯДИ УЗАГАЛЬНЕНИХ РОЗПОДІЛІВ НАЛЕЖАЛИ ДО ПІДКЛАСІВ ОДНОЛИСТИХ ФУНКЦІЙ

We establish a relationship between the subclasses of univalent functions and generalized distribution series. The main aim of our investigation is to obtain necessary and sufficient conditions for the generalized distribution series to belong to the classes $\mathcal{TF}(\rho,\vartheta)$, $\mathcal{TH}(\rho,\vartheta)$, $\mathcal{TJ}(\rho,\vartheta)$, and $\mathcal{TX}(\rho,\vartheta)$. In addition, we obtain some particular cases of our main results.

Встановлено зв'язок між підкласами однолистих функцій і рядами узагальнених розподілів. Основною метою цього дослідження є встановлення необхідних і достатніх умов для того, щоб ряди узагальнених розподілів належали до класів $\mathcal{TF}(\rho,\vartheta)$, $\mathcal{TH}(\rho,\vartheta)$, $\mathcal{TJ}(\rho,\vartheta)$ і $\mathcal{TX}(\rho,\vartheta)$. Крім того, отримано деякі окремі випадки наших основних результатів.

1. Introduction and preliminaries. Let $\mathcal{A}$ be the class of all analytic functions of the form

$$f(z) = z + \sum_{\ell=2}^{\infty} a_\ell z^\ell, \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{D} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ that contains the function of the form

$$f(z) = z - \sum_{\ell=2}^{\infty} a_\ell z^\ell, \quad a_\ell \geq 0. \tag{1.2}$$

Definition 1.1. *For functions* $f(z) \in \mathcal{A}$, *given by* (1.1), *and* $g(z) \in \mathcal{A}$, *defined by*

$$g(z) = z + \sum_{\ell=2}^{\infty} b_\ell z^\ell,$$

Hadamard product (*or convolution*) *of* $f(z)$ *and* $g(z)$ *is given by*

$$(f * g)(z) = z + \sum_{\ell=2}^{\infty} a_\ell b_\ell z^\ell = (g * f)(z), \quad z \in \mathcal{D}.$$

In [15], Lashin et al. defined and studied a new class as the following.

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Definition 1.2 [15]. *A function $f \in \mathcal{A}$ is said to be in the class $\mathfrak{G}(\Theta, \Phi, \Upsilon, \rho, \vartheta)$ if the following condition is satisfied*:

$$\operatorname{Re}\left\{\frac{(1-\rho)(f*\Theta)(z)+\rho(f*\Phi)(z)}{(f*\Upsilon)(z)}\right\} > \vartheta, \quad 0 \le \rho \le 1, \quad 0 \le \vartheta < 1,$$

where the functions Θ, Φ *and* Υ *denote by*

$$\Theta(z) = z + \sum_{\ell=2}^{\infty} \alpha_\ell z^\ell, \qquad \Phi(z) = z + \sum_{\ell=2}^{\infty} \beta_\ell z^\ell \qquad \text{and} \qquad \Upsilon(z) = z + \sum_{\ell=2}^{\infty} \gamma_\ell z^\ell,$$

$$\alpha_\ell, \beta_\ell, \gamma_\ell \ge 0, \quad z \in \mathcal{D}.$$

Also,

$$\mathcal{T}\mathfrak{G}(\Theta, \Phi, \Upsilon, \rho, \vartheta) = \mathfrak{G}(\Theta, \Phi, \Upsilon, \rho, \vartheta) \cap \mathcal{T}.$$

The class, defined above, includes several simpler subclasses. We point out here some of these special cases as follows:

If we put $\Theta(z) = \dfrac{z}{(1-z)^2}$, $\Phi(z) = \dfrac{z+z^2}{(1-z)^3}$ and $\Upsilon(z) = z$ in Definition 1.2, then we have the following subclass.

