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ON REFLECTED DIFFUSIONS IN CONES AND CYLINDERS

ПРО ВІДБИТІ ДИФУЗІЇ В КОНУСАХ І ЦИЛІНДРАХ

Let X be a diffusion in a cone with oblique reflection at the boundary. We study the question whether X reaches a vertex of the cone for a finite time with positive probability. We propose new probabilistic method of investigation connected with the long-term behavior of a diffusion reflected in a cylinder.

Нехай X — дифузія в конусі зі скісним відбиттям на межі. Вивчається питання про можливість досягнення дифузиею X вершини конуса за скінченний час з додатною ймовірністю. Запропоновано новий імовірнісний метод дослідження, що пов'язаний з довготривалою поведінкою дифузії, відбитої в циліндрі.

Introduction. Consider a stochastic differential equation with reflection (RSDE) at the boundary of an open set $K \subset \mathbb{R}^d$:

$$dX(t) = A(X(t)) dt + \sum_{k=1}^m B_k(X(t)) dw_k(t) + C(X(t)) dL(t), \quad t \geq 0, \quad (0.1)$$

where $X(t)$ belongs to the closure $\overline{K}$ for all $t \geq 0$, $L(t)$ is a continuous, adapted, nondecreasing process, $L(0) = 0$, such that

$$\int_0^\infty \mathbf{1}_{X(s) \notin \partial K} dL(s) = 0 \quad \text{a.s.} \quad (0.2)$$

It is well-known that if the boundary of K is smooth enough, coefficients of (0.1) are locally Lipschitz functions, and vector field C points inside K from the boundary, i.e., $\langle C(x), n(x) \rangle > 0$, $x \in \partial K$, where $n(x)$ is the inner normal vector, then there is a unique solution to (0.1), (0.2) (see, for example, [1, 10, 23]).

If the boundary ∂K is not smooth everywhere, say K is a wedge, a cone or a polyhedron, then construction of a reflected diffusion in K is a hard task. One way is based on investigation of the Skorokhod reflected problem in K . If the corresponding (deterministic) Skorokhod map is well defined and Lipschitz continuous, then a proof of existence and uniqueness to the RSDE is quite standard. Note that the study of properties of Skorokhod map or its extensions is a nontrivial problem even for reflecting problem in an orthant with constant reflection at its faces. See, for example, [4–9] for investigation of the Skorokhod map in various cases. Another technique is based on a solving a submartingale problem (see [3] for domains with smooth boundary, and [15, 16, 18–20] for wedges, cusps, cones, orthants, etc.). Notice the following unusual fact: it may be that a reflected Brownian motion in a domain with singular boundary is *not* a semimartingale [24–26]. Certainly, if coefficients

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of the equation are smooth, the semimartingale property may fail or other difficulties may arise only at the very moment when the diffusion visits a singular point of the boundary.

The central object studied in this paper is a reflected diffusion in a cone K with vertex at 0. We assume that the intersection of K and a unit sphere S_{d-1} is an open set D (in S_{d-1}) with sufficiently smooth boundary ∂D . It is also assumed that all coefficients of (0.1) are locally Lipschitz everywhere except possibly at 0, the vertex of K .

Construction and comprehensive study of a reflected Brownian motion in a cone, where the reflection vector field v is radially homogeneous, was done in [15, 16]. In these papers necessary and sufficient conditions of vertex accessibility are done in order to assure (i) a possibility to *exit* the vertex and (ii) the uniqueness of the solution to a submartingale problem. These results are based on the construction of harmonic and subharmonic functions for the reflected Brownian motion in the cone, and also on estimates for the mean time of hitting the vertex. The heuristic arguments are rather clear, but detailed proof requires hard computations and use of deep functional analysis results. It is unclear whether it is simple to generalize their proofs to the case where the reflection field v admits angular limits at the vertex 0. These latter limits are defined by $\lim_{r \rightarrow 0+, \varphi \rightarrow \varphi_0} C(x) =: \bar{C}(\varphi_0)$, where, as usual, $r := |x|$ and $\varphi := \frac{x}{|x|}$, $x \in \mathbb{R}^d \setminus \{0\}$. In [17] the case of variable radial component was treated but the spherical one remains constant.

In this paper, we adopt a completely different approach than in the cited papers to solve (0.1) whose coefficients admit angular limits at 0 that depend on the polar angle φ . We also give conditions of accessibility/nonaccessibility of the vertex 0. Our result admits the following probabilistic interpretation. We consider the RSDE in log-polar coordinates and make some time transformation. Then the new transformed equation becomes a RSDE in a cylinder $\mathbb{R} \times D$, where $\{-\infty\} \times D$ corresponds to the vertex of K . If the coefficients of the original RSDE are radially invariant, then the transformed coefficients are independent of the first coordinate, i.e., they depend only on the coordinate in D . Similar approach was used in [2] for investigation of properties of a Brownian motion on the plane with membranes on rays with a common endpoint.

We prove in Section 1 a strong ergodic limit in time for the first coordinate of an homogeneous SDE in a cylinder. In Section 2, we then consider small perturbations of RSDE of the form introduced in Section 1 and compare their time asymptotics behavior with deterministic constants. These results, obtained for dynamics in cylinders, are then applied to investigate a RSDE in the cone in Section 3.

1. Time asymptotics for an homogeneous SDE in a cylinder. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbf{P})$ be a filtered probability space satisfying the usual hypotheses, $\{w_k(t), t \geq 0\}_{1 \leq k \leq m}$ be independent one-dimensional $\mathcal{F}_t$ -Wiener processes, $D \subset \mathbb{R}^d$ be a bounded connected open set with sufficiently smooth boundary (say C^3).

Consider the SDE in $\bar{D}$ with reflection at the boundary:

$$dY(t) = a(Y(t))dt + \sum_{k=1}^m b_k(Y(t))dw_k(t) + v(Y(t))dL(t), \quad (1.1)$$

where

$$\begin{aligned} Y(t) &\in \bar{D}, \quad t \geq 0, \\ L &\text{ is nondecreasing, continuous, } \mathcal{F}_t\text{-adapted process, } L(0) = 0, \end{aligned} \quad (1.2)$$

$$L(t) = \int_0^t \mathbf{1}_{Y(s) \in \partial D} dL(s), \quad t \geq 0,$$

$$Y(0) = \xi \in \bar{D}.$$

Assume that the functions a , b_k , v are Lipschitz continuous and

$$\langle v(y), n(y) \rangle = 1, \quad y \in \partial D, \quad (1.3)$$

where $n(y)$ is the inner normal vector at the point $y \in \partial D$.

It is well-known that, for any $\mathcal{F}_0$ -measurable initial condition ξ , there exists a unique strong solution of (1.1), (1.2) (see, for example, [10]).

Define now X as the following process:

$$X(t) = X_0 + \int_0^t \alpha(Y(s)) ds + \sum_{k=1}^m \int_0^t \beta_k(Y(s)) dw_k(s) + \int_0^t \gamma(Y(s)) dL(s), \quad t \geq 0. \quad (1.4)$$

The pair $(X(t), Y(t))_{t \geq 0}$ can be considered as solution of a SDE in a cylinder $\mathbb{R} \times \bar{D}$ with coefficients that are invariant with respect to translations along the x -axis.

The aim of this section is to identify the limit behavior of $X(t)$ as $t \rightarrow \infty$.

Let us introduce some notations:

$$\begin{aligned} \sum_k b_k(y) b_k(y)^T &= \left(\sum_k b_{k,i}(y) b_{k,j}(y) \right)_{i,j=1}^d =: (\sigma_{i,j}(y))_{i,j=1}^d = \sigma(y), \\ \mathcal{A}f(y) &:= \sum_j a_j(y) \frac{\partial f(y)}{\partial y_j} + \frac{1}{2} \sum_{i,j} \sigma_{ij}(y) \frac{\partial^2 f(y)}{\partial y_i \partial y_j}, \\ \mathcal{A}^* f &= \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j} (\sigma_{i,j} f) - \sum_i \frac{\partial}{\partial y_i} (a_i f). \end{aligned} \quad (1.5)$$

We further assume that a strong ellipticity condition is satisfied in the SDE (1.1):

$$\exists c > 0 \quad \forall y \in \bar{D}: \sum_{k=1}^m b_k(y) b_k(y)^T \geq cI. \quad (1.6)$$

Condition (1.6) together with the smoothness of the coefficients ensures existence and uniqueness of a stationary distribution π_Y for the process Y . Moreover, an exponential convergence of the process Y towards its stationary distribution holds:

$$\sup_{\mu=P_{Y(0)}} \text{var}(\pi_Y - P_{Y(t)}) \leq C e^{-\lambda t}, \quad t \geq 0, \quad (1.7)$$

where $C > 0$ and $\lambda > 0$ are some constants. This follows from the *Dobrushin condition*: for some $t_1 > 0$,

$$\sup_{Y(0)=y, Y'(0)=y'} \text{var} \left(P_{Y(t_1)} - P_{Y'(t_1)} \right) < 2 \quad (1.8)$$

in a standard way (see, e.g., [27, Section 2.3]). The latter inequality is a direct consequence of the following *accessibility property*: for any given ball B , there exists t_0 such that $\inf_y P(Y(t_0) \in B) > 0$. Recall that the transition distribution density of the process Y , conditioned not to reach boundary, is the solution to the Dirichlet problem for the operator $\partial_t - \mathcal{A}^*$ and thus is a continuous function which is not the trivial constant 0. Combined with the Markov and accessibility properties this yields (1.8).

Let us justify why the above accessibility property holds. Since D is connected and bounded, the diffusion is nondegenerate. Without loss of generality we may restrict ourselves to prove that

$$\exists \delta > 0 \quad \exists t_1 > 0: \quad \inf_{\text{dist}(y, \partial D) < \delta} \mathbf{P}_y \left(\text{dist}(Y(t_1), \partial D) \geq \delta \right) > 0$$

or there exists a continuous positive function h on D which vanishes on ∂D such that

$$\exists \delta_1 > 0 \quad \exists t_1 > 0 \quad \exists c > 0: \quad \inf_{\text{dist}(y, \partial D) < \delta_1} \mathbf{P}_y \left(h(Y(t_1)) \geq c \right) > 0.$$

Select such a function h of class C^2 such that $\langle v(y), \nabla h(y) \rangle > 0$, $y \in \partial D$. For example, we may define $h(x) := \text{dist}(x, \partial D)$ in a small neighborhood of ∂D . Applying the Itô formula to $h(Y(t))$, we get

$$h(Y(t)) \geq h(Y(0)) + M_h(t) - Kt,$$

where $K := \sup_y |\mathcal{A}h(y)|$, $M_h(t) := \int_0^t \sum_j \langle \nabla h(Y(s)), b_j(Y(s)) \rangle dw_j(s)$.

