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DIFFUSION IN MEDIA WITH MEMBRANES AND SOME NONLOCAL PARABOLIC PROBLEMS

ДИФУЗІЯ В СЕРЕДОВИЩАХ З МЕМБРАНАМИ ТА ДЕЯКІ НЕЛОКАЛЬНІ ПАРАБОЛІЧНІ ЗАДАЧІ

We establish the classical solvability of a certain conjugation problem for one-dimensional (with respect to a spatial variable) Kolmogorov backward equation with discontinuous coefficients and some variants of the general nonlocal Feller – Wentzell boundary condition given on nonsmooth boundaries of considered curvilinear domains. In addition, we prove, that the two-parameter Feller semigroup defined by the solution of this problem describes some inhomogeneous diffusion process with moving membranes on the given region of the real line. We also show the relationship between the constructed process and the generalized diffusion in the sense of M. I. Portenko.

Встановлено класичну розв'язність однієї задачі спряження для одновимірного (за просторовою змінною) оберненого рівняння Колмогорова з розривними коефіцієнтами та заданими на негладких межах розглядуваних криволінійних областей деякими варіантами загальної нелокальної крайової умови типу Феллера – Вентцеля. Крім того, доведено, що визначена за допомогою розв'язку цієї задачі двопараметрична напівгрупа Феллера описує на заданому проміжку числової прямої деякий неоднорідний дифузійний процес з рухомими мембранами. Показано також зв'язок побудованого процесу з узагальненою дифузією в розумінні М. І. Портенка.

Introduction. The paper deals with two interrelated issues: first, a proof of a classical solvability of a certain nonlocal initial-boundary value problem of Feller – Wentzell's type for a linear one-dimensional parabolic equation of the second order with discontinuous coefficients; second, a construction, by using its solution, of the two-parameter Feller semigroup associated with the inhomogeneous Markov process on the given region of the real line. The union of these two issues represents the so-called problem on pasting together two diffusion processes, which are given by their generating differential operators in subdomains of the mentioned region. This problem can also be treated as the problem on mathematical modeling of the physical phenomenon of diffusion in medium with membranes (see [1, 2]). It is assumed that at the boundary points of the domains, where the moving membranes are placed, certain variants of the general boundary condition or the conjugation condition of Feller – Wentzell are given (see [3 – 5]). The term "moving membrane" means that the position of the membrane on the real line is determined by a given function which depends on the time variable.

In the paper, we study in detail the case, when the boundary conditions given on all the parts of boundaries of the considered domains correspond to such boundary effects of diffusion process as reflection (on the common boundary of the domains, we speak of partial reflection) and its leaving the boundary by jumps. Concerning the questions relating to the study of other variants of the general

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boundary condition and the conjugation condition of Feller – Wentzell, we restrict ourselves only to rather short general notes.

The main part of this paper consists in the investigation of the nonlocal parabolic conjugation problem, which is stated in Section 1. The problem, formulated in such a way, is considered presumably for the first time in the case when the domains, in which the equations are given, are curvilinear and the functions determining the boundaries of these domains satisfy only the Hölder condition with exponent greater than $\frac{1}{2}$ (cf. [6, 7]). The solution of this problem is obtained by the boundary integral equations method, and it is proved that it has a semigroup property. The availability of the integral representation for the constructed semigroup allows us to prove relatively easily the assertion that this semigroup yields some inhomogeneous Markov process in the given region of the real line (see Section 2). Furthermore, in partial cases, some additional properties of the constructed process are established. In particular, assuming that the domains, where the diffusion processes are given, are semibounded (this case is studied in detail in [8]), we prove that the constructed process can be treated as the generalized diffusion in the sense of M. I. Portenko [1, 9], provided the absence of nonlocal term in the Feller – Wentzell conjugation condition.

Note that the similar problem is considered in [10, 11] for the case of nonmoving membrane (see also [12, 13], where the other variants of general Feller – Wentzell boundary condition are considered), and the technique for investigation of the parabolic conjugation problem, which is developed in this paper, is promising in the sense that it can be used in construction of the corresponding classes of multidimensional diffusion processes with membranes (see [1, 2, 9, 14]) as well as in solving some actual physical problems (see, e.g., [15]). In addition to the publications listed above, we also mention [16 – 27] (see also the references given there) which reflect the development of other approaches to construction of the diffusion processes with boundary conditions, namely the analytical approaches basing on the application of methods of the semigroup theory and functional analysis, and the direct probabilistic approaches which allow to construct such processes as the solutions of the corresponding stochastic differential equations. We would add to this that more difficult one-dimensional stochastic differential equations (with the generalized drift of different forms) are studied in [28 – 31], and that some classes of stable stochastic processes with membranes appear for the first time in [32, 33].

1. Setting of the nonlocal parabolic conjugation problem, basic assumptions and auxiliary results. Consider the strip $\Pi = \{(s,x)\colon 0 \le s \le T; -\infty < x < \infty\}$ in Euclidean space $\mathbb{R}^2$ of variables (s,x) and two domains $S_t^{(i)} = \{(s,x)\colon 0 \le s < t \le T, h_i(s) < x < h_{i+1}(s)\}$, $i = 1,2$, in it, where $h_j(s)$, $s \in [0,T]$, $j = 1,2,3$, are given functions. Let $S_t = S_t^{(1)} \cup S_t^{(2)}$, $D_{is} = (h_i(s), h_{i+1}(s))$, $i = 1,2$, $D_s = D_{1s} \cup D_{2s}$, $\Delta_x^{\widetilde{x}} f(\cdot,x) = f(\cdot,x) - f(\cdot,\widetilde{x})$, $\Delta_t^{\widetilde{t}} f(t,\cdot) = f(t,\cdot) - f(\widetilde{t},\cdot)$. Denote by $\overline{G}$ the closure of a set G and by $C_b(\mathbb{R})$ the space of bounded continuous functions on $\mathbb{R}$ with the norm $\|\varphi\| = \sup_{x\in\mathbb{R}} |\varphi(x)|$. In what follows, the symbols C and c denote various different positive constants which do not depend on (s,x). Other notations will be explained where they first appear.

In the strip Π we consider two second order parabolic operators with the bounded continuous coefficients

$$\frac{\partial}{\partial s} + L_s^{(i)} \equiv \frac{\partial}{\partial s} + \frac{1}{2} b_i(s,x) \frac{\partial^2}{\partial x^2} + a_i(s,x) \frac{\partial}{\partial x}, \quad i = 1,2. \tag{1}$$

In D_t, we set the following nonlocal parabolic conjugation problem of Feller – Wentzell's type with respect to the unknown function $u(s,x,t)$ $((s,x) \in \overline{S}_t)$:

$$\frac{\partial u}{\partial s} + L_s^{(i)} u = 0, \quad (s,x) \in S_t^{(i)}, \quad i = 1,2, \tag{2}$$

$$\lim_{s \uparrow t} u(s,x,t) = \varphi(x), \quad x \in \overline{D}_t, \tag{3}$$

$$B_1 u \equiv u(s, h_2(s) - 0, t) - u(s, h_2(s) + 0, t) = 0, \quad 0 \le s \le t \le T, \tag{4}$$

$$B_2 u \equiv q_1(s) \frac{\partial u}{\partial x}(s, h_2(s) - 0, t) - q_2(s) \frac{\partial u}{\partial x}(s, h_2(s) + 0, t)$$
$$+ \int_{D_s} [u(s, h_2(s), t) - u(s,y,t)] \mu(s, dy) = 0, \quad 0 \le s < t \le T, \tag{5}$$

$$B_{3i-3} u \equiv (-1)^i p_i(s) \frac{\partial u}{\partial x}(s, h_{2i-1}(s), t)$$
$$+ \int_{D_{is}} [u(s, h_{2i-1}(s), t) - u(s,y,t)] \mu_i(s, dy) = 0, \quad 0 \le s < t \le T, \quad i = 1,2. \tag{6}$$

Here, $q_i(s)$ and $p_i(s)$ are nonnegative functions such that

$$q_1(s) + q_2(s) > 0, \quad s \in [0,T], \tag{7}$$

$$p_i(s) > 0, \quad s \in [0,T], \quad i = 1,2, \tag{8}$$

$\mu_2(s,\cdot)$ and $\mu_{2i-1}(s,\cdot)$, $i = 1,2$, are nonnegative measures on D_s and D_{is}, respectively, such that, for any $\delta > 0$,

$$\int_{D_s^\delta(h_2(s))} |y - h_2(s)| \mu(s, dy) + \mu(s, D_s \setminus D_s^\delta(h_2(s))) < \infty, \quad s \in [0,T], \tag{9}$$

$$\int_{D_{is}^\delta(h_{2i-1}(s))} |y - h_{2i-1}(s)| \mu_i(s, dy) + \mu_i(s, D_{si} \setminus D_{is}^\delta(h_{2i-1}(s))) < \infty, \quad s \in [0,T], \quad i = 1,2, \tag{10}$$

where $D_s^\delta(h_2(s)) = \{y : y \in D_s; |y - h_2(s)| < \delta\}$, $D_{is}^\delta(h_{2i-1}(s)) = \{y : y \in D_{is}; |y - h_{2i-1}(s)| < \delta\}$, $i = 1,2$, and where $u(s, h_2(s) - 0, t) \left(\frac{\partial u}{\partial x}(s, h_2(s) - 0, t) \right)$ and $u(s, h_2(s) + 0, t) \left(\frac{\partial u}{\partial x}(s, h_2(s) + 0, t) \right)$ denote the limits of function $u(s,x,t) \left(\frac{\partial u}{\partial x}(s,x,t) \right)$ at $(s, h_2(s))$ as the point (s,x) tends to $(s, h_2(s))$ from the side of the domains $S_t^{(1)}$ and $S_t^{(2)}$, respectively.

