DOI: 10.3842/umzh.v75i10.7255

UDC 517.5

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THE SHARP BOUND OF CERTAIN SECOND HANKEL DETERMINANTS FOR THE CLASS OF INVERSE OF STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

ТОЧНА ГРАНИЦЯ ДЕЯКИХ ДРУГИХ ВИЗНАЧНИКІВ ГАНКЕЛЯ ДЛЯ КЛАСУ ОБЕРНЕНИХ ЗІРКОПОДІБНИХ ФУНКЦІЙ ЩОДО СИМЕТРИЧНИХ ТОЧОК

We investigate the sharp bound of certain coefficient functionals associated with a Hankel determinant of second kind for the inverse function, when f belongs to the class of starlike functions with respect to symmetric points.

Досліджено точну границю деяких коефіцієнтних функціоналів, що пов'язані з визначником Ганкеля другого роду для оберненої функції, у випадку, коли f належить класу зіркоподібних функцій щодо симетричних точок.

1. Introduction. Let $\mathcal{A}$ represent the family of all analytic normalized mappings f of the type

$$f(z) = \sum_{t=1}^{\infty} a_t z^t, \quad a_1 := 1,$$

in $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$, denotes the open unit disc and $\mathcal{S}$ is the subfamily of $\mathcal{A}$, possessing univalent (schlicht) mappings. Pommerenke [10] characterized the rth-Hankel determinant of order n for f with $r,\ n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, namely

$$H_{r,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2} \end{vmatrix}. \tag{1.1}$$

The Fekete – Szegö functional is obtained for $r = 2$ and $n = 1$ in (1.1), denoted by $H_{2,1}(f)$. Further, for $r = n = 2$ and $r = 2,\ n = 3$ in (1.1), called as second order Hankel determinants, respectively given by

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \tag{1.2}$$

and

$$H_{2,3}(f) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix}. \tag{1.3}$$

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The exact estimate of $|H_{2,2}(f)|$ for the subfamilies of $\mathcal{S}$, were proved by Janteng et al. [5, 6]. Zaprawa et al. [16] obtained sharp bound of $|H_{2,3}(f)|$ for the class of bounded turning functions, also for the class of starlike and convex functions by taking $a_2 = 0$. For the class of starlike functions with respect to symmetric points, introduced by Sakaguchi [14], denoted as $\mathcal{S}_s^*$, satisfying the analytic condition $\mathrm{Re}\{(2zf'(z))/(f(z) - f(-z))\} > 0$, Ramreddy et al. [11] shown that $|H_{2,2}(f)| \leq 1$.

For $f \in \mathcal{S}_s^*$, Hern et al. [4] proved that $|H_{2,3}(f)| \leq 13/16$ and $|H_{2,3}(f)| \leq 1$, which is sharp, by assuming $a_2 = 0$.

For $f \in \mathcal{S}$, has an inverse f^{-1} given by

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n, \quad |w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}.$$

Ali [1] determined sharp bounds on the first four coefficients and sharp estimate for the Fekete–Szegö coefficient functional of the inverse functions belong to the class of strongly starlike functions. Recently Sim et al. [15] obtained sharp bound of $\left|H_{2,2}\big(f^{-1}\big)\right|$ for the class of strongly Ozaki functions.

Motivated by the results obtained by the authors mentioned above, in this paper we are making an attempt to estimate sharp bound for coefficient functionals, namely $\left|H_{2,1}\big(f^{-1}\big)\right|$, $\left|H_{2,2}\big(f^{-1}\big)\right|$ and $\left|H_{2,3}\big(f^{-1}\big)\right|$, when f belongs to the class of $\mathcal{S}_s^*$.

The collection $\mathcal{P}$ of all functions p, each one called as Carathéodory function [2] of the form

$$p(z) = 1 + \sum_{t=1}^{\infty} c_t z^t,$$

having a positive real part in $\mathbb{D}$. Buliding our analysis on the familiar formulas of coefficients c_2 (see [10, p. 166]), c_3 (see [8, 9]) and c_4 can be found in [12, 13].