Let B be a Brownian motion in order to represent the martingale M_h :

$$M_h(t) = B \left(\int_0^t \sum_j \langle \nabla h(Y(s)), b_j(Y(s)) \rangle^2 ds \right).$$

Note that ∇h is nondegenerate in a neighborhood of ∂D . Then there is $\delta > 0$ such that

$$c_- := \inf_{\text{dist}(y, \partial D) \leq \delta} \sum_j \langle \nabla h(y), b_j(y) \rangle^2 > 0.$$

Set

$$c_+ := \sup_{y \in D} \sum_j \langle \nabla h(y), b_j(y) \rangle^2.$$

Then, for any $\delta_1 > 0$, $t_1 > 0$, and $c > 0$,

$$\begin{aligned} & \inf_{\text{dist}(y, \partial D) < \delta_1} \mathbf{P}_y \left(h(Y(t_1)) \geq c \right) \\ & \leq \mathbf{P} \left(\sup_{s \in [0, t_1]} B(c_+ s) + Kt_1 + \delta_1 \leq \delta, \quad \inf_{s \in [c_- t_1, c_+ t_1]} B(s) - Kt_1 \geq c \right). \end{aligned} \quad (1.9)$$

It is easy to see that

$$\mathbf{P} \left(\sup_{s \in [0, t_1]} B(c_+ s) + Kt_1 + \delta_1 \leq \delta, \quad \inf_{s \in [c_- t_1, c_+ t_1]} B(s) - Kt_1 > 0 \right) > 0$$

if t_1 and δ_1 are small enough. So, the right-hand side of (1.9) is positive for sufficiently small $c > 0$. This completes the proof of (1.7).

The measure π_Y has a density $p(y)$, $y \in \bar{D}$, where p is solution of the following PDE (see [14, Corollary 3.4]):

$$\mathcal{A}^*p(y) = 0, \quad y \in D,$$

with boundary condition

$$\begin{aligned} & -2p(y)\langle n(y), a(y) \rangle + (n(y))^T \sigma(y) \nabla p(y) + p(y)K(y) \\ & - \nabla \cdot (p(y)(n(y)^T) \sigma(y) n(y) v(y) - p(y) \sigma(y) n(y)) = 0, \end{aligned}$$

where the function K is given by the formula

$$K(y) = \left\langle n(y), \sum_j \frac{\partial \sigma_{\cdot j}(y)}{\partial y_j} \right\rangle = \sum_k n_k(y) \sum_j \frac{\partial \sigma_{kj}(y)}{\partial y_j}.$$

The main result of this section is the following theorem.

Theorem 1.1. *Let α , β , γ be continuous functions, and assume that assumption (1.6) holds. Then the process X solution of (1.4) grows a.s. asymptotically linearly with a determinist rate. More precisely,*

$$\lim_{t \rightarrow +\infty} \frac{X(t)}{t} = \mathbb{A}_1 + \mathbb{A}_2 := \int_D \alpha(y) p(y) dy + \frac{1}{2} \int_{\partial D} \gamma(y) p(y) n^T(y) \sigma(y) n(y) dS(y), \quad (1.10)$$

where p is the stationary density of Y and S is a surface measure on ∂D .

Remark 1.1. We use values of v and γ on ∂D only. However, for our purposes it is convenient to assume that v, γ are defined on the whole $\mathbb{R}^d$ and possess the corresponding smoothness. This is not a loss of generality since we always are able to select the corresponding extension.

Proof. Observe that for any $t_0 > 0$ a random variable $Y(t_0)$ has a continuous and positive density on $\bar{D}$. So, the distribution of a process $\{Y(t_0 + t), t \geq 0\}$ is equivalent to the distribution of Y with a stationary initial condition. So we can assume without loss of generality that $Y(0)$ has a stationary distribution and $\{Y(t), t \geq 0\}$ is a stationary process (see arguments of Theorem 20.2 [22]).

Let us denote integrals in (1.4) by $I_1(t)$, $I_2(t)$, $I_3(t)$, respectively.

We first have that

$$\mathbf{P} \left(\frac{I_1(t)}{t} = \frac{\int_0^t \alpha(Y(s)) ds}{t} \xrightarrow{t \rightarrow +\infty} \mathbf{E}_\pi \alpha(Y(0)) = \int_D \alpha(y) p(y) dy \right) = 1. \quad (1.11)$$

For the stationary version of $Y(t)$ this follows by the ergodic Birkhoff–Khinchin theorem and the fact that, because of the stabilization rate (1.7), $Y(t)$ is ϕ -mixing at exponential rate and this has its invariant σ -algebra degenerate (see, e.g., [27, Sections 1.3 and 5.1]). For nonstationary $Y(t)$ we can use the coupling construction from the appendix. Namely, for any $\tau > 0$ we use Lemma B.2 with $T = \infty$ and arbitrary, but fixed $\tau > 0$. Then (1.11) holds true for stationary $Y = Z_2$, and thus for nonstationary $Y = Z_1$ the probability in (1.11) is

$$\geq 1 - \frac{1}{2}\delta(\tau) \geq 1 - \frac{C}{2}e^{-\lambda\tau},$$

in the last inequality we have used (1.7). Taking $\tau \rightarrow \infty$, we get the required statement.

The process $I_2(t)$, $t \geq 0$, is a continuous martingale with quadratic variation

$$\langle I_2 \rangle(t) = \int_0^t \sum_{k=1}^m \beta_k^2(Y(s)) ds \leq \sup_y |\beta(y)|^2 t.$$

Then there exists a Brownian motion B defined on some extension of the probability space such that $I_2(t) = B(\langle I_2 \rangle(t))$. Therefore, by the law of iterated logarithm,

$$\lim_{t \rightarrow +\infty} \frac{|I_2(t)|}{t} \leq \lim_{t \rightarrow +\infty} \sup_{s \in [0, t]} \frac{|B(\sup_y |\beta(y)|^2 s)|}{t} = 0 \quad \text{a.s.}$$

Consider the process $I_3(t)$. Let us prove the theorem supposing v , γ sufficiently smooth, say C^∞ . Select a function $f \in C^2(\bar{D})$ such that

$$\nabla f(y)v(y) = \gamma(y), \quad y \in \partial D. \quad (1.12)$$

Remark 1.2. There are several possibilities for such a function f . Recall that we assume that v and γ are C^∞ -functions defined on the whole $\mathbb{R}^d$. Since vector field v is transversal to ∂D , there is a solution of the first order PDE $\nabla u(y)v(y) = \gamma(y)$ defined for y from a neighborhood U of ∂D (see [11]). Let V be an open set such that $\partial D \subset V \subset \bar{V} \subset U$, and $h \in C^\infty$ be such that $h(y) = 1$ for $y \in V$, $h(y) = 0$ for $y \notin U$. Then $f(y) = u(y)h(y)$ satisfies (1.12).

We recall a method of characteristics used for a construction of a solution $\nabla u v = \gamma$ because we will need to control smoothness of u in terms of smoothness of v and γ later. Let S be a C^n compact manifold, $\mathbf{n}(x)$ is a normal vector at $x \in S$. Assume that a vector field $\mathbf{v}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and a function $\gamma: \mathbb{R}^d \rightarrow \mathbb{R}$ are C^n , and $\inf_{x \in S} \langle \mathbf{n}(x), \mathbf{v}(x) \rangle > 0$ (cf. (1.3)). Denote by $X_x(t)$, $x \in S$, $t \in \mathbb{R}$, the solution to the ordinary differential equation

$$\frac{dX_x(t)}{dt} = \mathbf{v}(X_x(t)), \quad t \in \mathbb{R},$$

$$X_x(0) = x, \quad x \in S.$$

It is well-known that the mapping $(x, t) \rightarrow X_x(t)$ is C^n . Moreover, the transversality assumption $\inf_{x \in S} \langle \mathbf{n}(x), \mathbf{v}(x) \rangle > 0$ and compactness of S yield that for a small $\varepsilon > 0$ the map

$$S \times (-\varepsilon, \varepsilon) \ni (x, t) \rightarrow X_x(t)$$

is C^n -diffeomorphism. By U_ε denote the image $\{X_x(t) : (x, t) \in S \times (-\varepsilon, \varepsilon)\}$ and $(x(y), t(y))$, $y \in U_\varepsilon$ the inverse map. Observe that continuously differentiable function $\mathbf{u}: U_\varepsilon \rightarrow \mathbb{R}$ satisfies the equation $\nabla \mathbf{u}(y)\mathbf{v}(y) = \gamma(y)$, $y \in U_\varepsilon$, if and only if

$$\frac{\partial}{\partial t} (\mathbf{u}(X_x(t))) = \gamma(X_x(t)), \quad x \in S, \quad |t| < \varepsilon.$$

Hence, an example of C^n solution to $\nabla \mathbf{u}(y)\mathbf{v}(y) = \gamma(y)$, $y \in U_\varepsilon$ is the function $\mathbf{u}(y) := \int_0^{t(y)} \gamma(X_{x(y)}(s)) ds$.

