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FEYNMAN – KAC REPRESENTATION OF PARABOLIC ANDERSON EQUATIONS WITH GENERAL GAUSSIAN NOISE

ЗОБРАЖЕННЯ ФЕЙНМАНА – КАЦА ДЛЯ ПАРАБОЛІЧНИХ РІВНЯНЬ АНДЕРСОНА ІЗ ЗАГАЛЬНИМ ГАУССОВИМ ШУМОМ

We provide the Feynman – Kac representation for the parabolic Anderson equations driven by a general Gaussian noise. As a feature of the idea, we can mention the argument of subadditivity in establishing the required exponential integrability.

Наведено зображення Фейнмана – Каца для параболічних рівнянь Андерсона, керованих загальним гауссовим шумом. Особливістю ідеї є застосування аргументу субадитивності при встановленні необхідної експоненціальної інтегровності.

1. Introduction. Consider the parabolic Anderson equation

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \dot W(t,x)u(t,x), \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ &u(0,x)=u_0(x), \quad x\in\mathbb{R}^d, \end{aligned} \tag{1.1}$$

run by a mean zero and possibly generalized time-space Gaussian noise $\dot W(t,x),\ (t,x)\in\mathbb{R}\times\mathbb{R}^d$, with the covariance function

$$\mathrm{Cov}\Big(\dot W(t,x),\dot W(s,y)\Big) = |t-s|^{-\alpha_0}\gamma(x-y), \quad x,y\in\mathbb{R}^d, \tag{1.2}$$

where $0<\alpha_0<1$. Throughout, we assume that $\gamma(\cdot)\ge 0$. With maximal generality this paper allowed, $\gamma(\cdot)$ can be a generalized function that is defined as a linear functional on $\mathcal{S}(\mathbb{R}^d)$, the set of all rapidly decreasing functions known as Schwartz space. Since $\gamma(\cdot)$ is nonnegative definite as covariance function, by Bochner's theorem there is a unique measure on $\mathbb{R}^d$, known as the spectral measure of $\gamma(\cdot)$, such that

$$\gamma(x) = \int\limits_{\mathbb{R}^d} e^{i\xi\cdot x}\mu(d\xi). \tag{1.3}$$

Further, $\mu(d\xi)$ is tempered in the sense that

$$\int\limits_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)^p\mu(d\xi)<\infty$$

for some $p>0$. In particular, $\mu(d\xi)$ is locally finite.

The singularity of the system does not make (1.1) a rigorous definition. Mathematically, a random field $u(t,x),\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^d$, is called a weak solution of (1.1) if

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$$\int_{\mathbb{R}^d} u(t,x)\varphi(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\varphi(x)\,dx$$
$$+\frac{1}{2}\int_0^t\int_{\mathbb{R}^d} u(s,x)\Delta\varphi(x)\,dx\,ds + \int_0^t\int_{\mathbb{R}^d} u(s,x)\varphi(x)W(ds\,dx) \quad \text{a.s.} \tag{1.4}$$

for every C^∞-function φ with compact support, where the stochastic integral on the right-hand side is known as Stratonovich integral, which is defined as

$$\int_0^t\int_{\mathbb{R}^d} v(s,x)W(ds\,dx) \triangleq \lim_{\epsilon\to 0^+}\int_0^t\int_{\mathbb{R}^d} v(s,x)\dot{W}_\epsilon(t,x)\,dx\,ds \quad \text{in probability}$$

(whenever the limit exists) for all random fields $v(t,x),\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^d$, satisfying

$$\int_0^t\int_{\mathbb{R}^d} |v(s,x)|\,dx\,ds < \infty \quad \text{a.s.},$$

and where $\dot{W}_\epsilon(t,x)$ is a smoothed version of $\dot{W}(t,x)$ (see (2.1) below).

In the case when

$$\dot{W}(t,x) = \frac{\partial^{d+1}W^H(t,x)}{\partial t\partial x_1\ldots\partial x_d}, \quad \text{where} \quad x=(x_1,\ldots,x_d), \tag{1.5}$$

is the formal derivative of a fractional Brownian sheet $W^H(t,x)$ with Hurst parameter $(H_0,\ldots,H_d)$, where $H_0>1/2$, and $H_1,\ldots,H_d\ge 1/2$, it is proved in [7] that under the condition

$$2H_0+\sum_{j=1}^d H_j > d+1 \tag{1.6}$$

the random field

$$u(t,x) \triangleq \mathbb{E}_x \exp\left\{\int_0^t \dot{W}(t-s,B_s)\,ds\right\}u_0(B_t), \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d, \tag{1.7}$$

provides a weak solution to the parabolic Anderson equation (1.1). Here B_s is a d-dimensional Brownian motion starting at x and independent of $\dot{W}$, $\mathbb{E}_x$ is the expectation with respect to the Brownian motion, and the time-integral on the right-hand side is properly defined by the way of approximation (see (2.1) below).

Formula (1.7) is known as Feynman – Kac representation in literature and it appeared first in the setting of deterministic heat equation (see, e.g., Theorem 2.2 in [5, p. 132]) with $\dot{W}(t,x)$ being replaced by a deterministic function with sufficient regularity.

Representation (1.7) has been extended (see Section 6 in [4]) to a class of Gaussian noises with spatial covariance of the homogeneity

$$\gamma(cx) = c^{-\alpha}\gamma(x), \quad x\in\mathbb{R}^d, \quad c>0, \tag{1.8}$$

with $0<\alpha<2(1-\alpha_0)$.

This paper is to solve the parabolic Anderson equation by establishing representation (1.7) for the Gaussian noise with the general spatial covariance $\gamma(\cdot)$.

Theorem 1.1. *Assume that $u_0(x)$ be a bounded and measurable function on $\mathbb{R}^d$ and*

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{1-\alpha_0} \mu(d\xi) < \infty. \tag{1.9}$$

The random field $u(t,x)$ given in (1.7) *is well-defined and is a weak solution of the parabolic Anderson equation* (1.1). *Further, $u(t,x) \in \mathcal{L}^m(\Omega, \mathcal{A}, \mathbb{P})$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$ with the representation*

$$\mathbb{E}u^m(t,x) = \mathbb{E}_x \exp\left\{ \sum_{j,k=1}^m \int_0^t \int_0^t \frac{\gamma(B_j(s) - B_k(r))}{|s-r|^{\alpha_0}} \, ds\, dr \right\} \prod_{j=1}^m u_0(B_j(t)), \tag{1.10}$$

where $B_1(t), \ldots, B_m(t)$ are independent d-dimensional Brownian motions with $B_j(0) = x$, $\mathbb{E}_x$ is the expectation with respect to the Brownian motions, and the time-Hamiltonians on the right-hand side are defined by an appropriate approximation (*see* (2.6) *and* (2.7) *below*).