The foundation for the proofs of our main results are the following lemmas, and we adopt the procedure framed through Libera and Zlotkiewicz [9].

Lemma 1.1 [3]. *If $p \in \mathcal{P}$, then $|c_i - \mu c_j c_{i-j}| \leq 2$ satisfies for the values $i, j \in \mathbb{N}$ with $i > j$ and $\mu \in [0, 1]$, which is same as $|c_{n+k} - \mu c_n c_k|$ for $n, k \in \mathbb{N}$ with $\mu \in [0, 1]$.*

Lemma 1.2 [10]. *If $p \in \mathcal{P}$, then $|c_t| \leq 2$ for $t \in \mathbb{N}$, equality occurs for the function $p_0 = \dfrac{1+z}{1-z}$, $z \in \mathbb{D}$.*

Lemma 1.3 [7]. *If $p \in \mathcal{P}$, then*

$$2c_2 = c_1^2 + t\zeta,$$

$$4c_3 = c_1^3 + 2c_1 t\zeta - c_1 t\zeta^2 + 2t\big(1 - |\zeta|^2\big)\eta,$$

and

$$8c_4 = c_1^4 + 3c_1^2 t\zeta\big(4 - 3c_1^2\big)t\zeta^2 + c_1^2 t\zeta^3 + 4t\big(1 - |\zeta|^2\big)\big(1 - |\eta|^2\big)\xi$$
$$+ 4t\big(1 - |\zeta|^2\big)\big(c_1\eta - c\zeta\eta - \bar{\zeta}\eta^2\big),$$

where $t := 4 - c_1^2$, for some ζ, η and ξ such that $|\zeta| \leq 1$, $|\eta| \leq 1$ and $|\xi| \leq 1$.

2. Main result.

Theorem 2.1. *If* $f \in \mathcal{S}_s^*$, *then*

$$\left|H_{2,1}\left(f^{-1}\right)\right| \le 1$$

and the result is sharp for $f_0 = z/\left(1 - z^2\right)$.

Proof. For $f \in \mathcal{S}_s^*$, as per Definition 1.1, there exists an analytic function $p(z)$ in $\mathbb{D}$ with $\operatorname{Re} p(z) > 0$ such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = p(z) \Leftrightarrow 2zf'(z) = \{f(z) - f(-z)\}p(z). \tag{2.1}$$

Using the series representation for $f'(z),\ f(-z),\ f(z)$ and $p(z)$ in (2.1), after simplifying, equating the coefficients of like powers of z, we obtain

$$a_2 = \frac{c_1}{2}, \quad a_3 = \frac{c_2}{2}, \quad a_4 = \frac{2c_3 + c_1c_2}{8}, \quad a_5 = \frac{2c_4 + c_2^2}{8}. \tag{2.2}$$

Since $f \in \mathcal{S}_s^*$, using the definition of inverse function of f, we have

$$w = f\left(f^{-1}\right) = f^{-1}(w) + \sum_{n=2}^{\infty} a_n\left(f^{-1}(w)\right)^n.$$

Further, we get

$$w = f\left(f^{-1}\right) = w + \sum_{n=2}^{\infty} t_n w^n + \sum_{n=2}^{\infty} a_n \left(w + \sum_{n=2}^{\infty} t_n w^n\right)^n.$$

Upon simplification we obtain

$$\begin{aligned}(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 + \left(t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4\right)w^4 \\ + \left(t_5 + 2a_2t_4 + 2a_2t_2t_3 + 3a_3t_3 + 3a_3t_2^2 + 4a_4t_2 + a_5\right)w^5 + \ldots = 0.\end{aligned} \tag{2.3}$$

Equating the coefficients of like power in (2.3), after simplifing, we get

$$\begin{aligned} t_2 &= -a_2, \\ t_3 &= -a_3 + 2a_2^2, \\ t_4 &= -a_4 + 5a_2a_3 - 5a_2^3, \\ t_5 &= -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned} \tag{2.4}$$