Let us continue the proof of the theorem. By Itô's formula,

$$df(Y(t)) = \mathcal{A}f(Y(t))dt + \nabla f(Y(t))b_k(Y(t))dw_k(t) + \nabla f(Y(t))v(Y(t))dL(t),$$

where $\mathcal{A}f$ is given in (1.5), that is,

$$\begin{aligned} I_3(t) &= \int_0^t \gamma(Y(s))dL(s) = \int_0^t \nabla f(Y(s))v(Y(s))dL(s) \\ &= f(Y(t)) - f(Y(0)) - \int_0^t \mathcal{A}f(Y(s))ds - \int_0^t \nabla f(Y(s))b_k(Y(s))dw_k(s). \end{aligned}$$

Since $f \in C^2(\bar{D})$, by similar reasoning as for I_1 and I_2 , we get

$$\frac{I_3(t)}{t} \xrightarrow{t \rightarrow +\infty} -E_\pi \mathcal{A}f(Y(0)) \quad \text{a.s.} \quad (1.13)$$

Let us apply the divergence theorem to the right-hand side of (1.13), see calculations in [14, p. 10, 11]. The second formula on [14, p. 11] yields

$$\begin{aligned} \int_{\bar{D}} \mathcal{A}f(y)p(y)dy &= -\frac{1}{2} \int_{\partial D} \nabla f(y)v(y)p(y)(n(y))^T \sigma(y)n(y)dS(y) \\ &= -\frac{1}{2} \int_{\partial D} \gamma(y)p(y)(n(y))^T \sigma(y)n(y)dS(y). \end{aligned}$$

Remark 1.3. Our case is a particular case of [14]. In [14] it was assumed that the boundary ∂D is not necessarily smooth, but can be piecewise smooth. Hence, the set I from [14] contains only one element. Moreover, in our case the first item of the right hand-side in the second formula on [14, p. 11] equals 0.

Theorem 1.1 is proved if there exists $f \in C^2$ satisfying (1.12) (for example, if v, γ were smooth enough).

Consider the general case. Assume that γ is only continuous. Let $\varepsilon > 0$ be arbitrary. It is not difficult to find $f \in C^2$ such that

$$\nabla f(x)v(x) = \gamma_\varepsilon(x), \quad x \in \partial D, \quad \text{where} \quad \sup_{x \in \partial D} |\gamma_\varepsilon(x) - \gamma(x)| \leq \varepsilon,$$

apply, for example, arguments of Remark 1.2 for smoothing of v and γ . Then

$$\lim_{t \rightarrow \infty} \frac{\int_0^t \gamma_\varepsilon(Y(s))dL(s)}{t} = \frac{1}{2} \int_{\partial D} \gamma_\varepsilon(y)p(y)(n(y))^T \sigma(y)n(y)dS(y) \quad \text{a.s.}$$

So

$$\limsup_{t \rightarrow \infty} \left| \frac{\int_0^t \gamma(Y(s)) dL(s)}{t} - \frac{1}{2} \int_{\partial D} \gamma(y) p(y) (n(y))^T \sigma(y) n(y) dS(y) \right| \leq C_1 \varepsilon + \varepsilon \limsup_{t \rightarrow \infty} \frac{L(t)}{t} \quad \text{a.s.},$$

where C_1 is a constant, that is, independent of ε .

Applying reasoning above for $\gamma \equiv 1$ and selecting $f \in C^2$ such that $\nabla f(x)v(x) \geq \frac{1}{2}$, $x \in \partial D$, we get

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{L(t)}{t} &\leq 2 \limsup_{t \rightarrow \infty} \frac{\int_0^t \nabla f(X(s))v(X(s)) dL(s)}{t} \\ &= \int_{\partial D} \nabla f(y)v(y)p(y)(n(y))^T \sigma(y)n(y) dS(y) \leq 3 \int_{\partial D} p(y)(n(y))^T \sigma(y)n(y) dS(y) \quad \text{a.s.} \end{aligned}$$

Hence, there is a constant C_2 such that, for any $\varepsilon > 0$, we have

$$\limsup_{t \rightarrow \infty} \left| \frac{\int_0^t \gamma(Y(s)) dL(s)}{t} - \frac{1}{2} \int_{\partial D} \gamma(y) p(y) (n(y))^T \sigma(y) n(y) dS(y) \right| \leq C_2 \varepsilon \quad \text{a.s.}$$

Theorem 1.1 is proved.

2. Small perturbations of homogeneous SDEs in a cylinder. Let $X(t), Y(t)$ be given by (1.1)–(1.4). Denote by $X_\varepsilon(t), Y_\varepsilon(t)$, $t \geq 0$, processes satisfying the following perturbed SDE reflected in a cylinder:

$$\begin{aligned} dX_\varepsilon(t) &= \alpha_\varepsilon(X_\varepsilon(t), Y_\varepsilon(t))dt + \beta_\varepsilon(X_\varepsilon(t), Y_\varepsilon(t))dw(t) + \gamma_\varepsilon(X_\varepsilon(t), Y_\varepsilon(t))dL_\varepsilon(t), \\ dY_\varepsilon(t) &= a_\varepsilon(X_\varepsilon(t), Y_\varepsilon(t))dt + b_\varepsilon(X_\varepsilon(t), Y_\varepsilon(t))dw(t) + v_\varepsilon(X_\varepsilon(t), Y_\varepsilon(t))dL_\varepsilon(t), \end{aligned} \quad (2.1)$$

where $Y_\varepsilon(t) \in \overline{D} \subset \mathbb{R}^d$, $X_\varepsilon(t)$ is a real-valued process, $L_\varepsilon(t)$ is a continuous nondecreasing $\mathcal{F}_t$ -adapted process such that $L_\varepsilon(0) = 0$ and

$$L_\varepsilon(t) = \int_0^t \mathbf{1}_{Y_\varepsilon(s) \in \partial D} dL_\varepsilon(s), \quad t \geq 0.$$

Here, $w = (w_1, \dots, w_n)^T$ is an n -dimensional Brownian motion, $\beta_\varepsilon = (\beta_{\varepsilon,1}, \dots, \beta_{\varepsilon,n})$, $b_\varepsilon = (b_{\varepsilon,1}, \dots, b_{\varepsilon,n})$.

We assume that $\langle v_\varepsilon(x, y), n(y) \rangle = 1$, $y \in \partial D$, and functions $\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, a_\varepsilon, b_\varepsilon, v_\varepsilon$ are such that a weak solution to (2.1) exists (for example, all coefficients are continuous).

Assume that, for all x, y ,

$$|\alpha_\varepsilon(x, y) - \alpha(y)| < \varepsilon, \dots, |v_\varepsilon(x, y) - v(y)| < \varepsilon, \quad (2.2)$$

where $\alpha, \dots, v$ satisfy assumptions of Section 1.

Let f be a Lipschitz function. The aim of this section is to find deterministic bounds for

$$\overline{\lim}_{t \rightarrow \infty} \left| \frac{1}{t} \int_0^t f(Y_\varepsilon(s)) ds - \mathbb{A}_1(f) \right| \quad \text{and} \quad \overline{\lim}_{t \rightarrow \infty} \left| \frac{1}{t} \int_0^t f(Y_\varepsilon(s)) dL_\varepsilon(s) - \mathbb{A}_2(f) \right|,$$

where $\mathbb{A}_1(f)$ and $\mathbb{A}_2(f)$ are from Theorem 1.1 (see (1.10)):

$$\mathbb{A}_1(f) = \int_D f(y) p(y) dy, \quad \mathbb{A}_2(f) = \frac{1}{2} \int_{\partial D} f(y) p(y) n^T(y) \sigma(y) n(y) dS(y).$$

Theorem 2.1. Assume that the coefficients of (2.1) are Lipschitz continuous and (2.2), (1.6) hold true. Then, for any continuous function f ,

$$\lim_{\varepsilon \rightarrow 0} \overline{\lim}_{t \rightarrow \infty} \left| t^{-1} \int_0^t f(Y_\varepsilon(s)) ds - \mathbb{A}_1(f) \right| = 0 \quad a.s.$$

Moreover, if f is Lipschitz continuous function, then there exists a constant C , depending only on $\|f\|_\infty = \sup_y |f(y)|$, the Lipschitz constant of f , and the coefficients of (1.1) such that

$$\overline{\lim}_{t \rightarrow \infty} \left| t^{-1} \int_0^t f(Y_\varepsilon(s)) ds - \mathbb{A}_1(f) \right| \leq \frac{C \log(\log \varepsilon^{-1})}{\log \varepsilon^{-1}} \quad a.s. \quad (2.3)$$

Theorem 2.2. Assume that coefficients of (2.1) are Lipschitz continuous and (2.2), (1.6) hold true. Then, for any continuous function g defined on ∂D ,

$$\lim_{\varepsilon \rightarrow 0} \overline{\lim}_{t \rightarrow \infty} \left| t^{-1} \int_0^t g(Y_\varepsilon(s)) dL_\varepsilon(s) - \mathbb{A}_2(g) \right| = 0 \quad a.s. \quad (2.4)$$

If $v, g \in C^3$, then

$$\overline{\lim}_{t \rightarrow \infty} \left| t^{-1} \int_0^t g(Y_\varepsilon(s)) ds - \mathbb{A}_2(g) \right| \leq \frac{C \log(\log \varepsilon^{-1})}{\log \varepsilon^{-1}} \quad a.s., \quad (2.5)$$

where the constant C depends only on the quantity $\sum_{k=0}^3 (\|\nabla^k v\|_\infty + \|\nabla^k g\|_\infty)$ and on the coefficients of (1.1).

Proof of Theorem 2.1. It is sufficient to prove (2.3) only.

Observe that, for any functions y_1 and y_2 ,

$$\begin{aligned} \forall \delta > 0: \left| \int_0^t f(y_1(s)) ds - \int_0^t f(y_2(s)) ds \right| &\leq 2\|f\|_\infty \int_0^t \mathbf{1}_{|y_1(s) - y_2(s)| > \delta} ds \\ &\quad + \int_0^t |f(y_1(s)) - f(y_2(s))| \mathbf{1}_{|y_1(s) - y_2(s)| \leq \delta} ds, \end{aligned}$$

where $\|f\|_\infty = \sup_y |f(y)|$.

So, for any bounded Lipschitz function f with Lipschitz constant L ,

$$\overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \left| \int_0^t (f(y_1(s)) - f(y_2(s))) ds \right| \leq 2\|f\|_\infty \overline{\lim}_{t \rightarrow \infty} \frac{\int_0^t \mathbf{1}_{|y_1(s) - y_2(s)| \geq \delta} ds}{t} + L\delta. \quad (2.6)$$

Let $\tau > 0$ be a constant. By $\bar{Y}$ denote the càdlàg process (depending on ε and τ) given by $\bar{Y}(k\tau) = Y_\varepsilon(k\tau)$, $k \geq 0$; $\bar{Y}$ satisfies (1.1), (1.2) on the time interval $[k\tau, (k+1)\tau)$.