For the purpose of comparison, let us mention a different regime in which the parabolic Anderson equation (1.1) is defined by

$$u(t,x) = (p_t * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) u(s,y) W(ds\, dy), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

where $p_t(x)$ is the Brownian semigroup defined as

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left\{ -\frac{1}{2t}|x|^2 \right\}, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

and the stochastic integral on the right-hand side is understood as Skorokhod integral. In the Skorokhod regime, it has been proved (Theorem 3.6 of [6]) that equation (1.1) has a solution under the Dalang condition

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \, \mu(d\xi) < \infty. \tag{1.11}$$

Contrary to (1.11), assumption (1.9) shows that the singularity from time-component (quantified by α_0) of the Gaussian noise $\dot{W}(t,x)$ contributes to the system singularity in the setting of weak solution.

Assumption (1.9) is necessary when $u_0(x) = 1$: by (2.5) and (2.8) below,

$$\begin{aligned} \mathbb{E} \otimes \mathbb{E}_x \left[\int_0^t \dot{W}(t-s, B_s)\, ds \right]^2 &= \mathbb{E}_0 \int_0^t \int_0^t \frac{\gamma(B(s) - B(r))}{|s-r|^{\alpha_0}} \, ds\, dr \\ &= \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \int_0^t |s-r|^{-\alpha_0} \exp\left\{ -\frac{|\xi|^2}{2} |s-r| \right\}. \end{aligned}$$

One can check (see the computation next to (2.5) below) that condition (1.9) is equivalent to

$$\mathbb{E}\otimes\mathbb{E}_x\left[\int_0^t \dot W(t-s,B_s)\,ds\right]^2<\infty \quad \text{for some} \quad t>0 \quad \text{or, equivalently, for every} \quad t>0.$$

Hence, condition (1.9) is necessary for a meaningful and integrable expression given in (1.7).

With homogeneity (1.8) and by Lemma 3.10 in [3]

$$\int_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)^{1-\alpha_0}\mu(d\xi)=\alpha\mu(B(0,1))\int_0^\infty\left(\frac{1}{1+\rho^2}\right)^{1-\alpha_0}\rho^{\alpha-1}\,d\rho.$$

Since $\mu(d\xi)$ is tempered, $\mu(B(0,1))<\infty$. Therefore, (1.9) holds if and only if $\alpha<2(1-\alpha_0)$.

***Corollary* 1.1.** *In assumption* (1.8) *with* $0<\alpha<2(1-\alpha_0)$, *all statements in Theorem* 1.1 *hold.*

As for the special case when $\dot W(t,x)$ is the fractional Gaussian noise given in (1.5), homogeneity (1.8) is satisfied with

$$\alpha_0=2-2H_0 \qquad \text{and} \qquad \alpha=2d-2\sum_{j=1}^d H_j.$$

Consequently, (1.6) is equivalent to $0<\alpha<2(1-\alpha_0)$.

The proof of Theorem 1.1 is given in the next section. It is worth of mentioning a striking fact that the exponential integrability (given in (2.10) below) of the Brownian Hamiltonian

$$\int_0^t\int_0^t\frac{\gamma(B_s-B_r)}{|s-r|^{\alpha_0}}\,ds\,dr$$

is determined by its local behavior near $t=0$ (Lemma 2.1), and the efficiency of subadditivity approach in proving this fact.

2. Proof of Theorem 1.1. The time-integral in representation (1.7) is defined as

$$\int_0^t \dot W(t-s,B_s)\,ds\triangleq\lim_{\epsilon\to0^+}\int_0^t \dot W_\epsilon(t-s,B_s)\,ds \quad \text{in} \quad \mathcal{L}^2(\Omega,\mathcal{A},\mathbb{P}_x\otimes\mathbb{P}), \tag{2.1}$$

where $\dot W_\epsilon$ is the point-wisely defined Gaussian field $\dot W_\epsilon(t,x)$ is given as

$$\dot W_\epsilon(t,x)\triangleq\int_{\mathbb{R}^{d+1}}\dot W(u,y)\left[(2\pi\epsilon)^{-\frac{d+1}{2}}\exp\left\{-\frac{(t-u)^2+|x-y|^2}{2\epsilon}\right\}\right]du\,dy, \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d.$$

To make it work, we need to show that the limit on the right-hand side exists. To this end all we need is to show that the limit

$$\lim_{\epsilon,\epsilon'\to0^+}\mathbb{E}_x\otimes\mathbb{E}\left(\int_0^t\dot W_\epsilon(t-s,B_s)\,ds\right)\left(\int_0^t\dot W_{\epsilon'}(t-s,B_s)\,ds\right)$$

exists.

Notice that

$$\mathrm{Cov}(W_\epsilon(s,x),W_\epsilon(r,y))=\gamma_{0,\epsilon+\epsilon'}(s-r)\gamma_{\epsilon+\epsilon'}(x-y),$$

where

$$\gamma_{0,\epsilon}(u)=\int\limits_{\mathbb{R}}\frac{1}{|v|^{\alpha_0}}\left[\frac{1}{\sqrt{2\pi\epsilon}}\exp\left\{-\frac{(u-v)^2}{2\epsilon}\right\}\right]dv, \tag{2.2}$$

$$\gamma_\epsilon(x)=\int\limits_{\mathbb{R}^d}\gamma(y)\left[\frac{1}{(2\pi\epsilon)^{d/2}}\exp\left\{-\frac{|x-y|^2}{2\epsilon}\right\}\right]dy. \tag{2.3}$$

We have

$$\begin{aligned}\mathbb{E}_x\otimes\mathbb{E}&\left(\int\limits_0^t\dot W_\epsilon(t-s,B_s)\,ds\right)\left(\int\limits_0^t\dot W_{\epsilon'}(t-s,B_s)\,ds\right)\\ &=\mathbb{E}_0\int\limits_0^t\int\limits_0^t\mathbb{E}\dot W_\epsilon(t-s,B_s)\dot W_{\epsilon'}(t-r,B_r)\,ds\,dr\\ &=\mathbb{E}_0\int\limits_0^t\int\limits_0^t\gamma_{0,\epsilon+\epsilon'}(s-r)\gamma_{\epsilon+\epsilon'}(B_s-B_r)\,ds\,dr.\end{aligned}$$