Definition 1.3 [6]. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{F}(\rho, \vartheta)$ if the following condition is satisfied*:

$$\operatorname{Re}\left\{f'(z) + \rho z f''(z)\right\} > \vartheta, \quad 0 \le \rho \le 1, \quad 0 \le \vartheta < 1,$$

and (*see* [15, 20])

$$\mathcal{T}\mathcal{F}(\rho, \vartheta) = \mathcal{F}(\rho, \vartheta) \cap \mathcal{T}.$$

Lemma 1.1 [20 with $p = 1$]. *A function $f(z) \in \mathcal{T}$ is in the class $\mathcal{T}\mathcal{F}(\rho, \vartheta)$ if and only if*

$$\sum_{\ell=2}^{\infty} \ell(1-\rho+\rho\ell)a_\ell \le 1-\vartheta, \quad 0 \le \rho \le 1, \quad 0 \le \vartheta < 1.$$

If we put $\Theta(z) = \dfrac{z}{(1-z)^2}$, $\Phi(z) = \dfrac{z+z^2}{(1-z)^3}$ and $\Upsilon(z) = \dfrac{z}{1-z}$ in Definition 1.2, then we have the following subclass.

Definition 1.4 [19]. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{H}(\rho, \vartheta)$ if the following condition is satisfied*:

$$\operatorname{Re}\left\{\frac{zf'(z)+\rho z^2 f''(z)}{f(z)}\right\} > \vartheta, \quad 0 \le \rho \le 1, \quad 0 \le \vartheta < 1,$$

and (*see* [14, 15])

$$\mathcal{T}\mathcal{H}(\rho, \vartheta) = \mathcal{H}(\rho, \vartheta) \cap \mathcal{T}.$$

Lemma 1.2 [14]. *A function $f(z) \in \mathcal{T}$ is in the class $\mathcal{TH}(\rho,\vartheta)$ if and only if*

$$\sum_{\ell=2}^{\infty}((\ell-1)(\rho\ell+1)+1-\vartheta)a_\ell \le 1-\vartheta, \quad 0\le\rho\le1, \quad 0\le\vartheta<1.$$

If we put $\Theta(z)=\dfrac{z}{1-z}$, $\Phi(z)=\dfrac{z}{(1-z)^2}$ and $\Upsilon(z)=z$ in Definition 1.2, then we have the following subclass.

Definition 1.5 [7]. *A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{J}(\rho,\vartheta)$ if the following condition is satisfied*:

$$\operatorname{Re}\left\{\frac{(1-\rho)f(z)}{z}+\rho f'(z)\right\}>\vartheta, \quad 0\le\rho\le1, \quad 0\le\vartheta<1,$$

and $\left(\textit{see } [15] \textit{ and } \left[8 \textit{ with } h(z)=\dfrac{z}{1-z}\right]\right)$

$$\mathcal{TJ}(\rho,\vartheta)=\mathcal{J}(\rho,\vartheta)\cap\mathcal{T}.$$

Lemma 1.3 [15]. *A function $f(z)\in\mathcal{T}$ is in the class $\mathcal{TJ}(\rho,\vartheta)$ if and only if*

$$\sum_{\ell=2}^{\infty}(\rho(\ell-1)+1)a_\ell\le1-\vartheta, \quad 0\le\rho\le1, \quad 0\le\vartheta<1.$$

If we put $\Theta(z)=\dfrac{z}{1-z}$, $\Phi(z)=\dfrac{z+z^2}{(1-z)^3}$ and $\Upsilon(z)=\dfrac{z}{(1-z)^2}$ in Definition 1.2, then we have the following subclass.

Definition 1.6 [16]. *A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{X}(\rho,\vartheta)$ if the following condition is satisfied*:

$$\operatorname{Re}\left\{(1-\rho)\frac{f(z)}{zf'(z)}+\rho\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\vartheta, \quad 0\le\rho\le1, \quad 0\le\vartheta<1,$$

and

$$\mathcal{TX}(\rho,\vartheta)=\mathcal{X}(\rho,\vartheta)\cap\mathcal{T}.$$

Lemma 1.4 [16]. *A function $f(z)\in\mathcal{T}$ is in the class $\mathcal{TX}(\rho,\vartheta)$ if and only if*

$$\sum_{\ell=2}^{\infty}((\ell-1)(\rho(\ell+1)-1)+\ell(1-\vartheta))a_\ell\le1-\vartheta, \quad \frac{1}{3}\le\rho\le1, \quad 0\le\vartheta<1.$$

***Remark* 1.1.** (i) Putting $\rho=0$ in Definition 1.4, we obtain the class $\mathcal{S}^*(\vartheta)$ which contain the starlike function of order ϑ, $0\le\vartheta<1$, which studied by Robertson [26].