***Remark* 1.** Problem (2) – (6) can also be formulated as follows: find the function

$$u(s,x,t) = \begin{cases} u_1(s,x,t), & (s,x) \in S_t^{(1)}, \\ u_2(s,x,t), & (s,x) \in S_t^{(2)}, \end{cases}$$

such that u_i is a solution of equation (2) in $S_t^{(i)},\ i = 1, 2,$ and satisfies the “initial” condition (3) if $s = t,\ x \in \overline{D}_{it}$ as well as the boundary condition (6) if $x = h_{2i-1}(s),\ s \in [0, t),$ and such that, on the common boundary of the domains $S_t^{(1)},\ S_t^{(2)}$, the functions $u_1,\ u_2$ satisfy the conjugation conditions (4), (5), where $B_1 u$ and $B_2 u$ must be replaced by

$$B_1 u \equiv u_1(s, h_2(s), t) - u_2(s, h_2(s), t),$$

$$B_2 u \equiv q_1(s)\frac{\partial u_1}{\partial x}(s, h_2(s), t) - q_2(s)\frac{\partial u_2}{\partial x}(s, h_2(s), t)$$

$$+ \sum_{i=1}^{2} \int\limits_{D_{is}} [u_i(s, h_2(s), t) - u_i(s, y, t)]\mu(s, dy).$$

We once again turn our attention to the relationship between the parabolic initial-boundary value problem (2) – (6) and the problem on pasting together two diffusion processes on a line or, what amounts to the same, the problem of construction (using analytical methods) of one-dimensional diffusion process with membranes. In our case the membranes are moving. They are located at points $x = h_j(s),\ j = 1, 2, 3.$ If we suppose that the two-parameter family of operators (we denote it by $T_{st},\ 0 \le s \le t \le T$) yielded by a solution of problem (2) – (6) forms a Feller semigroup which describes some inhomogeneous Markov process (this fact is established in Section 3) the trajectories of which are contained in Π, then the fulfillment of the Kolmogorov equation (2) for the function $u(s, x, t) = T_{st}\varphi(x)$ indicates that this process coincides at the interior points of the domain D_{is} with the diffusion process given there by the operator $L_s^{(i)},\ i = 1, 2,$ and the “initial” condition (3) conforms with the equality $T_{ss} = E$, where E is the identity operator. Next, the conjugation condition (4) reflects the Feller property of the process, and the conjugation condition (5) and the boundary conditions (6) correspond to the extension of the process after the diffusing particle reaches any of the points where the membrane is placed. Note that these conditions, physical sense of which has been already commented in the Introduction, are only the partial cases of the general boundary condition and conjugation condition of Wentzell for one-dimensional diffusion processes (cf. [3 – 5]). Each of conditions (5) and (6) in the most general case contains two more terms, namely the unknown function and its derivative with respect to the time variable which correspond, respectively, to the properties of termination and delay of the process.

The classical solvability of problem (2) – (6) will be established in this paper by the methods of heat potential theory (see [1, 2, 6 – 9, 34 – 40]. To do this, we need the following conditions:

I. The operators $\dfrac{\partial}{\partial s} + L_s^{(i)},\ i = 1, 2,$ in (1) are uniformly parabolic in Π, i.e., there exist positive constants b and B such that $0 < b \le b_i(s, x) \le B < \infty,\ (s, x) \in \Pi,\ i = 1, 2.$

II. The coefficients of $L_s^{(i)},\ i = 1, 2,$ are bounded functions in Π and belong to the Hölder class $H^{\frac{\alpha}{2},\alpha}(\Pi),\ 0 < \alpha < 1$ (for the definition of Hölder classes, see [37, p. 16]).

III. The function φ in (3) is assumed to be defined on $\mathbb{R}$ and belongs to $C_b(\mathbb{R})$.

IV. The functions $h_j(s),\ j = 1, 2, 3,$ belong to the Hölder class $H^{\frac{1+\alpha}{2}}([0, T])$.

V. In (5) and (6), the functions $q_i(s)$ and $p_i(s),\ i = 1, 2,\ s \in [0, T]$, are nonnegative, continuous and satisfy inequalities (7) and (8), respectively; the measures $\mu(s, \cdot)$ and $\mu_i(s, \cdot),\ i = 1, 2,$ are nonnegative, satisfy inequality (9) and (10), respectively, and for any function $f \in C_b(\mathbb{R})$ and any number $\delta > 0$, the integrals $(i = 1, 2)$

$$F_f(s) = \int\limits_{D_s^\delta(h_2(s))} |y - h_2(s)| f(y)\mu(s, dy), \qquad G_f(s) = \int\limits_{D_s \setminus D_s^\delta(h_2(s))} f(y)\mu(s, dy),$$

$$F_f^{(i)}(s) = \int\limits_{D_{is}^\delta(h_{2i-1}(s))} |y - h_{2i-1}(s)| f(y)\mu_i(s, dy), \qquad G_f^{(i)}(s) = \int\limits_{D_{is} \setminus D_{is}^\delta(h_{2i-1}(s))} f(y)\mu_i(s, dy)$$

are continuous on $[0, T]$ as functions of s.

In what follows, we assume (without loss of generality) that $p_i(s) \equiv 1,\ s \in [0, T],\ i = 1, 2$.

Conditions I, II ensure the existence of the fundamental solution $G_i(s, x, t, y),\ 0 \le s < t \le T$, $x, y \in \mathbb{R}$, for each equation in (2). Recall that for fixed $t \in [0, T)$ and $y \in \mathbb{R}$ the function G_i satisfies equation (2) as a function of $(s, x) \in [0, t) \times \mathbb{R}$ and allows the representation

$$G_i(s, x, t, y) = Z_{i0}(s, x, t, y) + Z_{i1}(s, x, t, y), \quad i = 1, 2, \tag{11}$$

where

$$Z_{i0}(s, x, t, y) = [2\pi b_i(t, y)(t - s)]^{-\frac{1}{2}} \exp\left\{-\frac{(y - x)^2}{2b_i(t, y)(t - s)}\right\}, \tag{12}$$

$$Z_{i1}(s, x, t, y) = \int\limits_s^t d\tau \int\limits_{\mathbb{R}} Z_{i0}(s, x, \tau, z) Q_i(\tau, z, t, y) dz \tag{13}$$

(the density $Q_i(s, x, t, y)$ can be found from the condition that function (11) satisfies (2) with respect to (s, x)) and the estimates

$$|D_s^r D_x^p G_i(s, x, t, y)| \le C(t - s)^{-\frac{1+2r+p}{2}} \exp\left\{-c\frac{(y - x)^2}{t - s}\right\}, \tag{14}$$

$$|D_s^r D_x^p Z_{i1}(s, x, t, y)| \le C(t - s)^{-\frac{1+2r+p-\alpha}{2}} \exp\left\{-c\frac{(y - x)^2}{t - s}\right\} \tag{15}$$

hold. Here, r and p are the nonnegative integers for which $2r + p \le 2$, D_s^r and D_x^p are the symbols of partial derivative with respect to s of order r and with respect to x of order p, respectively. Note that estimate (14) with G_i replaced by Z_{i0} is valid for any nonnegative integers r and p.

Note also that (see [1, Chapter II, § 2], [9, Chapter II, § 2])

$$\int\limits_{\mathbb{R}} G_i(s, x, t, y) dy = 1, \quad i = 1, 2, \tag{16}$$

$$\int\limits_{\mathbb{R}} G_i(s, x, t, y)(y - x) dy = \int\limits_s^t d\tau \int\limits_{\mathbb{R}} G_i(s, x, \tau, z) a_i(\tau, z) dz, \quad i = 1, 2, \tag{17}$$

$$\int\limits_{\mathbb{R}} G_i(s, x, t, y)(y - x)^2 dy = \int\limits_s^t d\tau \int\limits_{\mathbb{R}} G_i(s, x, \tau, z) b_i(\tau, z) dz +$$

$$+ 2\int\limits_s^t d\tau \int\limits_{\mathbb{R}} G_i(s, x, \tau, z) a_i(\tau, z)(z - x) dz, \quad i = 1, 2. \tag{18}$$

Given the fundamental solution $G_i(s,x,t,y),\ i=1,2,$ and the function φ in (3), we can define the integral $(0\le s<t\le T,\ x\in\mathbb{R})$

$$u_{i0}(s,x,t)=\int\limits_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy,\quad i=1,2. \tag{19}$$

In the theory of parabolic equations the function $u_{i0}(s,x,t)$ is called the Poisson potential. Note some properties of this function. Assume that $\varphi\in C_b(\mathbb{R})$. Then, from the properties of the fundamental solution $G_i(s,x,t,y),\ i=1,2,$ it follows that the potential u_{i0} exists and, as a function of (s,x), for fixed $t\in(0,T]$, satisfies equation (2) in $(s,x)\in[0,t)\times\mathbb{R}$ with the "initial" condition

$$\lim_{s\uparrow t} u_{i0}(s,x,t)=\varphi(x),\quad x\in\mathbb{R},\quad i=1,2. \tag{20}$$

Furthermore, the function $u_{i0}(s,x,t),\ i=1,2,$ satisfies the inequality

$$\left|D_s^r D_x^p u_{i0}(s,x,t)\right|\le C(t-s)^{-\frac{2r+p}{2}}\|\varphi\|, \tag{21}$$

where $0\le s<t\le T,\ x\in\mathbb{R},\ r$ and p are the nonnegative integers such that $2r+p\le 2$.

Below, we will need the concept of the simple-layer potential. This concept is associated with the curves $h_j(s),\ j=1,2,3$. Let $h(s),\ s\in[0,T]$, be a curve from class $H^{\frac{1+\alpha}{2}}([0,T])$, which separates the strip $\Pi[0,T]$ into two parts: $\Pi^-=\big\{(s,x)\colon 0\le s\le T; -\infty<x<h(s)\big\}$ and $\Pi^+=\big\{(s,x)\colon 0\le s\le T; h(s)<x<\infty\big\}$.