Notice also that, for any $\delta>0$, $\gamma_\delta(\cdot)$ has the spectral measure $e^{-\delta|\xi|^2/2}\mu(d\xi)$. Let $\mu_0(d\lambda)$ be the spectral measure of $|\cdot|^{-\alpha_0}$ (one can easily check that $\mu_0(d\lambda)$ is a constant multiple of $|\lambda|^{-(1-\alpha_0)}d\lambda$). Then $\gamma_{0,\delta}(\cdot)$ has the spectral measure $e^{-\delta\lambda^2/2}\mu_0(d\lambda)$. By Fourier transform,

$$\begin{aligned}\int\limits_0^t\int\limits_0^t\gamma_{0,\epsilon+\epsilon'}(s-r)\gamma_{\epsilon+\epsilon'}(B_s-B_r)\,ds\,dr=\int\limits_{\mathbb{R}^{d+1}}&\exp\left\{-\frac{\epsilon+\epsilon'}{2}(\lambda^2+|\xi|^2)\right\}\\ &\times\left|\int\limits_0^t\exp\{i\lambda s+i\xi\cdot B_s\}\,ds\right|^2\mu_0(d\lambda)\mu(d\xi).\end{aligned}$$

Therefore, by dominated convergence theorem,

$$\begin{aligned}\lim_{\epsilon,\epsilon'\to0^+}\mathbb{E}_x\otimes\mathbb{E}&\left(\int\limits_0^t\dot W_\epsilon(t-s,B_s)\,ds\right)\left(\int\limits_0^t\dot W_{\epsilon'}(t-s,B_s)ds\right)\\ &=\int\limits_{\mathbb{R}^{d+1}}\mathbb{E}_0\left|\int\limits_0^t\exp\{i\lambda s+i\xi\cdot B_s\}\,ds\right|^2\mu_0(d\lambda)\mu(d\xi)\end{aligned}$$

provided that

$$\int_{\mathbb{R}^{d+1}} \mathbb{E}_0 \left| \int_0^t \exp\{i\lambda s + i\xi \cdot B_s\}\, ds \right|^2 \mu_0(d\lambda)\mu(d\xi) < \infty \quad \forall t > 0. \tag{2.4}$$

Here we have used the fact that the integral in (2.4) is independent of the starting point x of the Brownian motion (so we take $x = 0$). Indeed,

$$\int_{\mathbb{R}^{d+1}} \mathbb{E}_0 \left| \int_0^t \exp\{i\lambda s + i\xi \cdot B_s\}\, ds \right|^2 \mu_0(d\lambda)\mu(d\xi)$$

$$= \int_{\mathbb{R}^d} \mu(d\xi)\mathbb{E}_0 \int_0^t \int_0^t |s-r|^{-\alpha_0} \exp\{i\xi \cdot (B_s - B_r)\}\, ds\, dr$$

$$= \int_{\mathbb{R}^d} \mu(d\xi) \int_0^t \int_0^t |s-r|^{-\alpha_0} \exp\left\{-\frac{|\xi|^2}{2}\,|s-r|\right\} ds\, dr. \tag{2.5}$$

Notice that the right-hand side is monotonic in t. To establish (2.4), all we need is to prove that

$$\int_{\mathbb{R}^d} \mu(d\xi) \int_0^\infty dt e^{-t} \int_0^t \int_0^t |s-r|^{-\alpha_0} \exp\left\{-\frac{|\xi|^2}{2}\,|s-r|\right\} ds\, dr < \infty.$$

Indeed,

$$\int_{\mathbb{R}^d} \mu(d\xi) \int_0^\infty dt e^{-t} \int_0^t \int_0^t |s-r|^{-\alpha_0} \exp\left\{-\frac{|\xi|^2}{2}|s-r|\right\} ds\, dr$$

$$= 2\int_{\mathbb{R}^d} \mu(d\xi) \int_0^\infty dt e^{-t} \int_0^t \int_r^t (s-r)^{-\alpha_0} \exp\left\{-\frac{|\xi|^2}{2}\,(s-r)\right\} ds\, dr$$

$$= 2\int_{\mathbb{R}^d} \mu(d\xi) \int_0^\infty t^{-\alpha_0} \exp\left\{-\frac{|\xi|^2}{2}\,t\right\} e^{-t}\, dt$$

$$= 2\left(\int_0^\infty t^{-\alpha_0} e^{-t}\, dt\right) \int_{\mathbb{R}^d} \left(\frac{1}{1 + 2^{-1}|\xi|^2}\right)^{1-\alpha_0} \mu(d\xi),$$

where the last step follows from the integration substitution

$$t \mapsto \left(1 + 2^{-1}|\xi|^2\right)^{-1} t.$$

In summary, by condition (1.9) we have proved (2.4) and, therefore, justified the definition in (2.1).

Next, we clarify the time-Hamiltonians in (1.10) by making the definition

$$\int_0^t\int_0^t \frac{\gamma(B_s-B_r)}{|s-r|^{\alpha_0}}\,ds\,dr \triangleq \lim_{\epsilon\to 0^+}\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-B_r)\,ds\,dr \quad \text{in} \quad \mathcal{L}(\Omega,\mathcal{A},\mathbb{P}_x), \tag{2.6}$$

$$\int_0^t\int_0^t \frac{\gamma\Big(B_s-\widetilde{B}_r\Big)}{|s-r|^{\alpha_0}}\,ds\,dr \triangleq \lim_{\epsilon\to 0^+}\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon\Big(B_s-\widetilde{B}_r\Big)\,ds\,dr \quad \text{in} \quad \mathcal{L}(\Omega,\mathcal{A},\mathbb{P}_x) \tag{2.7}$$

for two independent Brownian motions B_t and $\widetilde{B}_t$, where $\gamma_{0,\epsilon}(\cdot)$ and $\gamma_\epsilon(\cdot)$ are given in (2.2) and (2.3), respectively.

Once again, notice that the problem is independent of the starting point of the Brownian motions, that

$$\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-B_r)\,ds\,dr = \int_{\mathbb{R}^{d+1}}\left|\int_0^t e^{i\lambda s+i\xi\cdot B_s}\,ds\right|^2 \exp\Big\{-\frac{\epsilon}{2}\left(\lambda^2+|\xi|^2\right)\Big\}\mu_0(d\lambda)\mu(d\xi)$$

and

$$\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon\Big(B_s-\widetilde{B}_r\Big)\,ds\,dr$$

$$= \int_{\mathbb{R}^{d+1}}\left[\int_0^t e^{i\lambda s+i\xi\cdot B_s}\,ds\right]\left[\int_0^t e^{-i\lambda s-i\xi\cdot \widetilde{B}_s}\,ds\right]\exp\Big\{-\frac{\epsilon}{2}\left(\lambda^2+|\xi|^2\right)\Big\}\mu_0(d\lambda)\mu(d\xi).$$