Using the values of $a_n,\ n = 2, 3, 4, 5,$ from (2.2) in (2.4), it simplifies to give

$$\begin{aligned} t_2 &= -\frac{c_1}{2}, \\ t_3 &= \frac{-c_2 + c_1^2}{2}, \\ t_4 &= \frac{-5c_1^3 + 9c_1c_2 - 2c_3}{8}, \\ t_5 &= \frac{7c_1^4 - 18c_1^2c_2 + 5c_2^2 + 6c_1c_3 - 2c_4}{8}. \end{aligned} \tag{2.5}$$

Now, based on $H_{2,1}(f)$, we have

$$H_{2,1}(f^{-1}) = \begin{vmatrix} 1 & t_2 \\ t_2 & t_3 \end{vmatrix} = t_3 - t_2^2. \tag{2.6}$$

Using the values of $t_j,\ j = 2, 3,$ from (2.5) in (2.6), it simplifies to give

$$H_{2,1}(f^{-1}) = -\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right).$$

Taking modulus and applying Lemma 1.1, we obtain

$$|H_{2,1}(f^{-1})| \le \frac{1}{2}\left|c_2 - \frac{c_1^2}{2}\right| = 1.$$

From f_0 we get $t_2 = 0$ and $t_3 = -1$.

Theorem 2.1 is proved.

Theorem 2.2. *If $f \in \mathcal{S}_s^*$, then*

$$\left|H_{2,2}\big(f^{-1}\big)\right| \le 1$$

and the inequality is sharp for the function f_o given in Theorem 2.1.

Proof. For $f \in \mathcal{S}_s^*$, in view of (1.2), we have

$$H_{2,2}(f^{-1}) = \begin{vmatrix} t_2 & t_3 \\ t_3 & t_4 \end{vmatrix} = t_2 t_4 - t_3^2. \tag{2.7}$$

Using the values of $t_j,\ j = 2, 3, 4,$ from (2.5) in (2.7), it simplifies to give

$$H_{2,2}\big(f^{-1}\big) = \frac{1}{16}\big(c_1^4 - c_1^2 c_2 - 4c_2^2 + 2c_1 c_3\big). \tag{2.8}$$

Applying Lemma 1.3 in expression (2.8), upon simplification it yields

$$H_{2,2}\big(f^{-1}\big) = \frac{t}{16}\left\{-\frac{3}{2}\,c_1^2\zeta - t\zeta^2 - \frac{1}{2}\,c_1^2\zeta^2 + c_1\big(1 - |\zeta|^2\big)\eta\right\}. \tag{2.9}$$

Taking modulus on both sides and then applying the triangle inequality by taking $c := c_1,\ t := 4 - c^2$ and $\mu := |\zeta|$, using $|\eta| \le 1$ in (2.9), we obtain

$$\begin{aligned}
\left|H_{2,2}\big(f^{-1}\big)\right| &\le \frac{4 - c^2}{16}\left\{\frac{3}{2}\,c^2 x + \big(4 - c^2\big)x^2 + \frac{1}{2}\,c^2 x^2 + c\big(1 - x^2\big)\right\} \\
&= \frac{4 - c^2}{16}\left\{c + \frac{3}{2}\,c^2 x + \frac{1}{2}\,(2 - c)(4 + c)x^2\right\} \\
&\le \frac{4 - c^2}{16}\left\{c + \frac{3}{2}\,c^2 + \frac{1}{2}\,(2 - c)(4 + c)\right\} \\
&= 1 - \frac{c^4}{16} \le 1.
\end{aligned}$$

Therefore, $\left|H_{2,2}\big(f^{-1}\big)\right| \le 1.$

From f_0, we obtain $t_2 = 0,\ t_3 = -1$ and $t_4 = 0.$

Theorem 2.2 is proved.

Theorem 2.3. *If* $f \in \mathcal{S}_s^*$, *then*

$$\left|H_{2,3}\left(f^{-1}\right)\right| \leq 2$$

and the inequality is sharp for f_0.