(ii) Putting $\rho=1$ in Definition 1.6, we obtain the class $\mathcal{C}(\vartheta)$ convex function of order ϑ, $0\le\vartheta<1$, which studied by Robertson [26].

(iii) Putting $\rho=0$ in Definition 1.3 and putting $\rho=1$ in Definition 1.5, we obtain the class $\mathcal{K}^*(\vartheta)$ which contain the close-to-convex function of order ϑ, $0\le\vartheta<1$, which studied by Libra [17].

The applications of hypergeometric functions [21, 27], generalized hypergeometric functions [13], Wright function [25], Fox – Wright function [5], generalized Bessel functions [4, 9, 24] are play an important role in geometric function theory. In 2014, Porwal [23] (see also [1, 10, 11, 18]) introduced Poisson distribution series and obtain necessary and sufficient conditions for certain classes of univalent functions and co-relates probability density function with geometric function theory.

In 2018, Porwal [22] introduced the generalized distribution series as follows.

Definition 1.7. *The generalized distribution series is defined as*

$$S = \sum_{\ell=0}^{\infty} t_\ell, \quad t_\ell \geq 0.$$

That series is convergent.

Porwal [22] gave the generalization of discrete probability distribution with its probability mass function is given by

$$\chi(\ell) = \frac{t_\ell}{S}, \quad \ell \in \mathbb{N} \cup \{0\}, \quad \mathbb{N} = \{1, 2, 3, \ldots\}.$$

Obviously $\chi(\ell) \geq 0$ and $\sum_\ell \chi(\ell) = 1$. Also, he defined the series

$$\psi(x) = \sum_{\ell=0}^{\infty} t_\ell x^\ell.$$

From the above equation can be verified easily that it is convergent when $-1 < x \leq 1$. By specializing the values of t_ℓ, we obtain several known discrete probability distributions (see [22]).

In [22], Porwal defined a power series whose coefficients are probabilities of the generalized distribution as follows:

$$\mathcal{G}_\psi(z) = z + \sum_{\ell=2}^{\infty} \frac{t_{\ell-1}}{S} z^\ell$$

and

$$\mathcal{T}\mathcal{G}_\psi(z) = z - \sum_{\ell=2}^{\infty} \frac{t_{\ell-1}}{S} z^\ell.$$

Now, we defined the functions $\mathcal{N}_\psi(\tau, z)$ and $\mathcal{N}_\psi^*(\tau, z)$ as follows:

$$\begin{aligned} \mathcal{N}_\psi(\tau, z) &= (1-\tau)\mathcal{G}_\psi(z) + \tau z(\mathcal{G}_\psi(z))' \\ &= z + \sum_{\ell=2}^{\infty} (1 + \tau(\ell-1)) \frac{t_{\ell-1}}{S} z^\ell, \quad 0 \leq \tau \leq 1, \quad z \in \mathcal{D}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}_\psi^*(\tau, z) &= 2z - \mathcal{N}_\psi(\tau, z) \\ &= z - \sum_{\ell=2}^{\infty} (1 + \tau(\ell-1)) \frac{t_{\ell-1}}{S} z^\ell, \quad 0 \leq \tau \leq 1, \quad z \in \mathcal{D}. \end{aligned}$$

In this paper, we establish a relation between subclasses of univalent functions and generalized distribution series. The main aim of the present investigation is to obtain the necessary and sufficient conditions for generalized distribution series belongs to the classes $\mathcal{TF}(\rho,\vartheta)$, $\mathcal{TH}(\rho,\vartheta)$, $\mathcal{TJ}(\rho,\vartheta)$ and $\mathcal{TX}(\rho,\vartheta)$.

2. Main results. In this section, we derive the sufficient and necessary conditions for the function $\mathcal{N}^*_\psi(\tau,z)$ to be in the above classes. We will assume throughout in this paper that $0\le\tau\le 1$, $0\le\rho\le 1$, $0\le\vartheta<1$ and $z\in\mathcal{D}$.