So, we have that

$$\mathbb{E}_0\left|\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-B_r)\,ds\,dr - \int_0^t\int_0^t \gamma_{0,\epsilon'}(s-r)\gamma_{\epsilon'}(B_s-B_r)\,ds\,dr\right|$$

$$\le \int_{\mathbb{R}^{d+1}}\left|\exp\Big\{-\frac{\epsilon}{2}\left(\lambda^2+|\xi|^2\right)\Big\}-\exp\Big\{-\frac{\epsilon'}{2}\left(\lambda^2+|\xi|^2\right)\Big\}\right|$$

$$\times\,\mathbb{E}_0\left|\int_0^t e^{i\lambda s+i\xi\cdot B_s}\,ds\right|^2\mu_0(d\lambda)\mu(d\xi)$$

and

$$\mathbb{E}_0\left|\int_0^t\int_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon\Big(B_s-\widetilde{B}_r\Big)\,ds\,dr - \int_0^t\int_0^t \gamma_{0,\epsilon'}(s-r)\gamma_{\epsilon'}\Big(B_s-\widetilde{B}_r\Big)\,ds\,dr\right|$$

$$\le \int_{\mathbb{R}^{d+1}}\left|\exp\Big\{-\frac{\epsilon}{2}\left(\lambda^2+|\xi|^2\right)\Big\}-\exp\Big\{-\frac{\epsilon'}{2}\left(\lambda^2+|\xi|^2\right)\Big\}\right|$$

$$\times \left\{ \mathbb{E}_0 \left| \int_0^t e^{i\lambda s + i\xi \cdot B_s}\, ds \right| \right\}^2 \mu_0(d\lambda)\mu(d\xi).$$

By (2.4) and dominated convergence, the right-hand sides tend to 0 as $\epsilon, \epsilon' \to 0^+$. That is the justification for (2.6) and (2.7). Further, from above argument we get

$$\int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s - r|^{\alpha_0}}\, ds\, dr = \int_{\mathbb{R}^{d+1}} \left| \int_0^t e^{i\lambda s + i\xi \cdot B_s}\, ds \right|^2 \mu_0(d\lambda)\mu(d\xi). \tag{2.8}$$

We now show that the random field $u(t, x)$ in (1.7) is well-defined by proving that

$$\mathbb{E}|u(t, x)| < \infty \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d. \tag{2.9}$$

By assumption, $|u_0(\cdot)| \le C$ for a constant $C > 0$. So, we obtain

$$\mathbb{E}|u(t, x)| \le C\mathbb{E} \otimes \mathbb{E}_x \exp\left\{ \int_0^t \dot{W}(t - s, B_s)\, ds \right\} = C\mathbb{E} \otimes \mathbb{E}_0 \exp\left\{ \int_0^t \dot{W}(t - s, B_s)\, ds \right\}.$$

From (2.1) and (2.6) we can see that conditioning on the Brownian motion, the random variable

$$\int_0^t \dot{W}(t - s, B_s)\, ds$$

is a mean zero normal with the variance

$$\int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s - r|^{\alpha_0}}\, ds\, dr.$$

So, we have

$$\mathbb{E} \exp\left\{ \int_0^t \dot{W}(t - s, B_s)\, ds \right\} = \exp\left\{ \frac{1}{2} \int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s - r|^{\alpha_0}}\, ds\, dr \right\} \quad \text{a.s.}$$

To establish the integrability requested for the definition in (1.7), therefore, all we need is the exponential integrability

$$\mathbb{E}_0 \exp\left\{ \theta \int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s - r|^{\alpha_0}}\, ds\, dr \right\} < \infty \quad \forall \theta, t > 0. \tag{2.10}$$

To this end we first establish the following lemma.

Lemma 2.1. *Under condition* (1.9),

$$\lim_{t \to 0^+} \frac{1}{t}\, \mathbb{E}_0 \int_{\mathbb{R}^{d+1}} \left| \int_0^t e^{i\lambda s + i\xi \cdot B_s}\, ds \right|^2 \mu_0(d\lambda)\mu(d\xi) = 0. \tag{2.11}$$

Proof. From (2.5) and variable substitution

$$\mathbb{E}_0 \int\limits_{\mathbb{R}^{d+1}} \left| \int\limits_0^t e^{i\lambda s + i\xi\cdot B_s}\, ds \right|^2 \mu_0(d\lambda)\mu(d\xi)$$

$$= \int\limits_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int\limits_0^{|\xi|^2 t}\int\limits_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} \exp\left\{-\frac{1}{2}|s-r|\right\} ds\, dr$$

$$= \int\limits_{\{|\xi|\le t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int\limits_0^{|\xi|^2 t}\int\limits_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} \exp\left\{-\frac{1}{2}\,|s-r|\right\} ds\, dr$$

$$+ \int\limits_{\{|\xi|> t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int\limits_0^{|\xi|^2 t}\int\limits_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} \exp\left\{-\frac{1}{2}\,|s-r|\right\} ds\, dr.$$

For the first term

$$\int\limits_{\{|\xi|\le t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int\limits_0^{|\xi|^2 t}\int\limits_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} \exp\left\{-\frac{1}{2}\,|s-r|\right\} ds\, dr$$

$$\le \int\limits_{\{|\xi|\le t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int\limits_0^{|\xi|^2 t}\int\limits_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}}\, ds\, dr$$

$$= \frac{2t^{2-\alpha_0}}{(1-\alpha_0)(2-\alpha_0)}\, \mu\Big(B\Big(0, t^{-1/2}\Big)\Big).$$

According to Kronecker lemma, (1.9) implies that

$$\lim_{t\to 0^+} t^{1-\alpha_0}\mu\Big(B\Big(0, t^{-1/2}\Big)\Big) = 0.$$

As for the second term in our decomposition, we use the simple bound

$$\int\limits_{\{|\xi|> t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int\limits_0^{|\xi|^2 t}\int\limits_0^{|\xi|^2 t} \frac{1}{|s-r|^{\alpha_0}} \exp\left\{-\frac{1}{2}\,|s-r|\right\} ds\, dr$$

$$\le 2 \int\limits_{\{|\xi|> t^{-1/2}\}} \frac{\mu(d\xi)}{|\xi|^{4-2\alpha_0}} \int\limits_0^{|\xi|^2 t}\int\limits_r^{\infty} \frac{1}{(s-r)^{\alpha_0}} \exp\left\{-\frac{1}{2}\,(s-r)\right\} ds\, dr$$

$$= 2\left(\int_0^\infty \frac{1}{s^{\alpha_0}} \exp\left\{-\frac{1}{2}\, s\right\} ds\right) t \int_{\left\{|\xi|>t^{-1/2}\right\}} \frac{\mu(d\xi)}{|\xi|^{2(1-\alpha_0)}}$$

and the obvious fact derived from (1.9):

$$\lim_{t\to 0^+} \int_{\left\{|\xi|>t^{-1/2}\right\}} \frac{\mu(d\xi)}{|\xi|^{2(1-\alpha_0)}} = 0.$$

The lemma is proved.