Assume also that $V(s,t)$ is a measurable function defined on $0\le s<t\le T$ with values in $\mathbb{R}$. Put

$$W_i(s,x,t)=\int\limits_s^t G_i(s,x,\tau,h(\tau))V(\tau,t)d\tau,\quad i=1,2, \tag{22}$$

for all $0\le s<t\le T,\ x\in\mathbb{R}$. The function $W_i(s,x,t)$ is called the simple-layer potential. If the function $V(\tau,t)$ is bounded, then the integral in the right-hand side of (22) exists. This follows from (11)–(15). As the function of $(s,x),\ W_i(s,x,t),\ i=1,2,$ is bounded and continuous in $0\le s<t\le T,\ x\in\mathbb{R}$, satisfies equation (2) for $(s,x)\in[0,t)\times(\mathbb{R}\setminus h(s))$ and the "initial" condition

$$\lim_{s\uparrow t} W_i(s,x,t)=0,\quad x\in\mathbb{R},\quad i=1,2. \tag{23}$$

An important property of the function W_i is described by the so-called theorem on the jump of conormal derivative of a parabolic simple-layer potential (see, e.g., [1, Chapter II, § 3], [6, 7, 39, 40], [36, Chapter V, § 2], [37, Chapter IV, § 15]). Regarding the potential $W_i(s,x,t)$, this theorem asserts that if the curve $h(s)$ belongs to $H^{\frac{1+\alpha}{2}}([0,T])$ for some $\alpha\in(0,1)$ and the function $V(\tau,t)$ is continuous in $0\le\tau\le t\le T$, then for every point $x=h(s)$ and $s\in[0,t)$ the function $W_i(s,x,t)$ satisfies the relation

$$\lim_{x\to h(s)\pm 0}\frac{\partial W_i}{\partial x}(s,x,t)=\mp\frac{V(s,t)}{b_i(s,h(s))}+\int\limits_s^t\frac{\partial G_i}{\partial x}(s,h(s),\tau,h(\tau))V(\tau,t)d\tau,\quad i=1,2. \tag{24}$$

The integral on the right-hand side of (24) is called the direct value of the simple-layer potential. Its existence follows from the inequality

$$\left|\frac{\partial G_i}{\partial x}(s,h(s),\tau,h(\tau))\right| \le C(\tau-s)^{-1+\frac{\alpha}{2}}, \quad i=1,2. \tag{25}$$

Note that all the properties of the potential W_i mentioned above, remain valid when the function $V(\tau,t)$ is continuous in $0\le\tau<t\le T$ and "weakly" bounded as $(t-\tau)^{-\mu}$, where $0\le\mu\le\frac{1}{2}$, if $\tau\to t$ (see [36, Chapter V, § 2–4], [34, Chapter XXII, § 7–9], [8]).

2. Solving the nonlocal parabolic conjugation problem. In this section, we solve problem (2)–(6) by the boundary integral equations method.

Theorem 2.1. *Let conditions* I–V *hold. Then there exists a unique classical solution of problem* (2)–(6), *which is continuous in* $\overline{S}_t$.

Proof. We look for a solution of problem (2)–(6) of the form

$$u(s,x,t)=u_i(s,x,t)=u_{i0}(s,x,t)+u_{i1}(s,x,t), \quad (s,x)\in S_t^{(i)}, \quad i=1,2, \tag{26}$$

where the function u_{i0} is the Poisson potential (19) and u_{i1} is the sum of simple-layer potentials

$$u_{i1}(s,x,t)=\sum_{j=0}^{1}\int\limits_s^t G_i(s,x,\tau,h_{i+j}(\tau))V_{2i+j-2}(\tau,t)d\tau, \quad i=1,2, \tag{27}$$

with the unknown densities $V_k(\tau,t),\ k=0,1,2,3,$ to be found.

Suppose a priori that the unknown functions $V_k(\tau,t),\ k=0,1,2,3,$ are continuous for $0\le s<t\le T$ and "weakly" bounded as $(\tau-s)^{-\mu}$, where $0\le\mu\le\frac{1}{2}$, if $\tau\to t$. These functions will be defined from conditions (4)–(6).

Consider first conditions (6). From (24) it is seen directly that these conditions lead to the following equalities:

$$V_{3i-3}(s,t)=\Psi_{3i-3}(s,t)+\sum_{j=0}^{1}\int\limits_s^t K_{ij}(s,\tau)V_{2i+j-2}(\tau,t)d\tau, \quad i=1,2, \tag{28}$$

where

$$\Psi_{3i-3}(s,t)=(-1)^{i-1}b_i(s,h_{2i-1}(s))\frac{\partial u_{i0}}{\partial x}(s,h_{2i-1}(s),t)$$
$$-\,b_i(s,h_{2i-1}(s))\int\limits_{D_{is}}(u_{i0}(s,h_{2i-1}(s),t)-u_{i0}(s,y,t))\mu_i(s,dy), \quad i=1,2,$$

$$K_{ij}(s,\tau)=(-1)^{i-1}b_i(s,h_{2i-1}(s))\frac{\partial G_i}{\partial x}(s,h_{2i-1}(s),\tau,h_{i+j}(\tau))$$
$$-\,b_i(s,h_{2i-1}(s))\int\limits_{D_{is}}[G_i(s,h_{2i-1}(s),\tau,h_{i+j}(\tau))$$
$$-\,G_i(s,y,\tau,h_{i+j}(\tau))]\mu_i(s,dy), \quad i=1,2, \quad j=0,1.$$

In the obtained Volterra integral equation of the second kind (28) we first estimate the functions Ψ_{3i-3}, $i = 1, 2$. For this purpose in the integral with respect to the measure μ_i we split the domain of integration D_{is} into D^1_{is} and $D_{is} \setminus D^1_{is}$, and in the first part of this integral we apply the mean value theorem to the difference $\Delta^y_{h_{2i-1}(s)} u_{i0}(s, h_{2i-1}(s), t)$. Then, using conditions I, III, V and inequalities (10) and (21), we find that $\Psi_{3i-3}(s, \tau)$, $0 \le s < t \le T$, $i = 1, 2$, are continuous functions satisfying the inequality

$$|\Psi_{3i-3}(s,t)| \le C\|\varphi\|(t-s)^{-\frac{1}{2}}, \quad i = 1, 2. \tag{29}$$

We get down to studying the kernels $K_{ij}(s, \tau)$, $i = 1, 2$, $j = 0, 1$, in the integral equation (28). Splitting the domain of integration D_{is} into D^δ_{is} and $D_{is} \setminus D^\delta_{is}$ ($\delta > 0$), we have

$$K_{ij}(s,\tau) = K^{(1)}_{ij}(s,\tau) + K^{(2)}_{ij}(s,\tau) + K^{(3)}_{ij}(s,\tau), \tag{30}$$

where

$$K^{(1)}_{ij}(s,\tau) = (-1)^{i-1} b_i(s, h_{2i-1}(s)) \frac{\partial G_i}{\partial x}(s, h_{2i-1}(s), \tau, h_{i+j}(\tau)),$$

$$K^{(2)}_{ij}(s,\tau) = -b_i(s, h_{2i-1}(s)) \int\limits_{D_{is}\setminus D^\delta_{is}} [G_i(s, h_{2i-1}(s), \tau, h_{i+j}(\tau)) - G_i(s, y, \tau, h_{i+j}(\tau))]\mu_i(s, dy),$$

$$K^{(3)}_{ij}(s,\tau) = -b_i(s, h_{2i-1}(s)) \int\limits_{D^\delta_{is}} [G_i(s, h_{2i-1}(s), \tau, h_{i+j}(\tau)) - G_i(s, y, \tau, h_{i+j}(\tau))]\mu_i(s, dy).$$

First of all, we note that in (30) the integrals with kernels $K^{(1)}_{10}$ and $K^{(1)}_{21}$ coincide with an accuracy to bounded factors, with the direct values of the conormal derivatives of the simple-layer potentials (see (24)). Thus, these kernels satisfy inequality (25) for any $0 \le s < t \le T$.

In view of conditions I, V, estimates (14), (15) and the inequalities

$$|h_i(\tau) - h_j(\tau)| \ge h_0 > 0 \quad (h_0 \text{ is a constant}), \quad i \ne j, \quad \tau \in [0, T], \tag{31}$$

$$\sigma^\nu e^{-c\sigma^2} \le \text{const} \quad (c > 0,\ 0 \le \sigma < \infty, \quad 0 \le \nu < \infty), \tag{32}$$

it is easy to verify that estimate (25) can be applied also to the kernels $K^{(1)}_{11}$ and $K^{(1)}_{20}$ and hence to all the functions $K^{(1)}_{ij}(s, \tau)$, $i = 1, 2$, $j = 0, 1$.

The kernels $K^{(2)}_{ij}(s, \tau)$ also allow the integrable singularity. This follows from the estimate

$$|K^{(2)}_{ij}(s,\tau)| \le C \int\limits_{D_{is}\setminus D^\delta_{is}} \big[|G_i(s, h_{2i-1}(s), \tau, h_{i+j}(\tau))| + |G_i(s, y, \tau, h_{i+j}(\tau))|\big]\mu_i(s, dy)$$

$$\le C(\tau - s)^{-\frac{1}{2}} \mu_i(D_{is} \setminus D^\delta_{is}) \le C(\delta)(\tau - s)^{-\frac{1}{2}}, \quad i = 1, 2, \quad j = 0, 1, \tag{33}$$

where $C(\delta)$ is a positive constant which depends on $\delta > 0$. It remains to estimate the last term in (30). Taking into account (11), we have

$$\begin{aligned}K_{ij}^{(3)}(s,\tau) &= -b_i(s,h_{2i-1}(s)) \int\limits_{D_{is}^{\delta}(h_{2i-1}(s))} \big[Z_{i0}(s,h_{2i-1}(s),\tau,h_{i+j}(\tau)) \\ &\quad - Z_{i0}(s,y,\tau,h_{i+j}(\tau))\big]\mu_i(s,dy) \\ &\quad - b_i(s,h_{2i-1}(s)) \int\limits_{D_{is}^{\delta}(h_{2i-1}(s))} \big[Z_{i1}(s,h_{2i-1}(s),\tau,h_{i+j}(\tau)) \\ &\quad - Z_{i1}(s,y,\tau,h_{i+j}(\tau))\big]\mu_i(s,dy) \\ &= K_{ij}^{(31)}(s,\tau) + K_{ij}^{(32)}(s,\tau). \end{aligned} \tag{34}$$