It follows from (2.6) that

$$\begin{aligned} & \overline{\lim}_{t \rightarrow \infty} t^{-1} \left| \int_0^t (f(Y_\varepsilon(s)) - f(\bar{Y}(s))) ds \right| \\ & \leq L\delta + \overline{\lim}_{n \rightarrow \infty} \frac{2\|f\|_\infty \sum_{k=0}^{n-1} \mathbf{1}_{\{\sup_{s \in [k\tau, (k+1)\tau]} |Y_\varepsilon(s) - \bar{Y}(s)| \geq \delta\}}}{n}. \end{aligned} \quad (2.7)$$

Lemma 2.1. Let $\mathcal{G}_k$, $k \geq 0$, be an increasing sequence of σ -algebras and take for any k a $\mathcal{G}_k$ -measurable random variable ξ_k . Assume that

- 1) $\mathbf{E}(\xi_{k+1} | \mathcal{G}_k) = 0$, $k \geq 0$;
- 2) $|\xi_k| \leq C$, $k \geq 1$, where C is a fixed constant.

Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \xi_k = 0 \quad a.s.$$

The proof follows from the strong law of large numbers for martingales (see, for example, Theorem 8b in Chapter II, §3 of [13]).

Corollary 2.1. Assume that a sequence of random variables $(\eta_k)_k$ satisfies assumption 2 of Lemma 2.1. Then

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \eta_k = \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mathbf{E}(\eta_k | \mathcal{G}_{k-1}) \quad a.s. \quad (2.8)$$

It follows from (2.8) that the right-hand side of (2.7) does not exceed

$$\begin{aligned} & L\delta + 2\|f\|_\infty \overline{\lim}_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} \mathbf{E}(\mathbf{1}_{\{\sup_{s \in [k\tau, (k+1)\tau]} |Y_\varepsilon(s) - \bar{Y}(s)| \geq \delta\}} | \mathcal{F}_{k\tau})}{n} \\ & \leq L\delta + 2\|f\|_\infty \overline{\lim}_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} \mathbf{E} \left(\sup_{s \in [k\tau, (k+1)\tau]} |Y_\varepsilon(s) - \bar{Y}(s)|^2 | \mathcal{F}_{k\tau} \right)}{\delta^2 n}. \end{aligned} \quad (2.9)$$

Lemma 2.2. Assume that Y and Y_ε are solutions on the interval $[s, s + \tau]$ of (1.1) and (2.1), respectively, with the same initial condition $Y_\varepsilon(s) = \bar{Y}(s) = \xi$, where ξ is $\mathcal{F}_s$ -measurable.

There exists a constant $c > 0$ depending only on a, b, v such that, for all sufficiently small $\varepsilon > 0$,

$$\forall s \quad \forall \xi \in \mathcal{F}_s : \mathbf{E} \left(\sup_{z \in [s, s+\tau]} |Y_\varepsilon(z) - Y(z)|^2 | \mathcal{F}_s \right) \leq c\varepsilon e^{c\tau} \quad a.s.$$

The proof is done in Appendix A.

It follows from (2.9) and Lemma 2.2 that

$$\overline{\lim}_{t \rightarrow \infty} t^{-1} \left| \int_0^t \left(f(Y_\varepsilon(s)) - f(\bar{Y}(s)) \right) ds \right| \leq L\delta + \frac{2c\|f\|_\infty \varepsilon e^{c\tau}}{\delta^2} \quad \text{a.s.}$$

Now we are going to construct a copy of Y that is close to the process $\bar{Y}$. Let $\tau_1 \in (0, \tau)$ be fixed. We can use iteratively Lemma B.2 in order to construct on certain enlargement of initial probability space, a process $\{\tilde{Y}(t), t \geq 0\}$ and a sequence of σ -algebras $\mathcal{G}_{k\tau+\tau_1}$, $k \geq 0$, such that

- 1) $\{\tilde{Y}(t), t \geq 0\} \stackrel{d}{=} \{Y(t), t \geq 0\}$;
- 2) $\mathcal{F}_{k\tau+\tau_1} \subset \mathcal{G}_{k\tau+\tau_1}$, $\mathcal{G}_{k\tau+\tau_1} \perp\!\!\!\perp \sigma(w(k\tau + \tau_1 + u) - w(k\tau + \tau_1), u \geq 0)$, $k \geq 0$;
- 3) $\mathbf{P}(\tilde{Y}(k\tau + \tau_1) \neq \bar{Y}(k\tau + \tau_1) \mid \mathcal{G}_{k\tau+\tau_1}) \leq Ce^{-\lambda\tau_1}$ a.s., $k \geq 0$;
- 4) $\mathbf{P}(\tilde{Y}(t) = \bar{Y}(t), t \in [k\tau + \tau_1, (k+1)\tau) \mid \tilde{Y}(k\tau + \tau_1) = \bar{Y}(k\tau + \tau_1)) = 1$.

Let us estimate how close the integral functionals of $\tilde{Y}$ and $\bar{Y}$:

$$\begin{aligned} & \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \left| \int_0^t f(\bar{Y}(s)) ds - \int_0^t f(\tilde{Y}(s)) ds \right| \\ & \leq \overline{\lim}_{n \rightarrow \infty} \frac{1}{n\tau} \sum_{k=0}^{n-1} \int_{k\tau}^{(k+1)\tau} |f(\bar{Y}(s)) - f(\tilde{Y}(s))| ds \\ & \leq 2\|f\|_\infty \left(\frac{\tau_1}{\tau} + \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{\bar{Y}(k\tau+\tau_1) \neq \tilde{Y}(k\tau+\tau_1)} \right). \end{aligned} \quad (2.10)$$

It follows from Corollary 2.1 and from the property 3 above that

$$\begin{aligned} & \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{\bar{Y}(k\tau+\tau_1) \neq \tilde{Y}(k\tau+\tau_1)} \\ & = \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{E}(\mathbf{1}_{\bar{Y}(k\tau+\tau_1) \neq \tilde{Y}(k\tau+\tau_1)} \mid \mathcal{G}_{k\tau+\tau_1}) \leq Ce^{-\lambda\tau_1} \quad \text{a.s.} \end{aligned} \quad (2.11)$$

Since $\tilde{Y} \stackrel{d}{=} Y$, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\tilde{Y}(s)) ds = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(Y(s)) ds = \mathbb{A}_1(f) \quad \text{a.s.} \quad (2.12)$$

It follows from (2.7), (2.9)–(2.12) that

$$\overline{\lim}_{t \rightarrow \infty} \left| \frac{1}{t} \int_0^t f(Y_\varepsilon(s)) ds - \mathbb{A}_1(f) \right|$$

$$\leq L\delta + \frac{2c\|f\|_{\infty}\varepsilon e^{c\tau}}{\delta^2} + \frac{2\|f\|_{\infty}\tau_1}{\tau} + 2\|f\|_{\infty}Ce^{-\lambda\tau_1} \quad \text{a.s.} \quad (2.13)$$

Let us find bounds for this expression minimizing subsequently on τ_1 , δ , and τ . We will always assume that $\delta > 0$ and $\varepsilon > 0$ are small enough, τ is large enough, etc.

Select $\tau_1^* := \frac{1}{\lambda} \log(\lambda\tau^*)$, $\delta^* = \left(\frac{4c\|f\|_{\infty}\varepsilon e^{c\tau^*}}{L}\right)^{1/3}$, $\tau^* = c^{-1} \log(\varepsilon^{-\beta})$, where $\beta \in \left(0, \frac{1}{3}\right)$ is an arbitrary fixed number.

Then the right-hand side of (2.13) does not exceed

$$\begin{aligned} L\delta^* + \frac{2c\|f\|_{\infty}\varepsilon e^{c\tau^*}}{(\delta^*)^2} + \frac{2\|f\|_{\infty}\tau_1^*}{\tau^*} + 2\|f\|_{\infty}Ce^{-\lambda\tau_1^*} \\ \leq \text{const} \left(\varepsilon^{1/3} \cdot \varepsilon^{-\beta} + \frac{\log(\log \varepsilon^{-\beta})}{\log \varepsilon^{-\beta}} \right) = O\left(\frac{\log(\log \varepsilon^{-1})}{\log(\varepsilon^{-1})}\right), \quad \varepsilon \rightarrow 0+. \end{aligned}$$

The estimate above implies (2.3).

Theorem 2.1 is proved.

Proof of Theorem 2.2. Let $f \in C^2(\overline{D})$. By Itô's formula,

$$\begin{aligned} df(Y_{\varepsilon}(t)) &= (\mathcal{A}f(Y_{\varepsilon}(t)) + \varepsilon_1^f(t))dt \\ &\quad + dM_{\varepsilon}^f(t) + \nabla f(Y_{\varepsilon}(t))(v(Y_{\varepsilon}(t)) + \varepsilon_2^f(t))dL_{\varepsilon}(t), \end{aligned} \quad (2.14)$$

where

$$M_{\varepsilon}^f(t) = \int_0^t \nabla f(Y_{\varepsilon}(s))b_{\varepsilon}(X_{\varepsilon}(s), Y_{\varepsilon}(s))dw(s),$$

$\{\varepsilon_i^f(t), t \geq 0\}$, $i = 1, 2$, are some $\mathcal{F}_t$ -adapted processes such that $|\varepsilon_i^f(t)| \leq C\varepsilon$, and the constant $C = C(f)$ does not depend on $t \geq 0$.

As in the previous section we get convergence of the martingale term $\lim_{t \rightarrow +\infty} \frac{M_{\varepsilon}^f(t)}{t} = 0$ a.s. because the integrand $\nabla f(Y_{\varepsilon}(s))b_{\varepsilon}(X_{\varepsilon}(t), Y_{\varepsilon}(t))$ is bounded.