Theorem 2.1. *The function $\mathcal{N}^*_\psi(\tau,z)$ is to be in the class $\mathcal{TF}(\rho,\vartheta)$ if and only if*

$$\left(\frac{1}{S}\right)\big(\rho\tau\psi'''(1)+(\tau+\rho(1+4\tau))\psi''(1)+(1+2\tau+2\rho(1+\tau))\psi'(1)+\psi(1)-\psi(0)\big)\le 1-\vartheta.$$

Proof. According to Lemma 1.1, we need to show that

$$\sum_{\ell=2}^{\infty}[\ell(1-\rho+\rho\ell)][1+\tau(\ell-1)]\frac{t_{\ell-1}}{S}\le 1-\vartheta.$$

Thus,

$$\sum_{\ell=2}^{\infty}[\ell(1-\rho+\rho\ell)][1+\tau(\ell-1)]\frac{t_{\ell-1}}{S}$$

$$=\left(\frac{1}{S}\right)\Bigg(\rho\tau\sum_{\ell=2}^{\infty}(\ell-1)(\ell-2)(\ell-3)t_{\ell-1}+(\tau+\rho(1+4\tau))\sum_{\ell=2}^{\infty}(\ell-1)(\ell-2)t_{\ell-1}$$

$$+(1+2\tau+2\rho(1+\tau))\sum_{\ell=2}^{\infty}(\ell-1)t_{\ell-1}+\sum_{\ell=2}^{\infty}t_{\ell-1}\Bigg)$$

$$=\left(\frac{1}{S}\right)\Bigg(\rho\tau\sum_{\ell=1}^{\infty}\ell(\ell-1)(\ell-2)t_\ell+(\tau+\rho(1+4\tau))$$

$$\times\sum_{\ell=1}^{\infty}\ell(\ell-1)t_\ell+(1+2\tau+2\rho(1+\tau))\sum_{\ell=1}^{\infty}\ell t_\ell+\sum_{\ell=1}^{\infty}t_\ell\Bigg)$$

$$=\left(\frac{1}{S}\right)\big(\rho\tau\psi'''(1)+(\tau+\rho(1+4\tau))\psi''(1)+(1+2\tau+2\rho(1+\tau))\psi'(1)+\psi(1)-\psi(0)\big)$$

$$\le 1-\vartheta.$$

Theorem 2.1 is proved.

Putting $\rho=0$ in Theorem 2.1, we obtain the following corollary.

***Corollary* 2.1.** *The function $\mathcal{N}^*_\psi(\tau,z)\in\mathcal{K}^*(\vartheta)$ if and only if*

$$\left(\frac{1}{S}\right)\Bigg(\tau\psi''(1)+(1+2\tau)\psi'(1)+\psi(1)-\psi(0)\Bigg)\le 1-\vartheta.$$

Putting $\rho=0$ and $\tau=0$ in Theorem 2.1, we obtain the following corollary.

***Corollary* 2.2.** *The function $\mathcal{TG}_\psi(z) \in \mathcal{K}^*(\vartheta)$ if and only if*

$$\left(\frac{1}{S}\right)\Big(\psi'(1) + \psi(1) - \psi(0)\Big) \leq 1 - \vartheta.$$

Theorem 2.2. *The function $\mathcal{N}^*_\psi(\tau, z)$ is to be in the class $\mathcal{TH}(\rho, \vartheta)$ if and only if*

$$\left(\frac{1}{S}\right)\Big(\rho\tau\psi'''(1) + (\rho + \tau(1+4\rho))\psi''(1)$$

$$+ ((1+2\rho)(1+\tau) + \tau(1-\vartheta))\psi'(1) + (1-\vartheta)(\psi(1) - \psi(0))\Big) \leq 1 - \vartheta.$$

Proof. According to Lemma 1.2, we need to show that

$$\sum_{\ell=2}^{\infty} \big[(\ell-1)(\rho\ell+1) + 1 - \vartheta][1 + \tau(\ell-1)\big] \frac{t_{\ell-1}}{S} \leq 1 - \vartheta.$$