We first estimate $K_{ij}^{(32)}(s,\tau)$. It is easily seen that as a consequence of the mean value theorem for the difference $\Delta^{y}_{h_{2i-1}(s)} Z_{i1}(s,h_{2i-1}(s),\tau,h_{i+j}(\tau))$, the condition I as well as inequalities (10) and (15) (with $r=0,\ p=1$),

$$\begin{aligned}|K_{ij}^{(32)}(s,\tau)| &\le C(\tau-s)^{-1+\frac{\alpha}{2}} \int\limits_{D_{is}^{\delta}(h_{2i-1}(s))} |y-h_{2i-1}(s)|\mu_i(s,dy) \\ &\le C(\delta)(\tau-s)^{-1+\frac{\alpha}{2}}, \quad i=1,2, \quad j=0,1. \end{aligned} \tag{35}$$

Next, after writing the integrand in $K_{ij}^{(31)}(s,\tau)$ in the form

$$\begin{aligned}&\Delta^{y}_{h_{2i-1}(s)} Z_{i0}(s,h_{2i-1}(s),\tau,h_{i+j}(\tau)) \\ &\qquad = [2\pi b_i(\tau,h_{i+j}(\tau))(\tau-s)]^{-\frac{1}{2}} \int\limits_0^1 \frac{\partial}{\partial\theta} \exp\left\{-\frac{A(\theta,y,h_{i+j}(\tau),h_{2i-1}(s))}{2b_i(\tau,h_{i+j}(\tau))(\tau-s)}\right\} d\theta,\end{aligned}$$

where $A(\theta,y,h_{i+j}(\tau),h_{2i-1}(s)) = (1-\theta)(y-h_{i+j}(\tau))^2 + \theta(h_{2i-1}(s)-h_{i+j}(\tau))^2$, we obtain

$$\begin{aligned}K_{ij}^{(31)}(s,\tau) &= \frac{-h_{2i-1}(s)+h_{i+j}(\tau)}{[2\pi b_i(\tau,h_{i+j}(\tau))(\tau-s)]^{\frac{1}{2}}(\tau-s)} \\ &\quad \times \int\limits_0^1 d\theta \int\limits_{D_{is}^{\delta}(h_{2i-1}(s))} (y-h_{2i-1}(s)) \exp\left\{-\frac{A(\theta,y,h_{i+j}(\tau),h_{2i-1}(s))}{2b_i(\tau,h_{i+j}(\tau))(\tau-s)}\right\}\mu_i(s,dy) \\ &\quad - \frac{1}{2[2\pi b_i(\tau,h_{i+j}(\tau))(\tau-s)]^{\frac{1}{2}}(\tau-s)} \\ &\quad \times \int\limits_0^1 d\theta \int\limits_{D_{is}^{\delta}} (y-h_{2i-1}(s))^2 \exp\left\{-\frac{A(\theta,y,h_{i+j}(\tau),h_{2i-1}(s))}{2b_i(\tau,h_{i+j}(\tau))(\tau-s)}\right\}\mu_i(s,dy) \\ &= K_{ij}^{(311)}(s,\tau) + K_{ij}^{(312)}(s,\tau). \end{aligned} \tag{36}$$

Using conditions I, IV, V, inequalities (31), (32) and

$$\sigma^{\nu}e^{-c\sigma^2} \le \left(\frac{\nu}{2ce}\right)^{\frac{\nu}{2}}, \quad c>0, \quad 0\le\sigma<\infty, \quad 0<\nu<\infty, \tag{37}$$

we deduce that $K_{ij}^{(311)}(s,\tau)$ satisfies inequality (35).

In order to estimate $K_{ij}^{(312)}(s,\tau)$, we note that

$$A(\theta,y,h_{i+j}(\tau),h_{2i-1}(s)) \ge \theta(1-\theta)(y-h_{2i-1}(s))^2. \tag{38}$$

Using condition I and inequalities (10), (37), (38), we get

$$\begin{aligned}|K_{ij}^{(312)}(s,\tau)| &\le \frac{1}{2\sqrt{2\pi b}(\tau-s)^{\frac{3}{2}}}\int_0^1 d\theta \int_{D_{is}^{\delta}(h_{2i-1}(s))} (y-h_{2i-1}(s))^2 \\ &\times \exp\left\{-\frac{\theta(1-\theta)}{2B}\frac{(y-h_{2i-1}(s))^2}{\tau-s}\right\}\mu_i(s,dy) \\ &\le m_{is}^{(1)}(\delta)(\tau-s)^{-1}, \quad i=1,2, \quad j=0,1,\end{aligned} \tag{39}$$

where b and B are the constants in condition I, $m_{is}^{(1)}(\delta) = \dfrac{1}{2}\left(\dfrac{\pi B}{2be}\right)^{\frac{1}{2}}\lambda_{is}^{(1)}(\delta)$, and $\lambda_{is}^{(1)}(\delta)$ denotes the integral of the function $|y-h_{2i-1}(s)|$ with respect to the measure μ_i over $D_{is}^{\delta}(h_{2i-1}(s))$.

From equalities (30), (34), (36) and estimates (25), (33), (35), (39) it follows that, for every $\delta>0$,

$$K_{ij}(s,\tau) = \widetilde{K}_{ij}^{(1)}(s,\tau) + \widetilde{K}_{ij}^{(2)}(s,\tau), \quad i=1,2, \quad j=0,1,$$

where $\widetilde{K}_{ij}^{(1)}(s,\tau) = K_{ij}^{(1)}(s,\tau) + K_{ij}^{(2)}(s,\tau) + K_{ij}^{(32)}(s,\tau) + K_{ij}^{(311)}(s,\tau)$, $\widetilde{K}_{ij}^{(2)}(s,\tau) = K_{ij}^{(312)}(s,\tau)$. Moreover, $\widetilde{K}_{ij}^{(1)}(s,\tau)$ and $\widetilde{K}_{ij}^{(1)}(s,\tau)$ satisfy inequalities (35) and (39), respectively.

Consider now the conjugation conditions (4), (5). Substituting the function u in (26) into (4), we obtain the equation

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\int_s^t (-1)^{i-1}G_i(s,h_2(s),\tau,h_{i+j}(\tau))V_{2i+j-2}(\tau,t)d\tau = \Phi_0(s,t), \tag{40}$$

where $\Phi_0(s,t) = u_{20}(s,h_2(s),t) - u_{10}(s,h_2(s),t)$.

Equation (40) is the Volterra integral equation of the first kind. Using the Holmgren method (see [6 – 8]), we reduce (40) to the equivalent Volterra integral equation of the second kind. For this purpose, we introduce the integro-differential operator $\mathcal{E}$ as follows:

$$\mathcal{E}(s,t)f = \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int_s^t (\rho-s)^{-\frac{1}{2}}f(\rho,t)d\rho, \quad 0\le s<t\le T.$$

Consider first the application of the operator $\mathcal{E}$ to the right-hand side of (40), i.e., to the function $\Phi_0(s,t)$. From the mean value theorem, condition IV for the function $h_2(s)$ and inequalities (21) it follows that

$$\big|\Phi_0(s,t)-\Phi_0(\widetilde{s},t)\big|\leq C\|\varphi\|(t-s)^{-\frac{1+\alpha}{2}}(s-\widetilde{s})^{\frac{1+\alpha}{2}} \tag{41}$$

for all $0\leq\widetilde{s}<s<t\leq T$.

In view of (41) and the condition that $\Phi_0(t,t)=0$, for the function $\Phi(s,t)=\mathcal{E}(s,t)\Phi_0$, we find the representation

$$\Phi(s,t)=\mathcal{E}(s,t)\Phi_0=\frac{1}{\sqrt{2\pi}}\int\limits_s^t(\rho-s)^{-\frac{3}{2}}[\Phi_0(\rho,t)-\Phi_0(s,t)]d\rho-\sqrt{\frac{2}{\pi}}(t-s)^{-\frac{1}{2}}\Phi_0(s,t) \tag{42}$$

and the estimate $(0\leq s<t\leq T)$

$$|\Phi(s,t)|\leq C\|\varphi\|(t-s)^{-\frac{1}{2}}.$$

We now apply the operator $\mathcal{E}$ to the left-hand side of equation (40), i.e., to the function $u_{11}(s,h_2(s),t)-u_{21}(s,h_2(s),t)$. We have

$$J(s,t)=\mathcal{E}(s,t)(u_{11}-u_{21})=\sum_{i=1}^{2}\sum_{j=0}^{1}J_{ij}(s,t), \tag{43}$$

where $(i=1,2,\ j=0,1)$

$$J_{ij}(s,t)=\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int\limits_s^t(\rho-s)^{-\frac{1}{2}}\left[\int\limits_\rho^t(-1)^{i-1}G_i(\rho,h_2(\rho),\tau,h_{i+j}(\tau))V_{2i+j-2}(\tau,t)d\tau\right]d\rho.$$

Interchanging the order of integration, we get

$$J_{ij}(s,t)=\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int\limits_s^t V_{2i+j-2}(\tau,t)\hat{J}_{ij}(s,\tau)d\tau, \tag{44}$$

where

$$\hat{J}_{ij}(s,\tau)=(-1)^{i-1}\int\limits_s^\tau(\rho-s)^{-\frac{1}{2}}G_i(\rho,h_2(\rho),\tau,h_{i+j}(\tau))d\rho,\quad i=1,2,\quad j=0,1.$$