Select $\tilde{f} \in C^{\infty}$ such that (cf. the proof of Theorem 1.1 and Remark 1.2)

$$\nabla \tilde{f}(y)v(y) \in [1, 2], \quad y \in \partial D.$$

Then it follows from (2.14) that, for some constants c_i and all sufficiently small $\varepsilon > 0$, we have

$$L_{\varepsilon}(t) \leq c_1 + c_2t + |M_{\tilde{f}}(t)|.$$

So, there exists $c_3 > 0$ such that, for all sufficiently small $\varepsilon > 0$,

$$\overline{\lim}_{t \rightarrow \infty} \frac{L_{\varepsilon}(t)}{t} \leq c_3 \quad \text{a.s.}$$

This yields, in particular, that

$$\overline{\lim}_{t \rightarrow \infty} t^{-1} \int_0^t |\nabla f(Y_{\varepsilon}(t))\varepsilon_2^f(t)|dL_{\varepsilon}(t) \leq c_3 \|\nabla f\|_{\infty} \varepsilon \quad \text{a.s.} \quad (2.15)$$

If $v, g \in C^3$, then we may select $f \in C^3$ such that $\nabla f(y)v(y) = g(y)$, $y \in \partial D$ (see Remark 1.2). The proof of (2.5) follows from (2.14), (2.15), and (2.3).

Assume now that g is only continuous. Fix $\delta > 0$. Let us select $f \in C^3$ (see Remark 1.2) such that

$$|\nabla f(y)v(y) - g(y)| \leq \delta, \quad y \in \partial D.$$

Denote $g_\delta(y) := \nabla f(y)v(y)$. Therefore

$$\begin{aligned} & \lim_{t \rightarrow \infty} \left| \frac{\int_0^t g(Y_\varepsilon(s)) dL_\varepsilon(s)}{t} - \mathbb{A}_2(g) \right| \\ & \leq \lim_{t \rightarrow \infty} \left| \frac{\int_0^t (g(Y_\varepsilon(s)) - g_\delta(Y_\varepsilon(s))) dL_\varepsilon(s)}{t} \right| \\ & \quad + \lim_{t \rightarrow \infty} \left| \frac{\int_0^t g_\delta(Y_\varepsilon(s)) dL_\varepsilon(s)}{t} - \mathbb{A}_2(g_\delta) \right| + |\mathbb{A}_2(g) - \mathbb{A}_2(g_\delta)| \\ & \leq \delta \lim_{t \rightarrow \infty} \frac{L_\varepsilon(t)}{t} + C(\delta) \frac{\log(\log \varepsilon^{-1})}{\log \varepsilon^{-1}} + |\mathbb{A}_2(g) - \mathbb{A}_2(g_\delta)| \\ & \leq c_3 \delta + C(\delta) \frac{\log(\log \varepsilon^{-1})}{\log \varepsilon^{-1}} + c_4 \delta = c_5 \delta + C(\delta) \frac{\log(\log \varepsilon^{-1})}{\log \varepsilon^{-1}} \quad \text{a.s.,} \end{aligned} \quad (2.16)$$

where the constant c_5 is independent of ε and δ .

Since $\delta > 0$ is arbitrary, inequality (2.16) yields (2.4).

Theorem 2.2 is proved.

3. Reflecting SDEs in cones and wedges. Let K be an open cone in $\mathbb{R}^d$ with boundary ∂K and a vertex at 0. Assume that the intersection of K and the unit sphere S_{d-1} is an open (in S_{d-1}) set D with smooth boundary ∂D .

Consider the following SDE with reflection at the boundary of K :

$$dX(t) = A(X(t))dt + \sum_k B_k(X(t))dw_k(t) + C(X(t))dL(t), \quad (3.1)$$

where $X(t) \in \overline{K}$, $L(t)$ is continuous, adapted, nondecreasing process,

$$L(0) = 0, \quad L(t) = \int_0^t \mathbf{1}_{X(s) \in \partial K} dL(s).$$

We assume that $\langle C(x), n(x) \rangle = 1$, $x \in \partial K$, where $n(x)$ is the inner normal vector at $x \in \partial K$.

Suppose also that functions A , B_k , C are locally Lipschitz continuous. Then there exists a unique strong solution of (3.1), defined up to the minimum of τ_0 , the hitting time of 0 or τ_∞ , the blow-up time.

Consider

$$\mathcal{A}f(x) := \sum_{i=1}^d A_i(x) \frac{\partial f(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d \left(\sum_k B_{k,i}(x) B_{k,j}(x) \right) \frac{\partial^2 f(x)}{\partial x_i \partial x_j},$$

$$\mathcal{L}f(x) := \sum_i C_i(x) \frac{\partial f(x)}{\partial x_i}.$$

For $x \in \mathbb{R}^d \setminus \{0\}$ define its polar coordinates by $r := |x|$ and φ , where φ is a some (smooth) parametrization of $D \subset S_{d-1}$. (The domain of φ will be also denoted by D .) One also define $\rho := \log r$.

We say that equation (3.1) is a model equation if operators $\mathcal{A}$ and $\mathcal{L}$ in the polar coordinates are of the form

$$\mathcal{A}f = r^{-2} \left(P_0(\varphi) \left(r \frac{\partial}{\partial r} \right)^2 + P_1(\varphi) \nabla_\varphi \left(r \frac{\partial}{\partial r} \right) + P_2(\varphi) \nabla_\varphi^2 + P_3(\varphi) \nabla_\varphi + P_4(\varphi) \left(r \frac{\partial}{\partial r} \right) \right),$$

$$\mathcal{L}f = r^{-1} \left(Q_0(\varphi) \left(r \frac{\partial}{\partial r} \right) + Q_1(\varphi) \nabla_\varphi \right),$$
(3.2)

where P_i (resp., Q_i) are functions on D (resp., ∂D).

Remark 3.1. If equation (3.1) is the model equation for some parametrization of D , then it is the model for any parametrization.

Remark 3.2. In PDE theory the pair $(\mathcal{A}, \mathcal{L})$ is sometimes called the model differential operator (cf. [21, Chapter 3, § 5.1]).

Remark 3.3. The operator $\mathcal{L}$ has a representation (3.2) if and only if $C(x) = C(x/|x|) = C(\varphi)$. Let us assume that the equation is the model equation. Make a change of variables and write the SDE for the pair of processes $\rho(t) = \log r(t)$ and $\varphi(t)$. Note the image of a cone K is a cylinder $\mathbb{R} \times D$, the vertex of K corresponds to $\{-\infty\} \times D$, and $\lim_{t \rightarrow \tau_0-} X(t) = 0 \Leftrightarrow \lim_{t \rightarrow \tau_0-} \rho(t) = -\infty$.

Assume that the operator $\mathcal{A}$ satisfies a strong ellipticity condition. Since the coefficients of the initial equation are Lipschitz continuous, there are (see, for example, [12]) Lipschitz functions $\alpha = \alpha(\varphi)$, $\beta_k = \beta_k(\varphi)$, $\gamma = \gamma(\varphi)$, $a = a(\varphi)$, $b_k = b_k(\varphi)$, $c = c(\varphi)$ and independent Wiener processes $\{\bar{w}_k(t), t \geq 0\}$ such that

$$d\rho(t) = e^{-2\rho(t)} \alpha(\varphi(t)) dt + e^{-\rho(t)} \beta_k(\varphi(t)) d\bar{w}_k(t) + e^{-2\rho(t)} \gamma(\varphi(t)) d\bar{L}(t),$$
(3.3)

$$d\varphi(t) = e^{-2\rho(t)} a(\varphi(t)) dt + e^{-\rho(t)} b_k(\varphi(t)) d\bar{w}_k(t) + e^{-2\rho(t)} c(\varphi(t)) d\bar{L}(t),$$
(3.4)

where $\bar{L}(t)$ is continuous, nondecreasing, adapted process, and $\bar{L}(0) = 0$, $\bar{L}(t) = \int_0^t \mathbf{1}_{\varphi(s) \in \partial D} d\bar{L}(s)$. This functions $\alpha, \dots, c$ naturally appears from the Itô formula. For example, $\alpha = P_4$, $a = P_3$, etc.

Without loss of generality we may assume that

$$\langle c(\varphi), n_D(\varphi) \rangle = 1, \quad \varphi \in \partial D,$$

where $n_D(\varphi)$ is the inner normal of D .

Make the change of time

$$\tilde{\rho}(t) = \rho(A_t), \quad \tilde{\varphi}(t) = \varphi(A_t), \quad \text{where} \quad A_t := \inf \left\{ s \geq 0 : \int_0^s e^{-2\rho(z)} dz = t \right\}.$$

The process $(\tilde{\rho}(t), \tilde{\varphi}(t))$ satisfies then a reflected SDE in a cylinder with a new Wiener processes $\tilde{w}_k(t)$ and a local time process $\tilde{L}(t)$:

$$d\tilde{\rho}(t) = \alpha(\tilde{\varphi}(t))dt + \beta_k(\tilde{\varphi}(t))d\tilde{w}_k(t) + \gamma(\tilde{\varphi}(t))d\tilde{L}(t), \quad (3.5)$$

$$d\tilde{\varphi}(t) = a(\tilde{\varphi}(t))dt + b_k(\tilde{\varphi}(t))d\tilde{w}_k(t) + c(\tilde{\varphi}(t))d\tilde{L}(t). \quad (3.6)$$

It follows from Theorem 1.1 that the following limit $\mathbb{A} = \lim_{t \rightarrow \infty} \frac{\tilde{\rho}(t)}{t}$ exists a.s., is finite and determinist.

Theorem 3.1. 1. If $\mathbb{A} < 0$, then $\mathbf{P}(\tau_0 < +\infty) = 1$.

2. If $\mathbb{A} > 0$, then $\mathbf{P}(\tau_0 = \tau_\infty = +\infty) = 1$.

Proof. The inverse change of time is $t \mapsto \inf \left\{ s \geq 0 : \int_0^s e^{2\tilde{\rho}(z)} dz = t \right\}$.

Assume that $\mathbb{A} < 0$. Then $\lim_{t \rightarrow \infty} \tilde{\rho}(t) = -\infty$ a.s. Hence $\lim_{t \rightarrow \tau_0-} \rho(t) = 0$ and $\tau_\infty = \infty$ a.s. Since $\tilde{\rho}(t) \sim_t \mathbb{A}t$ as $t \rightarrow +\infty$, the integral $\int_0^\infty e^{2\tilde{\rho}(s)} ds$ is finite a.s. Therefore, τ_0 is finite a.s.

If $\mathbb{A} > 0$, then $\lim_{t \rightarrow \tau_\infty-} \tilde{\rho}(t) = +\infty$ and $\int_0^\infty e^{2\tilde{\rho}(s)} ds = +\infty$. So, $\mathbf{P}(\tau_0 = \infty) = \mathbf{P}(\tau_\infty = \infty) = 1$.