Thus,

$$\sum_{\ell=2}^{\infty} \big[(\ell-1)(\rho\ell+1) + 1 - \vartheta][1 + \tau(\ell-1)\big] \frac{t_{\ell-1}}{S}$$

$$= \left(\frac{1}{S}\right)\Bigg(\sum_{\ell=2}^{\infty} \big\{(\ell-1)(\rho(\ell-2) + (1+2\rho)) + (1-\vartheta)\big\} t_{\ell-1}$$

$$+ \sum_{\ell=2}^{\infty} \big\{\tau\rho(\ell-1)(\ell-2)(\ell-3) + 2\rho\tau(\ell-1)(\ell-2)\big\} t_{\ell-1}$$

$$+ \sum_{\ell=2}^{\infty} \big\{\tau(1+2\rho)(\ell-1)(\ell-2) + \tau(1+2\rho)(\ell-1) + \tau(1-\vartheta)(\ell-1)\big\} t_{\ell-1}\Bigg)$$

$$= \left(\frac{1}{S}\right)\Bigg(\tau\rho\sum_{\ell=1}^{\infty} \big\{\ell(\ell-1)(\ell-2)\big\} t_\ell + (4\tau\rho + \tau + \rho)\sum_{\ell=1}^{\infty} \ell(\ell-1) t_\ell$$

$$+ \big[(1+2\rho)(1+\tau) + \tau(1-\vartheta)\big]\sum_{\ell=1}^{\infty} \ell t_\ell + (1-\vartheta)\sum_{\ell=1}^{\infty} t_\ell\Bigg)$$

$$= \left(\frac{1}{S}\right)\Big(\tau\rho\psi'''(1) + (4\tau\rho + \tau + \rho)\psi''(1)$$

$$+ \big[(1+2\rho)(1+\tau) + \tau(1-\vartheta)\big]\psi'(1) + (1-\vartheta)[\psi(1) - \psi(0)]\Big)$$

$$\leq 1 - \vartheta.$$

Theorem 2.2 is proved.

Putting $\rho = 0$ in Theorem 2.2, we obtain the following corollary.

***Corollary* 2.3.** *The function* $\mathcal{N}^*_{\psi}(\tau, z) \in \mathcal{S}^*(\vartheta)$ *if and only if*

$$\left(\frac{1}{S}\right)\Big(\tau\psi''(1) + (1+\tau(2-\vartheta))\psi'(1) + (1-\vartheta)(\psi(1)-\psi(0))\Big) \le 1-\vartheta.$$

Putting $\rho = 0$ and $\tau = 0$ in Theorem 2.2, we obtain the following corollary (see also [22, Theorem 10 with $\lambda = 0$]).

***Corollary* 2.4.** *The function* $\mathcal{TG}_{\psi}(z) \in \mathcal{S}^*(\vartheta)$ *if and only if*

$$\left(\frac{1}{S}\right)\Big(\psi'(1) + (1-\vartheta)(\psi(1)-\psi(0))\Big) \le 1-\vartheta.$$

Putting $\tau = 0$ in Theorem 2.2, we obtain the following corollary.

***Corollary* 2.5.** *The function* $\mathcal{TG}_{\psi}(z) \in \mathcal{TH}(\rho, \vartheta)$ *if and only if*

$$\left(\frac{1}{S}\right)\Big(\rho\psi''(1) + (1+2\rho)\psi'(1) + (1-\vartheta)(\psi(1)-\psi(0))\Big) \le 1-\vartheta.$$

***Remark* 2.1.** (i) If we take $t_\ell = \dfrac{m^\ell}{\ell!}$, $m > 0$, in Theorem 2.2, we get the results, which obtained by Lashin et al. [15, Theorem 7];

(ii) If we take $t_\ell = \dfrac{m^\ell}{\ell!}$, $m > 0$, in Corollary 2.5, we have the results, which obtained by Lashin et al. [15, Corollary 8].

Theorem 2.3. *The function* $\mathcal{N}^*_{\psi}(\tau, z)$ *is to be in the class* $\mathcal{TJ}(\rho, \vartheta)$ *if and only if*

$$\left(\frac{1}{S}\right)\Big(\rho\tau\psi''(1) + (\rho+\tau(1+\rho))\psi'(1) + \psi(1)-\psi(0)\Big) \le 1-\vartheta.$$

Proof. According to Lemma 1.3, we need to show that

$$\sum_{\ell=2}^{\infty}[1+\rho(\ell-1)][1+\tau(\ell-1)]\frac{t_{\ell-1}}{S} \le 1-\vartheta.$$