Put

$$\begin{aligned}G_i(\rho,h_2(\rho),\tau,h_{i+j}(\tau))&=G_i(\rho,h_2(\tau),\tau,h_{i+j}(\tau))\\&\quad+\big[G_i(\rho,h_2(\rho),\tau,h_{i+j}(\tau))-G_i(\rho,h_2(\tau),\tau,h_{i+j}(\tau))\big]\end{aligned} \tag{45}$$

and suppose that $i+j\neq 2$. Then, using the mean value theorem, condition IV and inequalities (14), (31), (37), we obtain

$$\big|G_i(\rho,h_2(\rho),\tau,h_{i+j}(\tau))\big|\leq C(\tau-\rho)^{-\frac{1}{2}+\frac{\alpha}{2}}. \tag{46}$$

In the case $i+j=2$, we use in addition representation (11) for the function

$$G_i(\rho,h_2(\tau),\tau,h_{i+j}(\tau))=G_i(\rho,h_2(\tau),\tau,h_2(\tau))$$

in (45). We have

$$G_i(\rho,h_2(\rho),\tau,h_2(\tau))=[2\pi b_i(\tau,h_2(\tau))(\tau-\rho)]^{-\frac{1}{2}}+Z_{i1}(\rho,h_2(\tau),\tau,h_2(\tau))$$
$$+[G_i(\rho,h_2(\rho),\tau,h_2(\tau))-G_i(\rho,h_2(\tau),\tau,h_2(\tau))].$$

Furthermore, the function $Z_{i1}(\rho,h_2(\tau),\tau,h_2(\tau))+[G_i(\rho,h_2(\rho),\tau,h_2(\tau))-G_i(\rho,h_2(\tau),\tau,h_2(\tau))]$ obviously allows inequality (46). From this, we find that

$$\lim_{s\to\tau}\hat{J}_{ij}(s,\tau)=0,\quad \text{if}\quad i=1,2,\quad j=0,1,\quad i+j\neq 2,$$

$$\lim_{s\to\tau}\hat{J}_{ij}(s,\tau)=(-1)^{i-1}\sqrt{\frac{\pi}{2b_i(\tau,h_2(\tau))}}\quad \text{if}\quad i=1,2,\quad j=0,1,\quad i+j=2.$$

Hence

$$J_{ij}(s,\tau)=\int\limits_s^t L_{ij}(s,\tau)V_{2i+j-2}(\tau,t)d\tau,\quad \text{if}\quad i=1,2,\quad j=0,1,\quad i+j\neq 2,$$

$$J_{ij}(s,\tau)=\frac{(-1)^iV_{2i+j-2}}{\sqrt{b_i(s,h_2(s))}}+\int\limits_s^t L_{ij}(s,\tau)V_{2i+j-2}(\tau,t)d\tau,\quad \text{if}\quad i=1,2,\quad j=0,1,\quad i+j=2,$$

where

$$L_{ij}(s,\tau)=(-1)^{i-1}\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int\limits_s^\tau(\rho-s)^{-\frac{1}{2}}G_i(\rho,h_2(\rho),\tau,h_{i+j}(\tau))d\rho,\quad i+j\neq 2,$$

$$L_{ij}(s,\tau)=(-1)^{i-1}\sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int\limits_s^\tau(\rho-s)^{-\frac{1}{2}}\big[Z_{i1}(\rho,h_2(\tau),\tau,h_{i+j}(\tau))$$
$$+(G_i(\rho,h_2(\rho),\tau,h_{i+j}(\tau))-G_i(\rho,h_2(\tau),\tau,h_{i+j}(\tau)))\big]d\rho,\quad i+j=2.$$

Furthermore,

$$|L_{ij}(s,\tau)|\leq C(\tau-s)^{-1+\frac{\alpha}{2}},\quad i=1,2,\quad j=0,1,\quad 0\leq s<\tau\leq t\leq T. \tag{47}$$

Since the proof of inequality (47) for $L_{ij}(s,\tau)$, $i=1,2$, $j=0,1$, is similar to the proof of inequality (2.10) in [8], we focus our attention on the proof of this inequality only in the special case where $i+j\neq 2$ putting $i=1$ and $j=0$.

In order to estimate the function $L_{10}(s,\tau)$ we first write it in the following form:

$$L_{10}(s,\tau)=\frac{1}{\sqrt{2\pi}}\int\limits_s^\tau(\rho-s)^{-\frac{3}{2}}[G_1(\rho,h_2(\rho),\tau,h_1(\tau))-G_1(s,h_2(\rho),\tau,h_1(\tau))]d\rho$$

$$+\frac{1}{\sqrt{2\pi}}\int\limits_s^\tau(\rho-s)^{-\frac{3}{2}}[G_1(s,h_2(\rho),\tau,h_1(\tau))-G_1(s,h_2(s),\tau,h_1(\tau))]d\rho$$

$$-\sqrt{\frac{2}{\pi}}(\tau-s)^{-\frac{1}{2}}G_1(s,h_2(s),\tau,h_1(\tau))=\sum_{k=1}^{3}L_{10}^{(k)}(s,\tau). \tag{48}$$

To prove the existence of the integral in $L_{10}^{(1)}$ we represent it in the form

$$L_{10}^{(1)}(s,\tau)=\frac{1}{\sqrt{2\pi}}\int\limits_s^{\frac{s+\tau}{2}}(\rho-s)^{-\frac{3}{2}}[G_1(\rho,h_2(\rho),\tau,h_1(\tau))-G_1(s,h_2(\rho),\tau,h_1(\tau))]d\rho$$

$$+\frac{1}{\sqrt{2\pi}}\int\limits_{\frac{s+\tau}{2}}^{\tau}(\rho-s)^{-\frac{3}{2}}[G_1(\rho,h_2(\rho),\tau,h_1(\tau))-G_1(s,h_2(\rho),\tau,h_1(\tau))]d\rho$$

$$=L_{10}^{(11)}(s,\tau)+L_{10}^{(12)}(s,\tau).$$

Consider $L_{10}^{(11)}(s,\tau)$. Using (11) and (45) as well as the the mean value theorem for the integrand in $L_{10}^{(11)}(s,\tau)$, we find that

$$(\rho-s)^{-\frac{3}{2}}[G_1(\rho,h_2(\rho),\tau,h_1(\tau))-G_1(s,h_2(\rho),\tau,h_1(\tau))]$$

$$=(\rho-s)^{-\frac{1}{2}}\left[\frac{\partial^2 Z_{10}}{\partial s\partial x}(s',x',\tau,h_1(\tau))(h_2(\rho)-h_2(\tau))\right.$$

$$\left.+\frac{\partial Z_{10}}{\partial s}(s',h_2(\tau),\tau,h_1(\tau))+\frac{\partial Z_{11}}{\partial s}(s',h_2(\rho),\tau,h_1(\tau))\right],$$

where s' is the intermediate value between s and p; x' is the intermediate value between $h_2(\tau)$ and $h_2(\rho)$. In view of (14), (15), (31), (32) and condition IV, we have

$$|L_{10}^{(11)}(s,\tau)|\le\int\limits_s^{\frac{s+\tau}{2}}(\rho-s)^{-\frac{1}{2}}\Big[(\tau-s')^{-2}(\tau-\rho)^{\frac{1+\alpha}{2}}$$

$$+(\tau-s')^{-\frac{3}{2}}\exp\left\{-c\frac{|h_2(\tau)-h_1(\tau)|^2}{\tau-s'}\right\}+(\tau-s')^{-\frac{3}{2}+\frac{\alpha}{2}}\Big]d\rho\le C(\tau-s)^{-1+\frac{\alpha}{2}}.$$

Next, using the equalities

$$G_1(\rho,h_2(\rho),\tau,h_1(\tau))=\Delta_{h_2(\rho)}^{h_2(\tau)}G_1(\rho,h_2(\rho),\tau,h_1(\tau))+G_1(\rho,h_2(\tau),\tau,h_1(\tau)),$$

$$G_1(s,h_2(\rho),\tau,h_1(\tau))=\Delta_{h_2(\rho)}^{h_2(\tau)}G_1(s,h_2(\rho),\tau,h_1(\tau))+G_1(s,h_2(\tau),\tau,h_1(\tau)),$$

the finite-increments formula, condition IV and inequalities (14), (31), (37), we estimate $L_{10}^{(12)}(s,\tau)$:

$$|L_{10}^{(12)}(s,\tau)| \le C \int\limits_{\frac{s+\tau}{2}}^{\tau} (\rho-s)^{-\frac{3}{2}} \big[|G_1(\rho,h_2(\rho),\tau,h_1(\tau))| + |G_1(s,h_2(\rho),\tau,h_1(\tau))|\big] d\rho$$

$$\le C \int\limits_{\frac{s+\tau}{2}}^{\tau} (\rho-s)^{-\frac{3}{2}} \big[(\tau-\rho)^{-\frac{1}{2}+\frac{\alpha}{2}} + (\tau-s)^{-\frac{1}{2}+\frac{\alpha}{2}}\big] d\rho \le C(\tau-s)^{-1+\frac{\alpha}{2}}.$$

Therefore,

$$|L_{10}^{(1)}(s,\tau)| \le |L_{10}^{(11)}(s,\tau)| + |L_{10}^{(12)}(s,\tau)| \le C(\tau-s)^{-1+\frac{\alpha}{2}}.$$

The terms $L_{10}^{(2)}$ and $L_{10}^{(3)}$ in (48) can be estimated in a similar way. Thus, inequality (47) holds for $L_{10}(s,\tau)$. After estimating all other kernels $L_{ij}(s,\tau)$, we find that they also allow inequality (47). This means that (47) holds for $L_{ij}(s,\tau),\ i=1,2,\ j=0,1$.

Substituting the expressions for $J_{ij}(s,\tau)$ in (44) into the right-hand side of (43) and then equating the obtained result to the function $\Phi(s,t)$ in (42), we obtain

$$\sum_{i=1}^{2} (-1)^{i-1} \frac{V_i(s,t)}{\sqrt{b_i(s,h_2(s))}} = -\Phi(s,t) + \sum_{i=1}^{2}\sum_{j=0}^{1} \int\limits_s^t L_{ij}(s,\tau) V_{2i+j-2}(\tau,t)d\tau. \tag{49}$$

Thus, the integral equation (40) is reduced to the equivalent Volterra integral equation of the second kind (49).