To establish (2.10), we use the argument by subadditivity. A stochastic process $Z_t,\ t \ge 0$, is said to be subadditive, if, for any $t_1,\ t_2 > 0$, there exists a random variable Z'_{t_2} such that $Z'_{t_2} \overset{d}{=} Z_{t_1}$, Z'_{t_2} is independent of $\{Z_s,\ s \le t_1\}$ and $Z_{t_1+t_2} \le Z_{t_1} + Z'_{t_2}$ a.s. An interested reader is referred to Section 1.3 in [1] for the discussion on this topic. Specifically, a nonnegative, nondecreasing and sample-path continuous subadditive process Z_t with $Z_0 = 0$ has the property ([1, p. 21], (1.3.7)) that

$$\mathbb{P}\{Z_t \ge a+b\} \le \mathbb{P}\{Z_t \ge a\}\mathbb{P}\{Z_t \ge b\} \quad \forall a,b,t>0. \tag{2.12}$$

We now exam the subadditivity for the process

$$Z_t \triangleq \left(\int_0^t\int_0^t \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}}\, ds\, dr\right)^{1/2}, \quad t \ge 0.$$

Indeed, by (2.8) and triangle inequality the subadditivity $Z_{t_1+t_2} \le Z_{t_1} + Z'_{t_2}$ holds with

$$Z'_{t_2} = \left(\int_{\mathbb{R}^{d+1}} \left|\int_{t_1}^{t_1+t_2} e^{i\lambda s + i\xi\cdot B_s}\, ds\right|^2 \mu_0(d\lambda)\mu(d\xi)\right)^{1/2}$$

$$= \left(\int_{\mathbb{R}^{d+1}} \left|\int_0^{t_2} e^{i\lambda s + i\xi\cdot(B_{t_1+s} - B_{t_1})}\, ds\right|^2 \mu_0(d\lambda)\mu(d\xi)\right)^{1/2}.$$

Clearly, $Z_0 = 0,\ Z_t$ is nondecreasing. By (2.8) Z_t is sample-path continuous (more precisely, the relation (2.8) provides a sample-path continuous modification of Z_t). So, Z_t satisfies (2.12).

For any $\theta > 0$, using (2.12) repeatably, we get

$$\mathbb{P}_0\left\{Z_t \ge m\theta^{-1}\sqrt{t}\right\} \le \left(\mathbb{P}_0\left\{Z_t \ge \theta^{-1}\sqrt{t}\right\}\right)^m, \quad m = 1,2,\ldots.$$

By Lemma 2.1, (2.8) and Chebyshev's inequality, there exists a possibly small $t_0 > 0$ such that

$$\sup_{t\le t_0} \mathbb{P}_0\left\{Z_t \ge \theta^{-1}\sqrt{t}\right\} \le e^{-2}.$$

Hence,

$$\mathbb{E}_0 \exp\Big\{\theta Z_t/\sqrt{t}\Big\} = 1 + \int\limits_0^\infty e^b\, \mathbb{P}_0\Big\{Z_t \ge b\theta^{-1}\sqrt{t}\Big\}\, db$$

$$\le 1 + e + \sum_{m=1}^\infty e^{m+1}\, \mathbb{P}_0\Big\{Z_t \ge m\theta^{-1}\sqrt{t}\Big\}$$

$$\le 1 + e + \sum_{m=0}^\infty e^{m+1}\, e^{-2m} = \frac{2e^2-1}{e-1} < \infty$$

for all $0 < t \le t_0$. Unfortunately, (2.13) is not even close to what is requested by (2.10). To improve it, first notice that the above estimation leads to the uniform bound

$$\mathbb{E}_0 Z_t^n \le \frac{2e^2-1}{e-1}\,\theta^{-n} n! t^{n/2}, \quad 0 < t < t_0, \quad n = 1, 2, \ldots. \tag{2.13}$$

By subadditivity, for any $t_1,\ t_2 > 0$ and integer $n \ge 1$,

$$\mathbb{E}_0 Z_{t_1+t_2}^n \le \mathbb{E}\big[Z_{t_1} + Z'_{t_2}\big]^n = \sum_{l=0}^n \binom{n}{l} \Big\{\mathbb{E} Z_{t_1}^l\Big\} \Big\{\mathbb{E} Z_{t_2}^{n-l}\Big\}.$$

For any $t > 0$ and integer $m \ge 1$, repeating the above inequality, we have

$$\mathbb{E} Z_t^n \le \sum_{l_1+\ldots+l_m=n} \frac{n!}{l_1!\ldots l_m!} \prod_{k=1}^m \mathbb{E} Z_{t/m}^{l_k} = \sum_{l_1+\ldots+l_m=n} \frac{n!}{l_1!\ldots l_m!} \prod_{k=1}^m \mathbb{E} Z_{t/m}^{l_k}.$$

Taking $m = n$ and $t \le t_0$, by (2.13) we get

$$\mathbb{E} Z_t^n \le \sum_{l_1+\ldots+l_n=n} \frac{n!}{l_1!\ldots l_n!} \prod_{k=1}^n \frac{2e^2-1}{e-1}\,\theta^{-l_j} l_j! \Big(\frac{t}{n}\Big)^{l_j/2}$$

$$= \bigg(\frac{\theta^{-1}\big(2e^2-1\big)}{e-1}\bigg)^n n! n^{-n/2} t^{n/2} \sum_{l_1+\ldots+l_n=n} 1.$$

A simple combinatorial argument gives

$$\sum_{l_1+\ldots+l_n=n} 1 = \binom{2n-1}{n} \le 4^n.$$

Thus, we obtain the following improved version of (2.13):

$$\mathbb{E}_0 Z_t^n \le \bigg(\frac{4\theta^{-1}\big(2e^2-1\big)}{e-1}\bigg)^n \sqrt{n!} t^{n/2}, \quad 0 < t \le t_0, \quad n = 1, 2, \ldots.$$

Replacing n by $2n$, we have

$$\mathbb{E}_0 Z_t^{2n} \le \bigg(\frac{4\theta^{-1}\big(2e^2-1\big)}{e-1}\bigg)^{2n} \sqrt{(2n)!} t^n \le \bigg(\frac{4\sqrt{2}\theta^{-1}\big(2e^2-1\big)}{e-1}\bigg)^{2n} n! t^n$$

for any $0 < t \le t_0$ and $n = 1, 2, \ldots$. Consequently, by Taylor expansion

$$\sup_{0<t\le t_0} \mathbb{E}_0 \exp\left\{ \left(\frac{(e-1)\theta}{8(2e^2-1)} \right)^2 \frac{Z_t^2}{t} \right\} < \infty. \tag{2.14}$$