Thus,

$$\begin{aligned}
&\sum_{\ell=2}^{\infty}[1+\rho(\ell-1)][1+\tau(\ell-1)]\frac{t_{\ell-1}}{S}\\
&\qquad = \left(\frac{1}{S}\right)\left(\sum_{\ell=2}^{\infty}t_{\ell-1} + (\tau+\rho)\sum_{\ell=2}^{\infty}(\ell-1)t_{\ell-1} + \tau\rho\sum_{\ell=2}^{\infty}(\ell-1)^2 t_{\ell-1}\right)\\
&\qquad = \left(\frac{1}{S}\right)\left(\sum_{\ell=1}^{\infty}t_{\ell} + (\tau+\rho+\tau\rho)\sum_{\ell=1}^{\infty}\ell t_{\ell} + \tau\rho\sum_{\ell=1}^{\infty}\ell(\ell-1)t_{\ell}\right)\\
&\qquad = \left(\frac{1}{S}\right)\Big(\rho\tau\psi''(1) + (\rho+\tau(1+\rho))\psi'(1) + \psi(1)-\psi(0)\Big) \le 1-\vartheta.
\end{aligned}$$

Theorem 2.3 is proved.

***Remark* 2.2.** (i) Putting $\rho = 1$ in Theorem 2.3, we get the result obtained in Corollary 2.1.
(ii) Putting $\rho = 1$ and $\tau = 0$ in Theorem 2.3, we have the result obtained in Corollary 2.2.
Putting $\tau = 0$ in Theorem 2.3, we obtain the following corollary.
***Corollary* 2.6.** *The function $\mathcal{TG}_{\psi}(z) \in \mathcal{TJ}(\rho, \vartheta)$ if and only if*

$$\left(\frac{1}{S}\right)\Big(\rho\psi'(1) + \psi(1) - \psi(0)\Big) \le 1 - \vartheta.$$

***Remark* 2.3.** (i) If we take $t_\ell = \dfrac{m^\ell}{\ell!}$, $m > 0$, in Theorem 2.3, we get the results, which obtained by Lashin et al. [15, Theorem 9].

(ii) If we take $t_\ell = \dfrac{m^\ell}{\ell!}$, $m > 0$, in Corollary 2.6, we have the results, which obtained by Frasin [12, Corollary 5.1].

Theorem 2.4. *The function $\mathcal{TG}_{\psi}(z)$ is to be in the class $\mathcal{TX}(\rho, \vartheta)$ if and only if*

$$\left(\frac{1}{S}\right)\Big(\rho\psi''(1) + (3\rho - \vartheta)\psi'(1) + (1 - \vartheta)(\psi(1) - \psi(0)\Big) \le 1 - \vartheta.$$

Proof. According to Lemma 1.4, we need to show that

$$\sum_{\ell=2}^{\infty}\big[(\ell - 1)(\rho(\ell + 1) - 1) + \ell(1 - \vartheta)\big]\frac{t_{\ell-1}}{S} \le 1 - \vartheta.$$

Thus,

$$\sum_{\ell=2}^{\infty}\big[(\ell - 1)(\rho(\ell + 1) - 1) + \ell(1 - \vartheta)\big]\frac{t_{\ell-1}}{S}$$

$$= \left(\frac{1}{S}\right)\left(\rho\sum_{\ell=2}^{\infty}(\ell - 1)(\ell - 2)t_{\ell-1} + (3\rho - \vartheta)\sum_{\ell=2}^{\infty}(\ell - 1)t_{\ell-1} + (1 - \vartheta)\sum_{\ell=2}^{\infty}t_{\ell-1}\right)$$

$$= \left(\frac{1}{S}\right)\left(\rho\sum_{\ell=1}^{\infty}\ell(\ell - 1)t_{\ell} + (3\rho - \vartheta)\sum_{\ell=1}^{\infty}\ell t_{\ell} + (1 - \vartheta)\sum_{\ell=1}^{\infty}t_{\ell}\right)$$

$$= \left(\frac{1}{S}\right)\Big(\rho\psi''(1) + (3\rho - \vartheta)\psi'(1) + (1 - \vartheta)(\psi(1) - \psi(0))\Big) \le 1 - \vartheta.$$

Theorem 2.4 is proved.

Putting $\rho = 1$ in Theorem 2.4, we obtain the following corollary (see also [22, Theorem 11 with $\lambda = 0$]).