The last integral equation which makes a connection between all four functions $V_k,\ k=0,1,2,3$, can be found from condition (5). Using relation (24), we get

$$\sum_{i=1}^{2} \frac{q_i(s)}{b_i(s,h_2(s))} V_i(s,t) = \Psi(s,t) + \sum_{i=1}^{2}\sum_{j=0}^{1} \int\limits_s^t M_{ij}(s,\tau) V_{2i+j-2}(\tau,t)d\tau, \tag{50}$$

where ($i=1,2,\ j=0,1$)

$$\Psi(s,t) = \sum_{i=1}^{2} \left[(-1)^i q_i(s) \frac{\partial u_{i0}}{\partial x}(s,h_2(s),t) + \int\limits_{D_{is}} \Delta_y^{h_2(s)} u_{i0}(s,y,t)\mu(s,dy) \right],$$

$$M_{ij}(s,\tau) = (-1)^i q_i(s) \frac{\partial G_i}{\partial x}(s,h_2(s),\tau,h_{i+j}(\tau)) + \int\limits_{D_{is}} \Delta_y^{h_2(s)} G_i(s,y,\tau,h_{i+j}(\tau))\mu(s,dy).$$

The estimation of the function Ψ and the kernels M_{ij} in (50) can be done similar to the estimation of the functions Ψ_{3i-3} and the kernels K_{ij} in (28). In particular, applying to Ψ the scheme of estimation of the functions Ψ_{3i-3}, we prove that the function $\Psi(s,t)$ is continuous in $0 \le s < t \le T$ and satisfies inequality (29) in this domain. In order to estimate the kernels M_{ij}, we consider the equality (cf. (30))

$$M_{ij}(s,\tau) = \widetilde{M}_{ij}^{(1)}(s,\tau) + \widetilde{M}_{ij}^{(2)}(s,\tau), \quad i=1,2, \quad j=0,1, \tag{51}$$

where

$$\widetilde{M}_{ij}^{(1)}(s,\tau) = (-1)^i q_i(s)\frac{\partial}{\partial x}G_i(s,h_2(s),\tau,h_{i+j}(\tau))$$

$$+ \int\limits_{D_{is}\setminus D_{is}^{\delta}(h_2(s))} \Delta_y^{h_2(s)} G_i(s,y,\tau,h_{i+j}(\tau))\mu(s,dy)$$

$$+ \int\limits_{D_{is}^{\delta}(h_2(s))} \Delta_y^{h_2(s)} Z_{i1}(s,y,\tau,h_{i+j}(\tau))\mu(s,dy) + \frac{h_{i+j}(\tau)-h_2(s)}{\sqrt{2\pi}[b_i(\tau,h_{i+j}(\tau))(\tau-s)]^{\frac{3}{2}}}$$

$$\times \int\limits_0^1 d\theta \int\limits_{D_{is}^{\delta}(h_2(s))} (y-h_2(s))\exp\left\{-\frac{A(\theta,y,h_{i+j}(\tau),h_2(s))}{2b_i(\tau,h_{i+j}(\tau))(\tau-s)}\right\}\mu(s,dy),$$

$$\widetilde{M}_{ij}^{(2)}(s,\tau) = -\frac{\pi}{[2\pi b_i(\tau,h_{i+j}(\tau))(\tau-s)]^{\frac{3}{2}}}\int\limits_0^1 d\theta \int\limits_{D_{is}^{\delta}(h_2(s))} \exp\left\{-\frac{A(\theta,y,h_{i+j}(\tau),h_2(s))}{2b_i(\tau,h_{i+j}(\tau))(\tau-s)}\right\}\mu(s,dy).$$

Using the same considerations and estimations as those used to estimate the kernels K_{ij} in (28), we find that for $\widetilde{M}_{ij}^{(1)}(s,\tau)$ the estimation (35) holds and the function $\widetilde{M}_{ij}^{(2)}(s,\tau)$ is bounded above (in absolute value) by

$$m_{is}^{(2)}(\delta)(\tau-s)^{-1}, \quad i=1,2, \quad j=0,1, \tag{52}$$

where $m_{is}^{(2)}(\delta) = \dfrac{1}{2b}\left(\dfrac{\pi B}{2be}\right)^{\frac{1}{2}}\lambda_{is}^{(2)}(\delta)$, and $\lambda_{is}^{(2)}(\delta)$ denotes the integral of the function $|y-h_2(s)|$ with respect to the measure μ over $D_{is}^{\delta}(h_2(s))$.

Consider the system of equations (49), (50). Solving it with respect to V_1 and V_2 and adding to the obtained equations two more equations from system (28), we finally find $(0 \le s < t \le T)$

$$V_i(s,t) = \Psi_i(s,t) + \sum_{j=0}^{3}\int\limits_s^t N_{ij}(s,\tau)V_j(\tau,t)d\tau, \quad j=0,1,2,3, \tag{53}$$

where Ψ_0 and Ψ_3 are defined in (28),

$$\Psi_i(s,t) = d_i(s)\left[\Psi(s,t) + (-1)^i\frac{q_{3-i}(s)}{\sqrt{b_{3-i}(s,h_2(s))}}\Phi(s,t)\right], \quad i=1,2,$$

$$d_i(s) = \frac{b_i(s,h_2(s))\sqrt{b_{3-i}(s,h_2(s))}}{q_1(s)\sqrt{b_2(s,h_2(s))}+q_2(s)\sqrt{b_1(s,h_2(s))}}, \quad i=1,2,$$

$$N_{00}(s,\tau) = K_{10}(s,\tau), \qquad N_{01}(s,\tau) = K_{11}(s,\tau),$$

$$N_{02}(s,\tau) = N_{03}(s,\tau) = N_{30}(s,\tau) = N_{31}(s,\tau) \equiv 0,$$

$$N_{32}(s,\tau) = K_{20}(s,\tau), \qquad N_{33}(s,\tau) = K_{21}(s,\tau),$$

$$N_{ij}(s,\tau) = d_i(s)\left[M_{1j}(s,\tau) + (-1)^{i-1}\frac{q_{3-i}(s)}{\sqrt{b_{3-i}(s,h_2(s))}}L_{1j}(s,\tau)\right], \quad i=1,2, \quad j=0,1,$$

$$N_{ij}(s,\tau) = d_i(s)\left[M_{2,j-2}(s,\tau) + (-1)^{i-1}\frac{q_{3-i}(s)}{\sqrt{b_{3-i}(s,h_2(s))}}L_{2,j-2}(s,\tau)\right], \quad i=1,2, \quad j=2,3.$$

Since the functions $\Psi_0,\ \Psi_3,\ \Psi$ and Φ are already estimated, in view of the relation

$$|d_i(s)| \le d_0 = \frac{B}{q_0}\sqrt{\frac{B}{b}}, \quad i=1,2, \qquad q_0 = \min_{s\in[0,T]}(q_1(s)+q_2(s)), \tag{54}$$

we find that the functions $\Psi_i,\ 0,1,2,3,$ are continuous in the domain $0 \le s < t \le T$ and for them the inequality

$$|\Psi_i(s,t)| \le C_0\|\varphi\|(t-s)^{-\frac{1}{2}} \tag{55}$$

holds, where C_0 is a constant.

Next, in accordance with (30) and (51), we represent the kernels N_{ij} of the integral equations in (53) as

$$N_{ij}(s,\tau) = N_{ij}^{(1)}(s,\tau) + N_{ij}^{(2)}(s,\tau), \quad i,j=0,1,2,3, \tag{56}$$

where $N_{0j}^{(k)}(s,\tau) = \widetilde{K}_{1j}^{(k)}(s,\tau),\ j=0,1,\ N_{0j}^{(k)}(s,\tau)=0,\ j=2,3,\ N_{3j}^{(k)}(s,\tau) = \widetilde{K}_{2,j-2}^{(k)}(s,\tau),$ $j=2,3,\ N_{3j}^{(k)}(s,\tau)=0,\ j=1,2,\ k=1,2,\ N_{ij}^{(2)}(s,\tau) = d_i(s)\widetilde{M}_{1,j}^{(2)}(s,\tau),\ j=0,1,\ N_{ij}^{(2)}(s,\tau) = d_i(s)\widetilde{M}_{2,j-2}^{(2)}(s,\tau),\ j=2,3,\ i=1,2,$ and $N_{ij}^{(1)}(s,\tau),\ i=1,2,\ j=0,1,2,3,$ are defined by the same formulas as $N_{ij}(s,\tau),\ i=1,2,\ j=0,1,2,3,$ with $M_{ik}(s,\tau)$ replaced by $\widetilde{M}_{ik}^{(1)}(s,\tau),\ i=1,2,$ $k=0,1.$ From this and from (35), (39), (52) and (54), we get

$$|N_{ij}^{(1)}(s,\tau)| \le C_1(\delta)(\tau-s)^{-1+\frac{\alpha}{2}}, \quad i,j=0,1,2,3, \tag{57}$$

where $C_1(\delta)$ is a constant, and the functions $N_{0j}^{(2)}(s,\tau),\ j=0,1,\ N_{3j}^{(2)}(s,\tau),\ j=2,3,\ N_{ij}^{(2)}(s,\tau),$ $j=0,1,\ i=1,2,\ N_{ij}^{(2)}(s,\tau),\ j=3,4,\ i=1,2,$ are bounded above (in absolute value) by $m_{1s}^{(1)}(\delta)(\tau-s)^{-1},\ m_{2s}^{(1)}(\delta)(\tau-s)^{-1},\ d_0m_{1s}^{(2)}(\delta)(\tau-s)^{-1},\ d_0m_{2s}^{(2)}(\delta)(\tau-s)^{-1}$, respectively.

As we see, all the nonzero kernels contain terms, which have nonintegrable singularities at $\tau=s$. Despite this, let us prove that the ordinary method of successive approximations can still be applied to the system of integral equations (53).