In addition, one can check that the process

$$S_t \triangleq \frac{Z_t^2}{t} = \frac{1}{t} \int_0^t \int_0^t \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}} \, ds \, dr, \quad t > 0,$$

is subadditive. Indeed, by (2.8) and Jensen's inequality one can establish the subadditivity $S_{t_1+t_2} \le S_{t_1} + S'_{t_2}$, where

$$S'_{t_2} = \frac{1}{t_2} \int_{\mathbb{R}^{d+1}} \left| \int_{t_1}^{t_1+t_2} e^{i\lambda s + i\xi\cdot B_s} \, ds \right|^2 \mu_0(d\lambda)\mu(d\xi)$$

$$= \frac{1}{t_2} \int_{\mathbb{R}^{d+1}} \left| \int_0^{t_2} e^{i\lambda s + i\xi\cdot(B_{t_1+s} - B_{t_1})} \, ds \right|^2 \mu_0(d\lambda)\mu(d\xi)$$

satisfies all requests for subadditivity[2]. Therefore,

$$\mathbb{E}_0 \exp\left\{ \left(\frac{(e-1)\theta}{8(2e^2-1)} \right)^2 S_{t_1+t_2} \right\} \le \mathbb{E}_0 \exp\left\{ \left(\frac{(e-1)\theta}{8(2e^2-1)} \right)^2 S_{t_1} \right\} \mathbb{E}_0 \exp\left\{ \left(\frac{(e-1)\theta}{8(2e^2-1)} \right)^2 S_{t_2} \right\}$$

for any $0 < t_1,\ t_2 < t_0$. By (2.14), the right-hand side is finite. Therefore, (2.14) can be extended to all $t > 0$:

$$\mathbb{E} \exp\left\{ \left(\frac{(e-1)\theta}{8(2e^2-1)} \right)^2 S_t \right\} < \infty \quad \forall t > 0.$$

In particular, take $t = 1$ and notice that $\theta > 0$ is arbitrary. We have reached the conclusion

$$\mathbb{E}_0 \exp\left\{ \theta \int_0^1 \int_0^1 \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}} \, ds \, dr \right\} < \infty \quad \forall \theta > 0.$$

This can be further extended to (2.10) since (by (2.8)), for any $t_1,\ t_2 > 0$,

$$\int_0^{t_1+t_2} \int_0^{t_1+t_2} \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}} \, ds \, dr \le 2 \int_0^{t_1} \int_0^{t_1} \frac{\gamma(B_s - B_r)}{|s-r|^{\alpha_0}} \, ds \, dr + 2 \int_0^{t_2} \int_0^{t_2} \frac{\gamma\left(\widetilde{B}_s - \widetilde{B}_r\right)}{|s-r|^{\alpha_0}} \, ds \, dr$$

with $\widetilde{B}(s) = B_{t_1+s} - B_{t_1}$ being a Brownian motion independent of $\{B_s,\ s \le t_1\}$.

More than (2.9), we now show that, for any integer $m \ge 1$, $u(t,x) \in \mathcal{L}^m(\Omega, \mathcal{A}, \mathbb{P})$ with representation (2.10). Indeed, conditioning on the Brownian motions, the random variable

[2] We do not have (2.12) this time for lack of monotonicity and for S_t not being defined at $t = 0$.

$$\sum_{j=1}^{m}\int_0^t \dot W(t-s,B_j(s))\,ds$$

is a mean-zero normal random variable with the variance

$$\sum_{j,k=1}^{m}\mathbb{E}\left[\int_0^t \dot W(t-s,B_j(s))\,ds\right]\left[\int_0^t \dot W(t-s,B_k(s))\,ds\right].$$

On the other hand, for any $\epsilon>0$,

$$\mathbb{E}\left[\int_0^t \dot W_\epsilon(t-s,B_j(s))\,ds\right]\left[\int_0^t \dot W_\epsilon(t-s,B_k(s))ds\right]$$

$$=\int_0^t\int_0^t \gamma_{0,2\epsilon}(s-r)\gamma_{2\epsilon}(B_j(s)-B_k(r))\,ds\,dr.$$

Therefore, by (2.1), (2.6) and (2.7),

$$\mathbb{E}\left[\int_0^t \dot W(t-s,B_j(s))\,ds\right]\left[\int_0^t \dot W\big(t-s,B_k(s)\big)ds\right]=\int_0^t\int_0^t \frac{\gamma(B_j(s)-B_k(r))}{|s-r|^{\alpha_0}}\,ds\,dr. \tag{2.15}$$

In summary,

$$\mathbb{E}\exp\left\{\sum_{j=1}^{m}\int_0^t \dot W(t-s,B_j(s))\,ds\right\}=\exp\left\{\frac{1}{2}\sum_{j,k=1}^{m}\int_0^t\int_0^t \frac{\gamma(B_j(s)-B_k(r))}{|s-r|^{\alpha_0}}\,ds\,dr\right\}.$$

On the other hand, from (1.9),

$$u^m(t,x)=\mathbb{E}_x\exp\left\{\sum_{j=1}^{m}\int_0^t \dot W(t-s,B_j(s))ds\right\}\prod_{j=1}^{m}u_0(B_j(t)).$$

By Fubini theorem,

$$\mathbb{E}u^m(t,x)=\mathbb{E}_x\left(\mathbb{E}\exp\left\{\sum_{j=1}^{m}\int_0^t \dot W(t-s,B_j(s))\,ds\right\}\right)\prod_{j=1}^{m}u_0(B_j(t))$$

$$=\mathbb{E}_x\exp\left\{\frac{1}{2}\sum_{j,k=1}^{m}\int_0^t\int_0^t \frac{\gamma(B_j(s)-B_k(r))}{|s-r|^{\alpha_0}}\,ds\,dr\right\}\prod_{j=1}^{m}u_0(B_j(t)).$$

This is (1.10). The integrability issue arising from its right-hand side is resolved by the boundedness of $u_0(\cdot)$, the relation (from (2.15)) that

$$\int_0^t \int_0^t \frac{\gamma(B_j(s)-B_k(r))}{|s-r|^{\alpha_0}}\,ds\,dr \le \frac{1}{2}\int_0^t \int_0^t \frac{\gamma(B_j(s)-B_j(r))}{|s-r|^{\alpha_0}}\,ds\,dr + \frac{1}{2}\int_0^t \int_0^t \frac{\gamma(B_k(s)-B_k(r))}{|s-r|^{\alpha_0}}\,ds\,dr, \quad j \neq k,$$

and (2.10).

We finally come to the step of showing that the random field $u(t,x)$ in (1.7) is a weak solution of the parabolic Anderson equation (1.1). This is done by Hu, Nualart and Song (Theorem 4.3 in [7]) in the setting of fractional noise. In their proof, system (1.1) is approximated by its smoothed version

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \dot W_\epsilon(t,x)u(t,x), \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d, \\ &u(0,x) = u_0(x), \quad x\in\mathbb{R}^d, \end{aligned} \tag{2.16}$$

where $\epsilon > 0$ is small but fixed (at least for a while) and $\dot W_\epsilon(t,x)$ is given in (2.1).