Proof. For $f \in \mathcal{S}_s^*$, based on (1.3), we have

$$H_{2,3}\left(f^{-1}\right) = \begin{vmatrix} t_3 & t_4 \\ t_4 & t_5 \end{vmatrix} = t_3 t_5 - t_4^2. \tag{2.10}$$

Using the values of $t_j,\ j = 3, 4, 5$ from (2.5) in (2.10), it simplifies to give

$$\begin{aligned} H_{2,3}\left(f^{-1}\right) = \frac{1}{64}\Big(&3c_1^6 - 10c_1^4c_2 + 4c_1^3c_3 + 11c_1^2c_2^2 + 8c_2c_4 \\ &- 8c_1^2c_4 + 12c_1c_2c_3 - 20c_2^3 - 4c_3^2\Big). \end{aligned} \tag{2.11}$$

In view of Lemma 1.3 and equation (2.11), we obtain

$$\begin{aligned} -10c_1^4c_2 &= -5\left[c_1^6 + c_1^4 t\zeta\right], \\ 4c_1^3c_3 &= \left[c_1^6 + 2c_1^4 t\zeta - c_1^4 t\zeta^2 + 2c_1^3 t\left(1 - |\zeta|^2\right)\eta\right], \\ -20c_2^3 &= -\frac{5}{2}\left[c_1^6 + 3c_1^4 t\zeta + 3c_1^2 t^2\zeta^2 + t^3\zeta^3\right], \\ 11c_1^2c_2^2 &= \frac{11}{4}\left[c_1^6 + 2c_1^4 t\zeta + c_1^2 t^2\zeta^2\right], \\ 12c_1c_2c_3 &= \frac{3}{2}\Big[c_1^6 + 3c_1^4 t\zeta + 2c_1^2 t^2\zeta^2 - c_1^4 t\zeta^2 - c_1^2 t^2\zeta^3 \\ &\quad + 2t\left(c_1^3 + c_1 t\zeta\right)\left(1 - |\zeta|^2\right)\eta\Big], \\ -4c_3^2 &= -\frac{1}{4}\Big[c_1^6 + 4c_1^4 t\zeta + 4c_1^2 t^2\zeta^2 - 2c_1^4 t\zeta^2 - 4c_1^2 t^2\zeta^3 + c_1^2 t^2\zeta^4 \\ &\quad + 4t\left(c_1^3 + 2ct\zeta - c_1 t\zeta^2\right)\left(1 - |\zeta|^2\right)\eta + 4t^2\left(1 - |\zeta|^2\right)^2\eta^2\Big], \\ 8c_2c_4 - 8c_1^2c_4 &= \frac{1}{2}\Big[- c_1^6 - 3c_1^4 t\zeta - c_1^2\left(4 - 3c_1^2\right)t\zeta^2 - c_1^4 t\zeta^3 - 4c_1^3 t(1 - \zeta)(1 - |\zeta|^2)\eta \\ &\quad + 4c_1^2 t\left(1 - |\zeta|^2\right)\bar{\zeta}\eta^2 - 4c_1^2 t\left(1 - |\zeta|^2\right)\left(1 - |\eta|^2\right)\xi + c_1^4 t\zeta + 3c_1^2 t^2\zeta^2 \\ &\quad + \left(4 - 3c_1^2\right)t^2\zeta^3 + c_1^2 t^2\zeta^4 + 4t^2 c_1\zeta(1 - \zeta)\left(1 - |\zeta|^2\right)\eta \\ &\quad - 4t^2\left(1 - |\zeta|^2\right)|\zeta|^2\eta^2 + 4t^2\left(1 - |\zeta|^2\right)\left(1 - |\eta|^2\right)\zeta\xi\Big]. \end{aligned} \tag{2.12}$$

From the expressions (2.11) and (2.12), we get

$$H_{2,3}\left(f^{-1}\right) = \frac{t}{64}\Bigg\{ - \frac{5}{2}\, c_1^4\zeta - \frac{1}{2}\, c_1^4\zeta^2 - 2c_1^2\zeta^2 - \frac{1}{2}\, c_1^4\zeta^3$$

$$+\,t\left[-2c_1^2\zeta^3+\frac{1}{4}\,c_1^2\zeta^4+2\zeta^3-\frac{5}{2}\,t\zeta^3-\frac{5}{4}\,c_1^2\zeta^2\right]$$

$$+\left[2(1+\zeta)c_1^3+c_1t\zeta(3-\zeta)\right]\left(1-|\zeta|^2\right)\eta$$

$$+\left[2c_1^2\bar{\zeta}-t\left(1+|\zeta|^2\right)\right]\left(1-|\zeta|^2\right)\eta^2$$

$$+\,2\left[t\zeta-c_1^2\right]\left(1-|\zeta|^2\right)\left(1-|\eta|^2\right)\xi\Bigg\}. \tag{2.13}$$