Thus, we look for solutions of the system of integral equations (53) of the form of the series

$$V_i(s,t) = \sum_{k=0}^{\infty} V_i^{(k)}(s,t), \quad i=0,1,2,3, \tag{58}$$

where

$$V_i^{(0)}(s,t) = \Psi_i(s,t),$$

$$V_i^{(k)}(s,t)=\sum_{j=0}^{3}\int\limits_s^t N_{ij}(s,\tau)V_j^{(k-1)}(\tau,t)d\tau,\quad i=0,1,2,3,\quad k=1,2,\ldots.$$

Let us show that the series (58) converges uniformly in $0\le s<t\le T$. For the function $V_i^{(0)}(s,t),\ i=0,1,2,3,$ we already have estimate (55). Let us now estimate $V_i^{(1)}(s,t)$. From (56) we have

$$V_i^{(1)}(s,t)=\sum_{j=0}^{3}\int\limits_s^t N_{ij}^{(1)}(s,\tau)V_j^{(0)}(\tau,t)d\tau$$
$$+\sum_{j=0}^{3}\int\limits_s^t N_{ij}^{(2)}(s,\tau)V_j^{(0)}(\tau,t)d\tau=V_i^{(11)}(s,t)+V_i^{(12)}(s,t),\quad i=0,1,2,3.$$

By using inequalities (55) and (57), we find that $(0\le s<t\le T)$

$$|V_i^{(11)}(s,t)|\le C_0\|\varphi\|4C_1(\delta)\int\limits_s^t(t-\tau)^{-\frac12}(\tau-s)^{-1+\frac{\alpha}{2}}d\tau$$
$$=C_0\|\varphi\|(t-s)^{-\frac12}\frac{4C_1(\delta)\Gamma\left(\dfrac{\alpha}{2}\right)\Gamma\left(\dfrac12\right)}{\Gamma\left(\dfrac{1+\alpha}{2}\right)}(t-s)^{\frac{\alpha}{2}},\quad i=1,4,\tag{59}$$

where Γ is the gamma function.

In order to estimate $V_i^{(12)}(s,t)$, consider two cases: $i=0,3$ and $i=1,2$. In particular, when estimating $V_0^{(12)}(s,t)$, we use the equalities $N_{0j}^{(2)}(s,\tau)=\widetilde{K}_{1j}^{(2)}(s,\tau),\ j=0,1,\ N_{0j}^{(2)}(s,\tau)=0,$ $j=3,4$ (see (56), (30)), inequalities (39), (55) as well as the relation

$$\int\limits_s^t(t-\tau)^{-\frac12}(\tau-s)^{-\frac32}\exp\left\{-\frac{\theta(1-\theta)}{2B}\frac{(y-h_k(s))^2}{(\tau-s)}\right\}d\tau$$
$$=\left(\frac{2\pi B}{\theta(1-\theta)(t-s)}\right)^{\frac12}\frac{1}{|y-h_k(s)|}\exp\left\{-\frac{\theta(1-\theta)(y-h_k(s))^2}{2B(t-s)}\right\},\quad k=1,2,3.$$

We obtain

$$|V_0^{(12)}(s,t)|\le C_0\|\varphi\|\frac{1}{\sqrt{2\pi b}}\int\limits_s^t(t-\tau)^{-\frac12}(\tau-s)^{-\frac32}d\tau$$
$$\times\int\limits_0^1 d\theta\int\limits_{D_{1s}^{(\delta)}(h_1(s))}(y-h_1(s))^2\exp\left\{-\frac{\theta(1-\theta)}{2B}\cdot\frac{(y-h_1(s))^2}{\tau-s}\right\}\mu_1(s,dy)$$

$$\leq C_0\|\varphi\|(t-s)^{-\frac{1}{2}}\pi\sqrt{\frac{B}{b}}\lambda_{1s}^{(1)}(\delta). \tag{60}$$

By similar considerations, we find that the functions $V_3^{(12)}(s,t)$ and $V_i^{(12)}(s,t),\ i=1,2,$ are bounded above (in absolute value) by

$$C_0\|\varphi\|(t-s)^{-\frac{1}{2}}\pi\sqrt{\frac{B}{b}}\lambda_{2s}^{(1)}(\delta) \qquad \text{and} \qquad C_0\|\varphi\|(t-s)^{-\frac{1}{2}}\frac{d_0}{b}\pi\sqrt{\frac{B}{b}}(\lambda_{1s}^{(2)}(\delta)+\lambda_{2s}^{(2)}(\delta)),$$

respectively. From this and (59), (60) it follows that

$$|V_i^{(1)}(s,t)| \leq C_0\|\varphi\|(t-s)^{-\frac{1}{2}}\left[\frac{4C_1(\delta)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)}(t-s)^{\frac{\alpha}{2}}+m(\delta)\right], \quad i=0,1,2,3,$$

where $m(\delta)=d\cdot\lambda(\delta),\ d=\max\left\{1,\dfrac{d_0}{b}\right\},\ \lambda(\delta)=\pi\sqrt{\dfrac{B}{b}}\max_{s\in[0,T]}\sum_{k,l=1}^{2}\lambda_{ls}^{(k)}(\delta).$

Note that the fulfillment of condition V for the measures $\mu_i,\ i=1,2,$ and μ guarantees the existence of the number $\delta>0$ for which $m(\delta)<1$. Let us fix one of these numbers $\delta=\delta_0$ and put $C_1(\delta_0)=C_1,\ m(\delta_0)=m_0<1.$

Further, in view of the last remark, it can be proved by induction on k that

$$|V_i^{(k)}(s,t)| \leq C_0\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{n=0}^{k}\binom{k}{n}h_{s,t}^{(k-n)}m_0^n,$$

where

$$h_{s,t}^{(k)}=\frac{\left[4C_1\Gamma\left(\frac{\alpha}{2}\right)\right]^k\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+k\alpha}{2}\right)}(t-s)^{\frac{k\alpha}{2}}.$$

Hence, we have $(0\leq s<t\leq T,\ i=0,1,2,3)$

$$\begin{aligned}\sum_{k=0}^{\infty}|V_i^{(k)}(s,t)| &\leq C_0\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\sum_{n=0}^{k}\binom{k}{n}h_{s,t}^{(k-n)}m_0^n\\ &= C_0\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}h_{s,t}^{(k)}\sum_{n=0}^{\infty}\binom{k+n}{n}m_0^n\\ &= C_0\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{h_{s,t}^{(k)}}{(1-m_0)^{k+1}}\\ &= C_0\|\varphi\|(t-s)^{-\frac{1}{2}}\sum_{k=0}^{\infty}\frac{\left(4C_1\Gamma\left(\frac{\alpha}{2}\right)\right)^k\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{k\alpha}{2}\right)(1-m_0)^{k-1}}(t-s)^{\frac{k\alpha}{2}}.\end{aligned} \tag{61}$$

Inequality (61) ensures the absolute and uniform convergence of series (58) for $t > s$ and leads to the estimate

$$|V_i(s,t)| \le C\|\varphi\|(t-s)^{-\frac{1}{2}}, \quad i = 1, 2, \tag{62}$$

where $0 \le s < t \le T$ and C is a constant.

Thus, formula (58) represents the unique solution of the system of integral equations (53) which is continuous in the domain $0 \le s < t \le T$ and satisfies inequality (62).

From estimates (14) (with $r = p = 0$) and (62) it follows that the integrals in representation (27) for the functions $u_{i1}(s,x,t)$, $i = 1, 2$, exist and satisfy condition (23) as well as the following inequality:

$$|u_{i1}(s,x,t)| \le C\|\varphi\|, \quad (s,x) \in \overline{D}_t, \quad i = 1, 2. \tag{63}$$

From (21) ($r = 0$, $p = 0$) we have that the inequality of the form (63) holds also for the functions $u_{i0}(s,x,t)$, $i = 1, 2$, in (26) and therefore for the function $u(s,x,t)$. Taking into account condition (20), we conclude that the function $u(s,x,t)$ defined by relations (26), (19), (27), (58) is the desired classical solution of problem (2)–(6). Concerning its uniqueness in the class of continuous functions, we obtain this assertion by repeating with appropriate changes the scheme of the proof of Theorem 2.2 in [8].

Theorem 2.1 is proved.

3. Construction of the Markov process and its connection with the processes of M. I. Portenko. Denote by $T_{st}\varphi(x)$ the value of solution $u(s,x,t)$ of problem (2)–(6) at point (s,x), $s \le t$, $x \in \overline{D}_s = [h_1(s), h_3(s)]$. If $s = t$, $T_{ss}\varphi(x) = \varphi(x)$, i.e., the operator $T_{ss} = E$, where E is the identity operator. If $s \le t$, the function $T_{st}\varphi(x)$ is continuous. Thus, the operator T_{st} maps $C(\overline{D}_t)$ into $C(\overline{D}_s)$. It is also obvious that this operator is linear.

Let us prove that the operator T_{st} is positivity preserving.