Remark 3.4. It can be shown that if $\mathbb{A} = 0$, then a process $\tilde{\rho}(t)$ is oscillating between $-\infty$ and ∞ , but we do not consider this specific case.

Example 3.1. Let $X(t)$ be a reflected Brownian motion in a cone K solution of the RSDE

$$dX(t) = dW(t) + C(X(t))dL(t),$$

where $W(t)$, $t \geq 0$ is a standard $\mathbb{R}^d$ -valued Brownian motion.

Recall that we always assume that the function C satisfies the condition $\langle C(x), n(x) \rangle = 1$, $x \in \partial K$, where $n(x)$ is the inner normal vector at the point x . Moreover, we assume that C is Lipschitz continuous and only depends on φ the direction of the vector x , i.e., $C(x) = C(x/|x|) = C(\varphi)$. Decompose $C(\varphi)$ in a sum of two orthogonal vectors

$$C(\varphi) = c_S(\varphi) + \gamma(\varphi)\varphi,$$

where $c_S(\varphi)$ belong to the tangent space $T_{S_{d-1}}(\varphi)$ and γ is a real-valued function.

The Laplace operator has the following representation in polar coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + (d-1)r^{-1}\frac{\partial}{\partial r} + r^{-2}\Delta_{d-1} = r^{-2} \left(\left(r \frac{\partial}{\partial r} \right)^2 + (d-2) \left(r \frac{\partial}{\partial r} \right) + \Delta_{d-1} \right), \quad (3.7)$$

where Δ_{d-1} is the Laplace–Beltrami operator on S_{d-1} .

Then equation (3.5) is the model equation and time-changed equation in log-polar coordinates, and RSDEs (3.5), (3.6) are of the form (compare with (3.7))

$$d\tilde{\rho}(t) = d\tilde{w}_\rho(t) + \frac{d-2}{2}dt + \gamma(\tilde{\varphi}(t))d\tilde{L}(t),$$

$$d\tilde{\varphi}(t) = d\tilde{w}_\varphi(t) + c_S(\tilde{\varphi}(t))d\tilde{L}(t),$$

where $\tilde{w}_\rho$ is a one-dimensional Brownian motion, $\tilde{w}_\varphi$ is a Brownian motion on a sphere S_{d-1} , $\tilde{w}_\rho$ and $\tilde{w}_\varphi$ are independent.

Remark 3.5. In Theorem 1.1 we considered a stochastic differential equation with reflection in a subset of Euclidean space. Nothing changes for an equation with reflection in a compact manifold with smooth boundary.

Let p be the invariant density for the process $\tilde{\varphi}$. Then the constant $\mathbb{A}_1$ from Theorem 1.1 calculated for the process $X(t) := \tilde{\rho}(t)$ equals $(d-2)/2$ and $\mathbb{A}_2$ equals $1/2 \int_{\partial D} \gamma(\varphi)p(y)dS(\varphi)$, where S is a surface measure.

Remark 3.6. A problem of vertex accessibility was considered in [16]. One can verify that the function ψ_0^* defined in Lemma 2.5 of [16] is equal to the invariant density p (compare carefully with the corresponding expression in Section 1) and that condition $\mathbb{A} = \mathbb{A}_1 + \mathbb{A}_2 < 0$ is equivalent to the condition of vertex accessibility $\alpha < 0$ obtained in Theorem 2.2 [16]. Note that, if the result of [16] was obtained by using an elegant machinery of functional analysis, our approach has more direct probabilistic interpretation.

Simpler formula for $\mathbb{A} < 0$ is obtained if $c_S(\varphi) = n(\varphi)$ is the inner normal vector at $\varphi \in \partial D$. Then the invariant distribution of $\tilde{\varphi}(t)$ is the uniform distribution on D . Hence the constant $\mathbb{A}$ from Theorem 1.1 equals

$$\mathbb{A} = \frac{1}{2|D|} \left((d-2)|D| + \int_{\partial D} \gamma(\varphi)dS(\varphi) \right).$$

The condition $\mathbb{A} < 0$ of hitting the vertex is equivalent to

$$\frac{\int_{\partial D} \gamma(\varphi)dS(\varphi)}{|D|} < 2 - d.$$

This formula coincides with the one in [17, p. 357].

Example 3.2. Consider the previous example in two-dimensional case, $d = 2$. Then K is a wedge. Without loss of generality we may assume that

$$K = \{x \in \mathbb{R}^2 : 0 < \varphi < \xi\},$$

where $\xi \in (0, 2\pi)$ is fixed. Now $D = (0, \xi)$, $\partial D = \{0; \xi\}$.

Define rays $l_1 = \{\varphi = 0\}$, $l_2 = \{\varphi = \xi\}$, and denote $C(0)$ by c_1 , $C(\xi)$ by c_2 . So, the equation for $X(t)$ is of the form

$$dX(t) = dW(t) + c_1 dL_1(t) + c_2 dL_2(t), \quad (3.8)$$

where the process $L_k(t)$ may increase only when $X(t) \in l_k$, $k = 1, 2$.

Let n_1 and n_2 be the inner normals for D at rays l_1 , l_2 , respectively. Recall that we assume

$$\langle c_1, n_1 \rangle = \langle c_2, n_2 \rangle = 1.$$

Denote $e_1 = (1, 0)$, $e_2 = (\cos \xi, \sin \xi)$ the direction vectors of rays l_1 and l_2 . Then $\gamma(0) = \langle c_2, e_1 \rangle$, $\gamma(\xi) = \langle c_2, e_2 \rangle$, the surface measure is a counting measure, and the condition $\mathbb{A} < 0$ that ensures hitting of the vertex is equivalent to

$$\langle c_1, e_1 \rangle + \langle c_2, e_2 \rangle < 0. \quad (3.9)$$

This formula coincides with the criterion obtained by Varadhan and Williams in [15].

Example 3.3. Let K be as in Example 3.2 and $X(t)$ be the solution to (3.8), where $W(t) = (W_1(t), W_2(t))$ is a Wiener process with correlated coordinates:

$$\langle W \rangle(t) = Bt,$$

for some symmetric, positive definite 2x2 matrix B .

Let us make a linear change of coordinates

$$\bar{X}(t) = AX(t),$$

where $A = \sqrt{B^{-1}}$. Then $\bar{X}(t)$ is a reflected Brownian motion in a wedge, where the driving noise $\bar{W}(t) = AW(t)$ is a standard Brownian motion. Direction vectors of the new rays are $\bar{e}_k = \frac{Ae_k}{|Ae_k|}$, $k = 1, 2$. The new reflecting vectors are

$$\bar{c}_k = \frac{Ac_k}{|Ac_k - \langle Ac_k, \bar{e}_k \rangle \bar{e}_k|}, \quad k = 1, 2.$$

So (3.9) is of the form

$$\sum_{k=1}^2 \frac{\langle Ac_k, Ae_k \rangle}{|Ac_k - \langle Ac_k, \bar{e}_k \rangle \bar{e}_k| |A\bar{e}_k|} < 0.$$

Consider now a reflected SDE in a multidimensional cone K depending on a small parameter ε :

$$dX_\varepsilon(t) = A^{(\varepsilon)}(X_\varepsilon(t))dt + \sum_k B_k^{(\varepsilon)}(X_\varepsilon(t))dw_k(t) + C^{(\varepsilon)}(X_\varepsilon(t))dL_\varepsilon(t)$$

and assume that its coefficients are small perturbations of coefficients of a model equation.

Let us write the associate equations for log-polar coordinates (cf. (3.3), (3.4)):

$$\begin{aligned} d\rho_\varepsilon(t) &= e^{-2\rho_\varepsilon(t)} \alpha^{(\varepsilon)}(\rho_\varepsilon(t), \varphi_\varepsilon(t))dt \\ &\quad + e^{-\rho_\varepsilon(t)} \beta_k^{(\varepsilon)}(\rho_\varepsilon(t), \varphi_\varepsilon(t))d\bar{w}_k(t) + e^{-2\rho_\varepsilon(t)} \gamma^{(\varepsilon)}(\rho_\varepsilon(t), \varphi_\varepsilon(t))d\bar{L}(t), \\ d\varphi_\varepsilon(t) &= e^{-2\rho_\varepsilon(t)} a^{(\varepsilon)}(\rho_\varepsilon(t), \varphi_\varepsilon(t))dt \\ &\quad + e^{-\rho_\varepsilon(t)} b_k^{(\varepsilon)}(\rho_\varepsilon(t), \varphi_\varepsilon(t))d\bar{w}_k(t) + e^{-2\rho_\varepsilon(t)} c^{(\varepsilon)}(\rho_\varepsilon(t), \varphi_\varepsilon(t))d\bar{L}(t). \end{aligned}$$

Assume that, uniformly in $x \in \bar{K}$,

$$\begin{aligned} |\alpha^{(\varepsilon)}(\rho, \varphi) - \alpha(\varphi)| &< \varepsilon, & |\beta_k^{(\varepsilon)}(\rho, \varphi) - \beta_k(\varphi)| &< \varepsilon, \\ |a^{(\varepsilon)}(\rho, \varphi) - a(\varphi)| &< \varepsilon, & |b_k^{(\varepsilon)}(\rho, \varphi) - b_k(\varphi)| &< \varepsilon, \\ |\gamma^{(\varepsilon)}(\rho, \varphi) - \gamma(\varphi)| &< \varepsilon, & |c^{(\varepsilon)}(\rho, \varphi) - c(\varphi)| &< \varepsilon, \end{aligned} \quad (3.10)$$

where α, β_k, γ are continuous functions and a, b, c are Lipschitz continuous functions.

Making a time transformation, applying Theorem 2.1 and similar argumentation than in the proof of Theorem 3.1, we obtain the following result.