For $c_1 := c$ and $t := 4-c^2$ in (2.13), it takes the form

$$H_{2,3}\left(f^{-1}\right)=\frac{4-c^2}{64}\left\{-\frac{5}{2}\,c^4\zeta-\frac{1}{2}\,c^4\zeta^2-2c^2\zeta^2-\frac{1}{2}\,c^4\zeta^3\right.$$

$$+\left(4-c^2\right)\left[-\frac{5}{4}\,c^2\zeta^2+\frac{1}{4}\,c^2\zeta^4-\left(8-\frac{c^2}{2}\right)\zeta^3\right]$$

$$+\left[2(1+\zeta)c^3+c\zeta\left(4-c^2\right)(3-\zeta)\right]\left(1-|\zeta|^2\right)\eta$$

$$+\left[2c^2\bar{\zeta}-\left(4-c^2\right)\left(1+|\zeta|^2\right)\right]\left(1-|\zeta|^2\right)\eta^2$$

$$\left.+\,2\left[\left(4-c^2\right)\zeta-c^2\right]\left(1-|\zeta|^2\right)\left(1-|\eta|^2\right)\xi\right\}. \tag{2.14}$$

Taking modulus on both sides in expression (2.14) with $|\zeta|=x\in[0,1],\ |\eta|=y\in[0,1],\ c_1=c\in[0,2]$ and $|\xi|\le 1,$ we obtain

$$\left|H_{2,3}\left(f^{-1}\right)\right|\le\frac{\Phi(c,x,y)}{64}, \tag{2.15}$$

where $\Phi(c,x,y)\colon\mathbb{R}^3\to\mathbb{R}$ is defined as

$$\Phi(c,x,y)=\left(4-c^2\right)\left\{\frac{5}{2}\,c^4x+\frac{1}{2}\,c^4x^2+2c^2x^2+\frac{1}{2}\,c^4\zeta^3\right.$$

$$+\left(4-c^2\right)\left[\frac{5}{4}\,c^2x^2+\frac{1}{4}\,c^2x^4+\left(8-\frac{c^2}{2}\right)x^3\right]$$

$$+\left[2(1+x)c^3+cx\left(4-c^2\right)(3+x)\right]\left(1-x^2\right)y$$

$$+\left[2c^2x+\left(4-c^2\right)\left(1+x^2\right)\right]\left(1-x^2\right)y^2$$

$$\left.+\,2\left[\left(4-c^2\right)x+c^2\right]\left(1-x^2\right)\left(1-y^2\right)\right\}. \tag{2.16}$$

Now, we will maximize the function $\Phi(c,x,y)$ in the region of the parallelepiped formed by $[0,2]\times[0,1]\times[0,1]$, where $c\in[0,2],\ x\in[0,1]$ and $y\in[0,1]$.