To follow Hu – Nualart – Song's footstep, we set

$$u_\epsilon(t,x) = \mathbb{E}_x \exp\left\{\int_0^t \dot W_\epsilon(t-s,B_s)\,ds\right\} u_0(B_t), \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d.$$

The smoothed Gaussian field $\dot W_\epsilon(t,x)$ has a continuous but unbounded path. Pointed out by the referee, the unboundedness of $\dot W_\epsilon(t,x)$ makes the legitimacy of $u_\epsilon(t,x)$ as solution of (2.16) a questionable issue. On the other hand, the argument used by Hu, Nualart and Song (Proof of Theorem 4.3 in [7]) requires $u_\epsilon(t,x)$ to be a weak solution of (2.16). That is,

$$\int_{\mathbb{R}^d} u_\epsilon(t,x)\varphi(x)\,dx = \int_{\mathbb{R}^d} u_0(x)\varphi(x)\,dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} u_\epsilon(s,x)\Delta\varphi(x)\,dx\,ds + \int_0^t \int_{\mathbb{R}^d} u_\epsilon(s,x)\varphi(x)\dot W_\epsilon(s,y)\,dy\,ds \quad \text{a.s.} \tag{2.17}$$

for every C^∞-function φ with compact support. By Lemma 3.1 below (conditionally on $\dot W$), (2.17) holds if

$$\begin{aligned} &\int_D \mathbb{E}_x \exp\left\{\int_0^t \dot W_\epsilon(t-s,B_s)\,ds\right\} dx < \infty \quad \text{a.s.}, \\ &\int_0^t \int_D \mathbb{E}_x \exp\left\{\int_0^s \dot W_\epsilon(s-r,B_r)\,dr\right\} dx\,ds < \infty \quad \text{a.s.} \end{aligned} \tag{2.18}$$

for any bounded $D\subset\mathbb{R}^d$ and $t > 0$.

The first inequality in (2.18) follows from the fact that

$$\mathbb{E}\int\limits_D \mathbb{E}_x \exp\left\{\int\limits_0^t \dot{W}_\epsilon(t-s,B_s)\,ds\right\}dx$$

$$=\int\limits_D \mathbb{E}_0 \exp\left\{\frac{1}{2}\int\limits_0^t\int\limits_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-B_r)\,ds\,dr\right\}dx$$

$$=|D|\mathbb{E}_0 \exp\left\{\frac{1}{2}\int\limits_0^t\int\limits_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-B_r)\,ds\,dr\right\}<\infty.$$

A further computation leads to

$$\mathbb{E}\int\limits_0^t\int\limits_D \mathbb{E}_x \exp\left\{\int\limits_0^s \dot{W}_\epsilon(s-r,B_r)\,dr\right\}dx\,ds$$

$$=|D|\int\limits_0^t \mathbb{E}_0 \exp\left\{\frac{1}{2}\int\limits_0^s\int\limits_0^s \gamma_{0,\epsilon}(r_1-r_2)\gamma_\epsilon(B_{r_1}-B_{r_2})\,dr_1\,dr_2\right\}ds$$

$$\le |D|t\mathbb{E}_0 \exp\left\{\frac{1}{2}\int\limits_0^t\int\limits_0^t \gamma_{0,\epsilon}(s-r)\gamma_\epsilon(B_s-B_r)\,ds\,dr\right\}<\infty,$$

where the second step follows from the time-monotonicity of the integrand. We have proved the second inequality in (2.18).

Based on the exponential integrability (2.10) and its consequence on the moment integrability of $u(t,x)$ given in (1.7), on equation (2.17), and on the square integrability stated in Lemma 2.2 below, an argument by approximation via Malliavin calculus given in the proof of Theorem 4.3 in [7] validates the Feynman – Kac representation (1.7) as a weak solution of (1.1).

Theorem 1.1 is proved.

The following lemma is a generalization of Lemma A.4 in [7] and allows us to follow the argument in Step 5, Proof of Theorem 4.3 in [7] (see (4.15), (4.16) in [7] for its relevance).

Lemma 2.2. *Under assumption* (1.9),

$$\mathbb{E}_0\left[\int\limits_0^t \frac{\gamma(B_s)}{s^{\alpha_0}}\,ds\right]^2<\infty,\quad t>0.$$

Proof. By monotonicity in time, all we need is to show that

$$\int\limits_0^\infty e^{-t}\mathbb{E}_0\left[\int\limits_0^t \frac{\gamma(B_s)}{s^{\alpha_0}}\,ds\right]^2 dt<\infty. \tag{2.19}$$

Write

$$\left[\int_0^t \frac{\gamma(B_s)}{s^{\alpha_0}}\,ds\right]^2 = 2\int_0^t dr\frac{\gamma(B_r)}{r^{\alpha_0}}\int_r^t \frac{\gamma(B_s)}{s^{\alpha_0}}\,ds \le 2\int_0^t dr\frac{\gamma(B_r)}{r^{\alpha_0}}\int_r^t \frac{\gamma(B_s)}{(s-r)^{\alpha_0}}\,ds.$$

By Markov property

$$\mathbb{E}_0\left[\int_0^t \frac{\gamma(B_s)}{s^{\alpha_0}}ds\right]^2 \le 2\mathbb{E}_0\int_0^t dr\,\frac{\gamma(B_r)}{r^{\alpha_0}}\int_r^t \frac{\mathbb{E}_{B_r}\gamma(B_{s-r})}{(s-r)^{\alpha_0}}\,ds.$$

By (1.3), for any $x\in\mathbb{R}^d$,

$$\mathbb{E}_x\gamma(B_{s-r}) = \mathbb{E}_0\gamma(x+B_{s-r}) = \mathbb{E}_0\int_{\mathbb{R}^d} e^{i\xi\cdot(x+B_{s-r})}\mu(d\xi)$$

$$= \int_{\mathbb{R}^d} e^{i\xi\cdot x}\exp\left\{-\frac{1}{2}\,|\xi|^2(s-r)\right\}\mu(d\xi) \le \int_{\mathbb{R}^d}\exp\left\{-\frac{1}{2}|\xi|^2(s-r)\right\}\mu(d\xi).$$

Hence,

$$\mathbb{E}_0\left[\int_0^t \frac{\gamma(B_s)}{s^{\alpha_0}}\,ds\right]^2 \le 2\int_0^t dr\frac{\mathbb{E}_0\gamma(B_r)}{r^{\alpha_0}}\int_r^t \frac{ds}{(s-r)^{\alpha_0}}\int_{\mathbb{R}^d}\exp\left\{-\frac{1}{2}\,|\xi|^2(s-r)\right\}\mu(d\xi).$$