Lemma 3.1. *If $\varphi \in C(\overline{D}_t)$ and $\varphi(x) \ge 0$ for all $x \in \overline{D}_t$, then $T_{st}\varphi(x) \ge 0$ for all $0 \le s \le t$, $x \in \overline{D}_s$.*

Proof. First, we note that in the case $\varphi(x) \equiv 0$, $x \in \overline{D}_t$, the assertion of the lemma is obvious, since in this case the unique solution of problem (2)–(6) (regardless of the extension of φ to $\mathbb{R}$) is $T_{st}\varphi(x) \equiv 0$, $s \le t$, $x \in \overline{D}_s$. Therefore we suppose that the function $\varphi \ge 0$ is nonzero. Denote by γ the minimum of the function $T_{st}\varphi(x)$ in the domain $(s,x) \in \overline{S}_t$ and assume that $\gamma < 0$. In view of the fact that $T_{st}\varphi(x)$ satisfies equation (2) in $(s,x) \in (0,t) \times D_{is}$, $i = 1, 2$, and the "initial" condition (3) as well as the well-known assertions belonging to the maximum principle for parabolic equations (see [36, Chapter II]), we deduce that this minimum is attained on $\bigcup_{j=1}^{3}\{(0,t) \times \{h_j(s)\}\}$. Suppose, for example, that the value γ is attained on $(0,t) \times \{h_1(s)\}$. Let $s = s_0$ and $x = h_1(s_0)$ for which $T_{s_0t}\varphi(x_0) = \gamma$. Since the function $T_{st}\varphi(x)$ is obviously nonconstant, there exists a neighborhood U of (s_0, x_0) such that $T_{st}\varphi(x) > \gamma$ if $(s,x) \in U \cap \{(0,t) \times D_{1s}\}$. But then

$$\frac{\partial T_{s_0t}\varphi}{\partial x}(x_0) \ge 0, \qquad \int\limits_{D_{1s}} [T_{s_0t}\varphi(x_0) - T_{s_0t}\varphi(y)]\mu_1(s_0, dy) \le 0. \tag{64}$$

Moreover, from the corollary to Theorem 1 in [41] (cf. Theorem 14 in [36, Chapter II, § 5]), it follows that in (64), the equal sign is not allowed. This contradicts condition (6) with $i = 1$. Similarly, we exclude the cases where the value $\gamma < 0$ is attained on the boundaries $(0,t) \times \{h_3(s)\}$ and

$(0,t)\times\{h_2(s)\}$. These contradictions are caused by the assumption that $\gamma<0$. Therefore, $\gamma\geq 0$, and the assertion of the lemma follows.

It is not difficult to see that the operators T_{st} are contractive, i.e., they do not increase the norm of the element. This property is an easy consequence of the assertion of Lemma 3.1 and the relation

$$T_{st}\varphi_0(x)\equiv 1,\quad 0\leq s\leq t,\quad x\in\overline{D}_s, \tag{65}$$

where $\varphi_0(x)\equiv 1,\ x\in\overline{D}_t$, because function (65) satisfies all conditions (2)–(6).

Finally, to find $u(s,x,t)$ when $u(t,x,t)=\varphi(x)$ one can do the following: solve equation (2) in the time interval $[\tau,t]$ and then solve it in the time interval $[s,\tau]$ starting with $u(s,x,\tau)|_{s=\tau}=u(\tau,x,t)$ which was obtained. In the other words, $T_{st}\varphi=T_{s\tau}(T_{\tau t}\varphi),\ \varphi\in C(\overline{D}_t)$, or $T_{st}=T_{s\tau}T_{\tau t}$.

From the above properties of the solution of problem (2)–(6), it follows that (see [38, Chapter II, § 1], [1, Chapter I]) the family of operators $(T_{st})_{0\leq s\leq t\leq T}$ defined by this solution is a semigroup associated with some one-dimensional inhomogeneous Feller process. If we denote by $P(s,x,t,dy)$ its transition function, then $T_{st}\varphi(x)$ can be represented as $T_{st}\varphi(x)=\int_{\overline{D}_t}\varphi(y)P(s,x,t,dy),\ 0\leq s\leq t\leq T,\ x\in\overline{D}_s$.

So, we have proved the following theorem.

Theorem 3.1. *Let the conditions of Theorem* 2.1 *hold. Then the two-parameter family of operators* $(T_{st})_{0\leq s\leq t\leq T}$ *defined by the solution of problem* (2)–(6) *is the semigroup associated with one-dimensional inhomogeneous Markov process, the trajectories of which coincide at the interior points of the curvilinear domains* $S_t^{(1)}$ *and* $S_t^{(2)}$ *with the trajectories of the diffusion processes given by* $L_s^{(1)}$ *and* $L_s^{(2)}$*, respectively, and their behavior at the boundary points of these domains is described by the boundary conditions* (6) *and the conjugation conditions* (4), (5).

Finally, we note that the results formulated in Theorems 2.1 and 3.1 can be considered as the nontrivial generalization of the corresponding results obtained in our earlier paper [8], where the similar problem was studied for the case when $S_t^{(1)}$ and $S_t^{(2)}$ are the semibounded strips separated by the curve $x=h(s)$, i.e.,

$$S_t^{(1)}=\{(s,x):0\leq s<t\leq T,-\infty<x<h(s)\},$$

$$S_t^{(2)}=\{(s,x):0\leq s<t\leq T,h(s)<x<\infty\},$$

$$S_t=S_t^{(1)}\cup S_t^{(2)},\qquad D_{1s}=(-\infty,h(s)),\qquad D_{2s}=(h(s),\infty),\qquad D_s=D_{1s}\cup D_{2s},$$

$$D_{is}^{\delta}=\big\{y:y\in D_{is},\ |y-h(s)|<\delta\big\},\quad i=1,2,\qquad D_s^{\delta}=D_{1s}^{\delta}\cup D_{2s}^{\delta},\quad \delta>0.$$

In this case, in order to define the two-parameter operator family $T_{st},\ 0\leq s\leq t\leq T$, associated with the desired Markov process on $\mathbb{R}$, we have the parabolic conjugation problem (2)–(5), where in the "initial" condition (3), we write $x\in\mathbb{R}$ instead of $x\in\overline{D}_t$ and in the conjugation conditions (4), (5), we replace $h_2(s)$ by $h(s)$. The conditions imposed on the output data of the problem are similar to I–V. If these conditions hold, then one can prove (see Theorems 2.1 and 2.2 in [8]) that there exists a unique classical solution $u(s,x,t)=T_{st}\varphi(x)$ of problem (2)–(5), continuous in $\overline{S}_t$, which, furthermore, allows the integral representation of the form of the sum of Poisson potentials $u_{i0}(s,x,t),\ (s,x)\in S_t^{(i)},\ i=1,2$, in (26) and the parabolic simple-layer potentials

$$u_{i1}(s,x,t) = \int\limits_s^t G_i(s,x,\tau,h(\tau))V_i(\tau,t)d\tau, \quad (s,x) \in S_t^{(i)}, \quad i=1,2,$$

where $V_i,\ i=1,2,$ can be defined from the conjugation conditions (4), (5). Note that the densities $V_i,\ i=1,2,$ here, as in the case of problem (2)–(6), can be formally determined from formulas (58), if we put $V_i(s,t)=\Psi_i(s,t)\equiv 0$ for $i=0,\ 3,$ and make corresponding obvious changes in the expressions for the functions $\Psi_i(s,t),\ i=1,2,$ and $N_{ij}(s,\tau),\ i=1,2,\ j=1,2.$

It is also proved (see Theorem 3.2 in [8]) that the family of linear operators $T_{st},\ 0\le s\le t\le T,$ defined by the solution of problem (2)–(5), which act in the space $C_b(\mathbb{R})$, yields some inhomogeneous Markov process on $\mathbb{R}$. Denoting its transition probability by $P(s,x,t,dy)$, we obtain the relation

$$T_{st}\varphi(x) = \int\limits_{\mathbb{R}} \varphi(y)P(s,x,t,dy). \tag{66}$$

Furthermore, a group of important properties of the constructed process was additionally established in [8]. Let us mention some of them in the present paper. For this purpose, we assume that in problem (2)–(5), in its conjugation condition (5), the measure μ, which is responsible for the possibility of the process exiting the boundary of the domain by jumps, is identically equal to zero. Then, relying on the integral representation of the constructed semigroup $T_{st}\varphi(x)$ and formulas (16)–(18), it is easy to prove that the probability of transition $P(s,x,t,dy)$ of the constructed process, determined by equality (66), satisfies the following relations:

a) for all $s\in[0,T),\ x\in\mathbb{R}$,

$$\lim_{t\downarrow s}\frac{1}{t-s}\int\limits_{\mathbb{R}}|y-x|^4P(s,x,t,dy)=0; \tag{67}$$

b) for all $s\in[0,T)$ and any continuous on $\mathbb{R}$ function f with compact support,

$$\lim_{t\downarrow s}\int\limits_{\mathbb{R}} f(x)\left[\frac{1}{t-s}\int\limits_{\mathbb{R}}(y-x)P(s,x,t,dy)\right]dx = \int\limits_{\mathbb{R}} a(s,x)f(x)dx + a_0(s)f(h(s)), \tag{68}$$

$$\lim_{t\downarrow s}\int\limits_{\mathbb{R}} f(x)\left[\frac{1}{t-s}\int\limits_{\mathbb{R}}(y-x)^2P(s,x,t,dy)\right]dx = \int\limits_{\mathbb{R}} b(s,x)f(x)dx, \tag{69}$$

where

$$b(s,x) = \begin{cases} b_i(s,x) & \text{for} \quad s\in[0,T], \quad x\in D_{is}, \quad i=1,2, \\ \sum_{j=1}^2 l_j(s)b_j(s,h(s)) & \text{for} \quad s\in[0,T], \quad x=h(s), \end{cases}$$

$$a(s,x) = \begin{cases} a_i(s,x) & \text{for} \quad s\in[0,T], \quad x\in D_{is}, \quad i=1,2, \\ \sum_{j=1}^2 l_j(s)a_j(s,h(s)) & \text{for} \quad s\in[0,T], \quad x=h(s), \end{cases}$$

$$l_j(s) = \frac{q_j(s)\sqrt{b_{3-j}(s,h(s))}}{q_1(s)\sqrt{b_2(s,h(s))}+q_2(s)\sqrt{b_1(s,h(s))}}, \quad j=1,2, \quad l_1(s)+l_2(s)=1,$$

$$a_0(s) = \frac{1}{2}(d_1(s) + d_2(s))(q_2(s) - q_1(s)),$$

and the functions $d_i(s),\ i = 1, 2,$ are defined in (53) (in the above expressions for $d_i(s)$ we write $h(s)$ instead of $h_2(s)$).

Relation (67) means that the Markov process with the probability transition $P(s, x, t, dy)$ can be considered continuous. Equalities (68) and (69) show that this process is a generalized diffusion process in the sense of M. I. Portenko (see [1, 9]) and for it, the diffusion coefficient is equal to $b(s, x)$ and the drift coefficient is equal to $a(s, x) + a_0(s)\delta(x - h(s))$, where $\delta(x - h(s))$ is the Dirac delta function concentrated at the point $x = h(s)$.

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