A. On the vertices of the parallelepiped, we obtain

$$\Phi(0,0,0)=\Phi(2,0,0)=\Phi(2,1,0)=\Phi(2,1,1)=\Phi(2,0,0)=0,$$

$$\Phi(0,0,1) = 16, \quad \Phi(0,1,0) = \Phi(0,1,1) = 128.$$

B. Now, considering the eight edges of the parallelepiped:
(i) for $c = 0,\ x = 0,\ 0 < y < 1$ in (2.16), we have

$$\Phi(0,0,y) = 16y^2 \le 16;$$

(ii) for $c = 0,\ x = 1,\ 0 < y < 1$ in (2.16), we get

$$\Phi(0,1,y) = 128;$$

(iii) for $c = 0,\ y = 0,\ 0 < x < 1$, we obtain

$$\Phi(0,x,0) = 32x + 96x^3 \le 128;$$

(iv) for $c = 0,\ y = 1,\ 0 < x < 1$, we have

$$\Phi(0,x,1) = 16 + 128x^3 - 16x^4 \le 128;$$

(v) for $x = 0,\ y = 0,\ 0 < c < 2$, we get

$$\Phi(c,0,0) = 8c^2 - 2c^4 \le 32;$$

(vi) for $x = 0,\ y = 1,\ 0 < c < 2$, we obtain

$$\Phi(c,0,1) = 16 - 8c^2 + 8c^3 + c^4 - 2c^5 \le 96;$$

(vii) for $x = 1,\ y = 0,\ 0 < c < 2$ and $x = 1,\ y = 1,\ 0 < c < 2$, we have

$$\Phi(c,1,y) = 128 - 40c^2 + 12c^4 - \frac{5c^6}{2} \le 128;$$

(viii) for $c = 2,\ x = 0,\ 0 < y < 1,\ c = 2,\ x = 1, 0 < y < 1,\ c = 2,\ y = 0,\ 0 < x < 1$ and $c = 2,\ y = 1,\ 0 < x < 1$, we get

$$\Phi(2,x,y) = 0.$$

C. Now, we consider the six faces of the parallelepiped:
(i) for $c = 2$, we obtain $\Phi(2,x,y) = 0$ for $x,y \in (0,1)$;
(ii) if $c = 0$ in (2.16), then, for $x,y \in (0,1)$, we have

$$\begin{aligned}\Phi(0,x,y) &= 4\big(32x^3 + 4\big(1-x^2\big)\big(1+x^2\big)y^2 + 8x\big(1-x^2\big)\big(1-y^2\big)\big)\\ &= 32x + 96x^3 + 16(1-x)^3(1+x)y^2\\ &\le 32x + 96x^3 + 16(1-x)^3(1+x) \le 128;\end{aligned}$$

(iii) if $x = 0$ in (2.16) then, for $c \in (0,2),\ x \in (0,1)$, we get

$$\begin{aligned}\Phi(c,0,y) &= \big(4-c^2\big)\big[2c^3y + \big(4-c^2\big)y^2 + 2c^2\big(1-y^2\big)\big]\\ &= \big(4-c^2\big)\big[2c^3y + 4y^2 + c^2\big(2-3y^2\big)\big]\\ &\le \big(4-c^2\big)\big[4 + 2c^2 + 2c^3\big] \le 96;\end{aligned}$$

(iv) on the face $x = 1,\ c \in (0,2),\ y \in (0,1)$ from (2.16), we observe that function $\Phi(c,1,y)$ is independent of y, and from **B** (vii), we obtain

$$\Phi(c,1,y) \le 128;$$

(v) on the face $y = 0,\ c \in (0,2),\ x \in (0,1)$ from (2.16), we have

$$\Phi(c,x,0) = \left(4-c^2\right)\left\{8x + 24x^3 + c^4\left(\frac{5x}{2} - \frac{3x^2}{4} + x^3 - \frac{x^4}{4}\right)\right.$$

$$\left. + c^2\left(2 - 2x + 5x^2 - 8x^3 + x^4\right)\right\} \le \left(4-c^2\right)\left\{32 + 2c^2 + \frac{5c^4}{2}\right\} \le 128;$$

(vi) on the face $y = 1,\ c \in (0,2),\ x \in (0,1)$ from (2.16), we get

$$\Phi(c,x,1) = \left(4-c^2\right)\left\{\frac{5}{2}\,c^4x + \frac{1}{2}\,c^4x^2 + 2c^2x^2 + \frac{1}{2}\,c^4\zeta^3\right.$$

$$+ \left(4-c^2\right)\left[\frac{5}{4}\,c^2x^2 + \frac{1}{4}\,c^2x^4 + \left(8 - \frac{c^2}{2}\right)x^3\right]$$

$$+ \left[2(1+x)c^3 + cx\left(4-c^2\right)(3+x)\right]\left(1-x^2\right)$$

$$\left. + \left[2c^2x + \left(4-c^2\right)\left(1+x^2\right)\right]\left(1-x^2\right)\right\}$$

$$:= g(c,x) \quad \text{with} \quad c \in (0,2) \quad \text{and} \quad x \in (0,1).$$