Taking Laplace transform, we have

$$\int_0^\infty e^{-t}\mathbb{E}_0\left[\int_0^t \frac{\gamma(B_s)}{s^{\alpha_0}}\,ds\right]^2 dt \le 2\left(\int_0^\infty e^{-t}\frac{\mathbb{E}_0\gamma(B_t)}{t^{\alpha_0}}\,dt\right)\left(\int_0^\infty \frac{dt}{t^{\alpha_0}}\,e^{-t}\int_{\mathbb{R}^d}\exp\left\{-\frac{1}{2}\,|\xi|^2 t\right\}\mu(d\xi)\right).$$

Using (1.3) again, we obtain

$$\mathbb{E}_0\gamma(B_t) = \int_{\mathbb{R}^d}\exp\left\{-\frac{1}{2}|\xi|^2 t\right\}\mu(d\xi).$$

So, we get

$$\int_0^\infty e^{-t}\mathbb{E}_0\left[\int_0^t \frac{\gamma(B_s)}{s^{\alpha_0}}\,ds\right]^2 dt \le 2\left(\int_0^\infty \frac{dt}{t^{\alpha_0}}\,e^{-t}\int_{\mathbb{R}^d}\exp\left\{-\frac{1}{2}\,|\xi|^2 t\right\}\mu(d\xi)\right)^2.$$

Finally, (2.19) follows from the following computation:

$$\int_0^\infty \frac{dt}{t^{\alpha_0}}\,e^{-t}\int_{\mathbb{R}^d}\exp\left\{-\frac{1}{2}\,|\xi|^2 t\right\}\mu(d\xi) = \int_{\mathbb{R}^d}\mu(d\xi)\int_0^\infty \frac{1}{t^{\alpha_0}}\exp\left\{-\left(1+\frac{1}{2}\,|\xi|^2\right)t\right\}dt$$

$$= \left(\int_0^\infty t^{-\alpha_0}e^{-t}\,dt\right)\int_{\mathbb{R}^d}\left(1+\frac{1}{2}|\xi|^2\right)^{-(1-\alpha_0)}\mu(d\xi) < \infty.$$

The lemma is proved.

3. Appendix. Let $c(t,x),\ (t,x)\in\mathbb{R}^+\times\mathbb{R}^d$, be a continuous function and consider the deterministic heat equation

$$\begin{aligned}&\frac{\partial u}{\partial t}(t,x)=\frac{1}{2}\Delta u(t,x)+c(t,x)u(t,x),\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\\ &u(0,x)=u_0(x),\quad x\in\mathbb{R}^d.\end{aligned}\tag{3.1}$$

As before, $u_0(x)$ is bounded and measurable. Write the correspondent Feynman – Kac representation

$$u(t,x)=\mathbb{E}_x\exp\left\{\int_0^t c(t-s,B_s)ds\right\}u_0(B_t),\quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\tag{3.2}$$

whenever the right-hand side expression makes sense. It is not clear weather or not $u(t,x)$ in (3.2) is a path-wise solution of (3.1) if $c(t,x)$ is unbounded on $\mathbb{R}^+\times\mathbb{R}^d$. In the following lemma, we claim that it is at least a weak solution of (3.1).

Lemma 3.1. *Assume that the Feynman – Kac representation in* (3.2) *is well-defined on* $\mathbb{R}^+\times\mathbb{R}^d$ *and*

$$\begin{aligned}&\int_D \mathbb{E}_x\exp\left\{\int_0^t c(t-s,B_s)\,ds\right\}dx<\infty,\\ &\int_0^t\int_D \mathbb{E}_x\exp\left\{\int_0^s c(s-r,B_r)dr\right\}dx\,ds<\infty\end{aligned}\tag{3.3}$$

for every bounded $D\subset\mathbb{R}^d$ *and* $t>0$. *Then the Feynman – Kac representation* $u(t,x)$ *in* (3.2) *is a weak solution of* (3.1) *in the sense that*

$$\begin{aligned}\int_{\mathbb{R}^d} u(t,x)\varphi(x)\,dx&=\int_{\mathbb{R}^d}u_0(x)\varphi(x)\,dx\\ &+\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}u(s,x)\Delta\varphi(x)\,dx\,ds+\int_0^t\int_{\mathbb{R}^d}u(s,x)\varphi(x)c(s,y)\,dy\,ds\end{aligned}\tag{3.4}$$

for every C^∞*-function* φ *with compact support.*

Proof. For any $R>0$ write $D_R=\{x\in\mathbb{R}^d,\ |x|<R\}$. Consider the heat equation of zero boundary condition:

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x)&=\frac{1}{2}\Delta u(t,x)+c(t,x)u(t,x),\quad (t,x)\in\mathbb{R}^+\times D_R,\\ u(0,x)&=u_0^R(x),\quad x\in D_R,\\ u(t,\partial D_R)&=0,\quad t\in\mathbb{R}^+,\end{aligned}\tag{3.5}$$

where $u_0^R(x)$ is a bounded function that supported on D_R such that $\left|u_0^R(x)\right|\le|u_0|(x)$ and $u_0^R(x)\to u_0(x)$ for every $x\in\mathbb{R}^d$ as $R\to\infty$.

Set the Brownian exit time
$$\tau_R = \inf\{t > 0,\ B_t \notin D_R\}.$$
According to Theorem 2.3 in [5, p. 133], the Feynman – Kac representation
$$u^R(t,x) = \mathbb{E}_x \exp\left\{\int_0^t c(t-s,B_s)\,ds\right\} u_0^R(B_t) 1_{\{\tau_R \geq t\}}, \quad (t,x) \in \mathbb{R}^+ \times D_R,$$
is a path-wise solution of (3.5). Given a C^∞-function φ with a compact support D, we take R sufficiently large so $D \subset D_R$. Then we have
$$\int_{\mathbb{R}^d} u^R(t,x)\varphi(x)\,dx = \int_{\mathbb{R}^d} u_0^R(x)\varphi(x)\,dx$$
$$+ \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} u^R(s,x)\Delta\varphi(x)\,dx\,ds + \int_0^t \int_{\mathbb{R}^d} u^R(s,x)\varphi(x)c(s,y)\,dy\,ds. \quad (3.6)$$
Notice that $u^R(t,x) \longrightarrow u(t,x)$ point-wise as $R \to \infty$. In addition,
$$\left|u^R(t,x)\right| \leq \|u_0\|_\infty \mathbb{E}_x \exp\left\{\int_0^t c(t-s,B_s)\,ds\right\}, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d.$$
Let $R \to \infty$ in (3.6). In view of assumption (3.3), applying dominated convergence theorem to every term of (3.6) properly, we have (3.4).

The lemma is proved.

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