Theorem 3.2. *Assume that (3.10), (1.6) hold true. Let $\mathbb{A}$ be the constant calculated for the model equation.*

If $\mathbb{A} > 0$, then

$$\exists \varepsilon_0 > 0 \quad \forall \varepsilon \in (0, \varepsilon_0) : \mathbf{P}(X_\varepsilon \text{ hits } 0 \text{ in a finite time}) = 0.$$

If $\mathbb{A} < 0$, then

$$\exists \varepsilon_0 > 0 \quad \forall \varepsilon \in (0, \varepsilon_0) : \mathbf{P}(X_\varepsilon \text{ hits } 0 \text{ in a finite time}) = 1.$$

It is well-known that if coefficients and initial conditions of two SDEs coincide in some domain and are locally Lipschitz, then the solutions coincide until the exit from this domain. This observation and some minor details lead to the following result.

Theorem 3.3. *Assume that coefficients of the model equation satisfy assumptions of Theorem 1.1 and the estimates (3.10) are fulfilled for all $x \in \bar{K}_{r_0} = \{x \in \bar{K} : |x| \leq r_0\}$, where $r_0 > 0$ is a constant.*

If $\mathbb{A} > 0$, then

$$\exists \varepsilon_0 > 0 \quad \forall \varepsilon \in (0, \varepsilon_0) : \mathbf{P}(X_\varepsilon \text{ hits } 0 \text{ in a finite time}) = 0.$$

If $\mathbb{A} < 0$ and $\mathbf{P}(|X_\varepsilon(0)| < r_0) > 0$, then

$$\exists \varepsilon_0 > 0 \quad \forall \varepsilon \in (0, \varepsilon_0) : \mathbf{P}(X_\varepsilon \text{ hits } 0 \text{ in a finite time}) > 0.$$

Moreover, if, for any x ,

$$\mathbf{P}(X_\varepsilon \text{ visits } \bar{K}_{r_0} \mid X_\varepsilon(0) = x) = 1,$$

then

$$\exists \varepsilon_0 > 0 \quad \forall \varepsilon \in (0, \varepsilon_0) : \mathbf{P}(X_\varepsilon \text{ hits } 0 \text{ in a finite time}) = 1.$$

Example 3.4. Assume that coefficients A, B_k of (3.1) are globally Lipschitz, and we have the uniform convergence

$$C(r, \varphi) \xrightarrow{\varphi \in \partial D} \tilde{C}(\varphi), \quad r \rightarrow 0+,$$

where $\tilde{C}$ is a Lipschitz function. Set $\tilde{B}_k := B_k(0)$, $\tilde{A}(x) := 0$.

Then the equation

$$dX_\infty(t) = \tilde{B}_k dW_k(t) + \tilde{C}(X_\infty(t)) dL_\infty(t)$$

is the model equation.

If the coefficient $\mathbb{A}$ for this equation is positive, then

$$\mathbf{P}(X(t) \text{ hits } 0 \text{ in a finite time}) = 0.$$

If $\mathbb{A} < 0$, then

$$\lim_{x \rightarrow 0} \mathbf{P}_x(X(t) \text{ hits } 0 \text{ in a finite time}) = 1.$$

Moreover, if, for any x ,

$\mathbf{P}_x(X(t) \text{ visits any neighborhood of } 0) = 1$,

then, for any $X(0)$,

$\mathbf{P}(X(t) \text{ hits } 0 \text{ in a finite time}) = 1$.

Remark 3.7. The arguments used in Theorem 3.3 can be applied also for a reflected SDE in a product space $\bar{K} \times \mathbb{R}^n$. Let a pair $(X(t), Z(t))$, $t \geq 0$, satisfying the SDEs

$$dX(t) = A(X(t), Z(t))dt + \sum_k B_k(X(t), Z(t))dw_k(t) + C(X(t), Z(t))dL(t),$$

$$dZ(t) = \bar{A}(X(t), Z(t))dt + \sum_k \bar{B}_k(X(t), Z(t))dw_k(t) + \bar{C}(X(t), Z(t))dL(t),$$

where $X(t) \in \bar{K}$, $Z(t) \in \mathbb{R}^n$, $L(t)$ is nondecreasing, adapted process, $L(0) = 0$, $L(t) = \int_0^t \mathbf{1}_{X(s) \in \partial D} dL(s)$.

Assume that

- 1) functions A , B_k , $\bar{A}$, $\bar{B}_k$, $\bar{C}$ are globally Lipschitz;
- 2) function C is locally Lipschitz, $\langle C(x, z), n(x) \rangle = 1$, $x \in \partial K$, $z \in \mathbb{R}^n$, and $C(r\varphi, z)$ uniformly converges to a Lipschitz function $C_\infty(\varphi, z)$ as $r \rightarrow 0+$. Let $\mathbb{A}(z)$, $z \in \mathbb{R}^n$, be the constant for equation (3.1) whose coefficients are given by

$$A_\infty \equiv 0, \quad B_{k,\infty}(x, z) := B_k(0, z), \quad C_\infty(\varphi, z).$$

The analog of Theorem 3.3 is the following theorem.

Theorem 3.4. Let $U \subset \mathbb{R}^n$ be a bounded open set.

1. Assume that $\mathbb{A}(z) > 0$ for any $z \in \bar{U}$. Then

$$\mathbf{P}((X, Z) \text{ hits } \{0\} \times U \text{ in a finite time}) = 0.$$

2. Assume that $\mathbb{A}(z) < 0$ for any $z \in \bar{U}$. Then, for any closed set $V \subset U$ and any sequence $\{(x_n, z_n), n \geq 1\} \subset \bar{K} \times V$ such that $\lim_{n \rightarrow \infty} x_n = 0$,

$$\mathbf{P}((X, Z) \text{ hits } \{0\} \times U \text{ in finite time} \mid X(0) = x_n, Z(0) = z_n) \rightarrow 1, \quad n \rightarrow \infty.$$

Appendix A. Reflected SDEs: moments, convergence. In this section, we recall some basic facts about reflected SDEs (see, for example, [10, 23]), sketch an idea how to get moments estimates, and prove Lemma 2.2.

At first let us recall few facts about deterministic one-dimensional Skorokhod reflecting problem on a half-line $[0, \infty)$. Let $h \in C([0, T])$, $h(0) \geq 0$. A pair of functions $f, g \in C([0, T])$ is a solution of the Skorokhod problem for h if

$$f(t) = g(t) + h(t), \quad t \in [0, T],$$

$f(t) \geq 0, t \geq 0$, $g(0) = 0$, g is nondecreasing, and

$$g(t) = \int_0^t \mathbf{1}_{f(z)=0} dg(z), \quad t \in [0, T].$$

It is well-known that there is a unique solution of the Skorokhod problem. This solution is given by the formula

$$g(t) = - \min_{s \in [0, t]} h(s) \wedge 0, \quad f(t) = g(t) + h(t) = - \min_{s \in [0, t]} h(s) \wedge 0 + h(t).$$

Moreover, it is easy to verify that if (f_1, g_1) and (f_2, g_2) are solutions of Skorokhod's problems for h_1, h_2 , then

$$\max_{t \in [0, T]} |f_1(t) - f_2(t)| \leq \max_{t \in [0, T]} |h_1(t) - h_2(t)|, \quad (\text{A.1})$$

$$\max_{t \in [0, T]} |g_1(t) - g_2(t)| \leq \max_{t \in [0, T]} |h_1(t) - h_2(t)|, \quad (\text{A.2})$$

$$\max_{t \in [0, T]} |g_1(t)| \leq \max_{t \in [0, T]} |h_1(t) - h_1(0)|, \quad (\text{A.3})$$

$$\max_{t \in [0, T]} |f_1(t) - f_1(0)| \leq \max_{t \in [0, T]} |h_1(t) - h_1(0)|. \quad (\text{A.4})$$

Consider equations (1.1), (1.2) in a domain $D = \mathbb{R}_+^d = \mathbb{R}^{d-1} \times [0, \infty)$. Assume that functions a, b_k, v satisfy Lipschitz condition, and $\langle v(y), n \rangle = 1$, $y \in \partial \mathbb{R}_+^d = \mathbb{R}^{d-1} \times [0, \infty)$, where $n = (0, \dots, 0, 1)$. The general case can be obtained via localization techniques and transformation of space arguments. We will also suppose that all functions appearing further are bounded. Note that existence of moments for all considered processes is well-known. So, we will not mention that the corresponding moments are finite when we apply Gronwall's lemma or take an expectation of stochastic integral.

Denote $\hat{Y}(t) = (Y_1(t), \dots, Y_{d-1}(t))$, $\hat{v}(y) = v(\hat{y}, y_d) = (v_1(y), \dots, v_{d-1}(y))$, $\hat{a}(y) = (a_1(y), \dots, a_{d-1}(y))$, $\hat{b}_k(y) = (b_{1,k}(y), \dots, b_{d-1,k}(y))$. Recall that $v_d(y) = 1$.

Equation (1.1) has the following coordinate form:

$$\hat{Y}(t) = \hat{Y}(0) + \int_0^t \hat{a}(Y(z)) dz + \int_0^t \hat{b}_k(Y(z)) dw_k(z) + \int_0^t \hat{v}(Y(z)) dL(z), \quad (\text{A.5})$$

$$Y_d(t) = Y_d(0) + \int_0^t a_d(Y(z)) dz + \int_0^t b_{d,k}(Y(t)) dw_k(t) + L(t). \quad (\text{A.6})$$

Observe that for a fixed ω a pair $(Y_d(t), L(t))$ is a solution of one-dimensional Skorokhod problem, where the function $h(t)$ equals $Y_d(0) + \int_0^t a_d(Y(z)) dz + \int_0^t b_{d,k}(Y(t)) dw_k(t)$. It follows from boundedness of coefficients, Burkholder inequality, (A.3) and (A.4) that

$$\begin{aligned} & \mathbf{E} \sup_{s \in [0, t]} (Y_d(s) - Y_d(0))^2 \vee \mathbf{E} \sup_{s \in [0, t]} (L(s))^2 \\ & \leq \mathbf{E} \sup_{s \in [0, t]} \left(\int_0^s a_d(Y(z)) dz + \int_0^s b_{d,k}(Y(t)) dw_k(t) \right)^2 \leq c_1(t + t^2) \end{aligned} \quad (\text{A.7})$$

with a universal constant c_1 . Applying this estimate to (A.5), we get

$$\mathbf{E} \sup_{s \in [0, t]} |Y(s) - Y(0)|^2 \leq c_2(t + t^2). \quad (\text{A.8})$$

Let us give a proof of Lemma 2.2 only for the case, when $D = \mathbb{R}_+^d$ is a hyperplane, $s = 0$, $\xi = x$ is nonrandom, $v(y) = n = (0, \dots, 0, 1)$ is the normal vector to the hyperplane $\partial \mathbb{R}_+^d$, and take the usual expectation instead of conditional. The general case can be considered similarly with the routine application of transformation of space and localization technique.