***Corollary* 2.7.** *The function $\mathcal{TG}_{\psi}(z) \in \mathcal{C}(\vartheta)$ if and only if*

$$\left(\frac{1}{S}\right)\Big(\psi''(1) + (3 - \vartheta)\psi'(1) + (1 - \vartheta)(\psi(1) - \psi(0)\Big) \le 1 - \vartheta.$$

***Remark* 2.4.** If we take $t_\ell = \dfrac{m^\ell}{\ell!}$, $m > 0$, in Theorem 2.4, we obtain the results, which obtained by Lashin et al. [15, Theorem 11].

Theorem 2.5. *The function $\mathcal{N}_{\psi}^{*}(\tau, z)$ is to be in the class $\mathcal{TX}(\rho, \vartheta)$ if and only if*

$$\left(\frac{1}{S}\right)\Bigg(\rho\tau\psi'''(1) + (\rho + \tau(5\rho - \vartheta))\psi''(1) + ((3\rho - \vartheta)(\tau + 1) + \tau(1 - \vartheta))\psi'(1) + (1 - \vartheta)(\psi(1) - \psi(0))\Bigg) \le 1 - \vartheta.$$

Proof. According to Lemma 1.4, we need to show that

$$\sum_{\ell=2}^{\infty}\big[(\ell-1)(\rho(\ell+1)-1) + \ell(1-\vartheta)\big][1 + \tau(\ell-1)]\frac{t_{\ell-1}}{S} \le 1 - \vartheta.$$

Thus,

$$\sum_{\ell=2}^{\infty}\big[(\ell-1)(\rho(\ell+1)-1) + \ell(1-\vartheta)\big][1 + \tau(\ell-1)]\frac{t_{\ell-1}}{S}$$

$$= \left(\frac{1}{S}\right)\Bigg(\rho\tau\sum_{\ell=2}^{\infty}(\ell-1)(\ell-2)(\ell-3)t_{\ell-1} + (\rho + \tau(5\rho - \vartheta))\sum_{\ell=2}^{\infty}(\ell-1)(\ell-2)t_{\ell-1} + ((3\rho - \vartheta)(\tau + 1) + \tau(1 - \vartheta))\sum_{\ell=2}^{\infty}(\ell-1)t_{\ell-1} + (1 - \vartheta)\sum_{\ell=2}^{\infty}t_{\ell-1}\Bigg)$$

$$= \left(\frac{1}{S}\right)\Bigg(\rho\tau\sum_{\ell=1}^{\infty}\ell(\ell-1)(\ell-2)t_{\ell} + (\rho + \tau(5\rho - \vartheta))\sum_{\ell=1}^{\infty}\ell(\ell-1)t_{\ell} + ((3\rho - \vartheta)(\tau + 1) + \tau(1 - \vartheta))\sum_{\ell=1}^{\infty}\ell t_{\ell} + (1 - \vartheta)\sum_{\ell=1}^{\infty}t_{\ell}\Bigg)$$

$$= \left(\frac{1}{S}\right)\Big(\rho\tau\psi'''(1) + (\rho + \tau(5\rho - \vartheta))\psi''(1) + ((3\rho - \vartheta)(\tau + 1) + \tau(1 - \vartheta))\psi'(1) + (1 - \vartheta)(\psi(1) - \psi(0))\Big)$$

$$\le 1 - \vartheta.$$

Theorem 2.5 is proved.

Putting $\rho = 1$ in Theorem 2.5, we obtain the following corollary.

***Corollary* 2.8.** *The function $\mathcal{N}_{\psi}^{*}(\tau, z)$ is to be in the class $\mathcal{C}(\vartheta)$ if and only if*

$$\left(\frac{1}{S}\right)\Bigg(\tau\psi'''(1) + (1 + \tau(5 - \vartheta))\psi''(1) + ((3 - \vartheta)(\tau + 1) + \tau(1 - \vartheta))\psi'(1) + (1 - \vartheta)(\psi(1) - \psi(0)\Bigg) \le 1 - \vartheta.$$

***Remark* 2.5.** If we take $t_{\ell} = \dfrac{m^{\ell}}{\ell!}$, $m > 0$, in Theorem 2.5, we get the results, which obtained by Lashin et al. [15, Theorem 12].

3. Conclusion. Many application on the complex plane have been studied by several authors (see, e.g., [1 – 3, 9 – 13]). In our paper, we obtain some sufficient conditions of generalized distribution series for belonging to certain classes of univalent functions. By giving specific values on generalized distribution series we obtain sufficient conditions for Poisson distribution series.

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