Further,

$$\frac{\partial g}{\partial c} = -16c + 24c^2 + 4c^3 - 10c^4 + 48x + 16cx - 48c^2x + 32c^3x + 5c^4x - 15c^5x + 16x^2$$

$$+ 56cx^2 - 48c^2x^2 - 40c^3x^2 + 15c^4x^2 + \frac{9c^5x^2}{2} - 48x^3 - 160cx^3 + 48c^2x^3$$

$$+ 64c^3x^3 - 5c^4x^3 - 6c^5x^3 - 16x^4 + 24cx^4 + 24c^2x^4 - 12c^3x^4 - 5c^4x^4 + \frac{3c^5x^4}{2}$$

and

$$\frac{\partial g}{\partial x} = 48c + 8c^2 - 16c^3 + 8c^4 + c^5 - \frac{5c^6}{2} + 32cx + 56c^2x - 32c^3x - 20c^4x + 6c^5x$$

$$+ \frac{3c^6x}{2} + 384x^2 - 144cx^2 - 240c^2x^2 + 48c^3x^2 + 48c^4x^2 - 3c^5x^2 - 3c^6x^2$$

$$- 64x^3 - 64cx^3 + 48c^2x^3 + 32c^3x^3 - 12c^4x^3 - 4c^5x^3 + c^6x^3.$$

Upon solving the expressions $\dfrac{\partial g}{\partial c} = 0$ and $\dfrac{\partial g}{\partial x} = 0$ (by using MATHEMATICA 12.0), we obtain no critical in $(0,2) \times (0,1)$.

Therefore, there is no critical point in $y = 1,\ (c, x) \in (0, 2) \times (0, 1)$.

D. Now, considering the interior of the parallelepiped formed by $(0, 2) \times (0, 1) \times (0, 1)$. Differentiating $\Phi(c, x, y)$ partially with respect y, we have

$$\begin{aligned}\frac{\partial \Phi}{\partial y} &= 8c^3 - 2c^5 + 48cx - 16c^3x + c^5x + 16cx^2 - 16c^3x^2 + 3c^5x^2 - 48cx^3 + 16c^3x^3 \\ &- c^5x^3 - 16cx^4 + 8c^3x^4 - c^5x^4 + 32y - 32c^2y \\ &+ 6c^4y - 64xy + 48c^2xy - 8c^4xy + 16c^2x^2y - 4c^4x^2y \\ &+ 64x^3y - 48c^2x^3y + 8c^4x^3y - 32x^4y + 16c^2x^4y - 2c^4x^4y.\end{aligned}$$

Upon solving $\dfrac{\partial \Phi}{\partial y} = 0$, we get

$$y_0 = \frac{4cx(3 + x) - c^3\left(-2 + x + x^2\right)}{2(4 + c^2(-3 + x) - 4x)(-1 + x)}$$

for $y_0 \in (0, 1)$ iff the conditions

$$4cx(3 + x) - c^3\left(-2 + x + x^2\right) - 2\left(4 + c^2(-3 + x) - 4x\right)(-1 + x) \leq 0 \tag{2.17}$$

and

$$2\left(4 + c^2(-3 + x) - 4x\right) < 0$$

must hold. But the condition (2.17) is not true for all $x \in (0, 1)$ and $c \in (0, 2)$.

Hence, $\Phi(c, x, y)$ has no critical point in the interior of parallelepiped.

In review of cases **A**, **B**, **C** and **D**, we obtain

$$\max\{\Phi(c, x, y) : c \in [0, 2],\ x \in [0, 1],\ y \in [0, 1]\} = 128. \tag{2.18}$$

From expression (2.15) and (2.18), we get

$$\left|H_{2,3}\left(f^{-1}\right)\right| \leq 2.$$

Form f_0, we obtain $t_3 = -1,\ t_4 = 0,\ t_5 = 2$.

Theorem 2.3 is proved.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Received 08.07.22