Without loss of generality it can be assumed that $\varepsilon \in (0, 1)$, so all functions $\alpha_\varepsilon, \dots, v_\varepsilon$ are bounded by the same constant. Similarly to (A.7), (A.8) we have estimates

$$\mathbf{E} \sup_{s \in [0, t]} |Y_\varepsilon(s) - Y_\varepsilon(0)|^2 \leq c(t + t^2), \quad (\text{A.9})$$

$$\mathbf{E}(L_\varepsilon(t))^2 + \mathbf{E} \sup_{s \in [0, t]} |Y_\varepsilon(s) - Y_\varepsilon(0)|^2 \leq c(t + t^2), \quad (\text{A.10})$$

where a constant c is independent of ε .

Proof of Lemma 2.2. It follows from (2.2), Lipschitz property for functions a and b_k , and Ito's formula that

$$\begin{aligned} \mathbf{E}|Y_\varepsilon(t) - Y(t)|^2 &= 2\mathbf{E} \int_0^t \langle Y_\varepsilon(s) - Y(s), a_\varepsilon(X_\varepsilon(z), Y_\varepsilon(z)) - a(Y(z)) \rangle dz \\ &\quad + \mathbf{E} \int_0^t |b_\varepsilon(X_\varepsilon(z), Y_\varepsilon(z)) - b(Y(z))|^2 dz \\ &\quad + 2\mathbf{E} \int_0^t \langle Y_\varepsilon(s) - Y(s), v_\varepsilon(X_\varepsilon(z), Y_\varepsilon(z)) dL_\varepsilon(z) - n dL(z) \rangle \\ &\leq c_3 \mathbf{E} \left(\int_0^t (\varepsilon^2 + |Y_\varepsilon(s) - Y(s)|^2) ds + \int_0^t \langle Y_\varepsilon(s) - Y(s), n \rangle (dL_\varepsilon(z) - dL(z)) \right. \\ &\quad \left. + \int_0^t \langle Y_\varepsilon(s) - Y(s), (v_\varepsilon(X_\varepsilon(z), Y_\varepsilon(z)) - n) \rangle dL_\varepsilon(z) \right) \\ &\leq c_3 \mathbf{E} \left(\int_0^t (\varepsilon^2 + |Y_\varepsilon(s) - Y(s)|^2) ds \right. \\ &\quad \left. + \varepsilon \sup_{s \in [0, t]} (|Y_\varepsilon(s) - Y_\varepsilon(0)| + |Y(s) - Y(0)|) L_\varepsilon(t) \right). \end{aligned}$$

In the last inequality we used the fact that $Y_\varepsilon(0) = Y(0)$ and the integral

$$\int_0^t \langle Y_\varepsilon(s) - Y(s), (v_\varepsilon(X_\varepsilon(z), Y_\varepsilon(z)) - n) \rangle dL_\varepsilon(z)$$

is nonpositive due to the definition of the Skorokhod reflecting problem. Applying (A.10), we get

$$\mathbf{E}|Y_\varepsilon(t) - Y(t)|^2 \leq c_4 \left(\varepsilon(t + t^2) + \int_0^t \mathbf{E}|Y_\varepsilon(s) - Y(s)|^2 ds \right).$$

By Gronwall's lemma,

$$\mathbf{E}|Y_\varepsilon(t) - Y(t)|^2 \leq c_4 \varepsilon(t + t^2) e^{c_4 t} \leq c_5 \varepsilon e^{c_5 t}. \quad (\text{A.11})$$

It follows from (A.2), (A.11), and Burkholder's inequality that

$$\begin{aligned} \mathbf{E} \sup_{s \in [0, t]} |L_\varepsilon(s) - L(s)|^2 &\leq 2 \left(\mathbf{E} \sup_{s \in [0, t]} \left| \int_0^s (a_{d, \varepsilon}(X_\varepsilon(z), Y_\varepsilon(z)) - a_d(Y(z))) dz \right|^2 \right. \\ &\quad \left. + \mathbf{E} \sup_{s \in [0, t]} \left| \int_0^s (b_{d, \varepsilon}(X_\varepsilon(z), Y_\varepsilon(z)) - b_d(Y(z))) dw(z) \right|^2 \right) \\ &\leq c_6 \left(t \int_0^t (\varepsilon + |Y_\varepsilon(s) - Y(s)|^2) ds + \int_0^t (\varepsilon + |Y_\varepsilon(s) - Y(s)|^2) ds \right) \\ &\leq c_7 \varepsilon e^{c_7 t}. \end{aligned}$$

Applying (A.11) and the last inequality, we obtain

$$\begin{aligned} \mathbf{E} \sup_{s \in [0, t]} |Y_\varepsilon(s) - Y(s)|^2 &\leq 3 \left(\mathbf{E} \sup_{s \in [0, t]} \left| \int_0^s (a_\varepsilon(X_\varepsilon(z), Y_\varepsilon(z)) - a(Y(z))) dz \right|^2 \right. \\ &\quad \left. + \mathbf{E} \sup_{s \in [0, t]} \left| \int_0^s (b_\varepsilon(X_\varepsilon(z), Y_\varepsilon(z)) - b(Y(z))) dw(z) \right|^2 + \right. \\ &\quad \left. + \mathbf{E} \sup_{s \in [0, t]} \left| \int_0^s v_\varepsilon(X_\varepsilon(z), Y_\varepsilon(z)) dL_\varepsilon(z) - \int_0^t v(Y(z)) dL(z) \right|^2 \right) \\ &\leq c_1 \left(t \int_0^t (\varepsilon + |Y_\varepsilon(s) - Y(s)|^2) ds + \int_0^t (\varepsilon + |Y_\varepsilon(s) - Y(s)|^2) ds \right. \\ &\quad \left. + \varepsilon \mathbf{E} L_\varepsilon(t)^2 + \mathbf{E} \sup_{s \in [0, t]} |L_\varepsilon(s) - L(s)|^2 \right) \leq c_8 \varepsilon e^{c_8 t}. \end{aligned}$$

Appendix B. Measurable successful couplings.

Lemma B.1. Consider two Polish spaces $\mathbb{S}_1, \mathbb{S}_2$ and a family $\{\mu_y, y \in \mathbb{S}_1\}$ of probability measures on $\mathbb{S}_2$ such that

$$y \mapsto \mu_y(A)$$

is measurable for any Borel set A . Then there exists a measurable mapping

$$\Xi: \mathbb{S}_1 \times [0, 1] \rightarrow \mathbb{S}_2$$

such that, for any $y \in \mathbb{S}_1$, the random variable $\Xi(y, U)$ has the distribution μ_y , where U is a random variable uniformly distributed on $[0, 1]$.

Proof. Since any Polish space admits a Borel isomorphism to $[0, 1]$, it is enough to prove the statement only for the case $\mathbb{S}_2 = [0, 1]$. In this case the required mapping is given explicitly as follows:

$$\Xi(y, p) = \inf\{u \in \mathbb{Q} : \mu_y([0, u]) \geq p\}, \quad y \in \mathbb{S}_1, \quad p \in [0, 1].$$

Lemma B.1 is proved.

In what follows, we fix $T \in [0, \infty]$ and denote $J_T = [0, T]$ for $T < \infty$ and $J_\infty = [0, \infty)$.

Lemma B.2. Let $\mathbb{S}$ be a Polish space and an $\mathbb{S}$ -valued Markov process $Y(t)$, $t \in J_T$, with Feller property and càdlàg trajectory be given. Denote, for a given $\tau \in (0, T)$,

$$\delta(\tau) := \sup_{y_1, y_2} \text{var}(P_\tau(y_1, \cdot) - P_\tau(y_2, \cdot)).$$

Then there exists a measurable mapping $\Upsilon: \mathbb{S} \times \mathbb{S} \times [0, 1] \rightarrow \mathbf{D}(J_T, \mathbb{S} \times \mathbb{S})$ such that, for any $y_1, y_2 \in \mathbb{R}^d$ and for any random variable U uniformly distributed on $[0, 1]$, the random process with values in $\mathbb{S} \times \mathbb{S}$

$$\Upsilon(y_1, y_2, U)(t) := (Z_1(t), Z_2(t)), \quad t \in J_T,$$

has the following properties:

- 1) components Z_i , $i = 1, 2$, have the same laws than the process Y conditioned by $Y(0) = y_i$, $i = 1, 2$;
- 2) $P(Z_1(t) = Z_2(t), t \geq \tau) \geq 1 - \frac{1}{2}\delta(\tau)$.

Proof. We will construct the required mapping using Lemma B.1 repeatedly. First, we construct a mapping which, for any given y_1, y_2 , gives the pair of random elements ξ_1, ξ_2 in $\mathbb{S}$, which will be the values of the required process Z at the time instant τ . Namely, by the coupling lemma for probability kernels [27, Theorem 2.2.4] there exists a family $\{\mu_y\}$ satisfying the assumptions of the Lemma B.1 with $\mathbb{S}_1 = \mathbb{S}_2 = \mathbb{S} \times \mathbb{S}$ and such that

marginal distributions of μ_y , $y = (y_1, y_2)$ are equal to $P_\tau(y_1, \cdot)$, $P_\tau(y_2, \cdot)$,

$$\inf_y \mu_y\left(\{z = (z_1, z_2) : z_1 \neq z_2\}\right) \geq 1 - \frac{1}{2}\delta(\tau).$$

Let Ξ_1 be the corresponding measurable mapping from Lemma B.1. In what follows, we will take a uniformly distributed random variable U_1 and define $(\xi_1, \xi_2) = \Xi_1(y, U_1)$. This will be the value of the required process Z at the time instant τ .

Next, consider the law $\nu_{y,z}$ of the process Y , restricted to $[0, \tau]$ and conditioned by $Y(0) = y, Y(\tau) = z$. Since $\mathbf{D}([0, \tau], \mathbb{S})$ is a Polish space, such a family can be chosen in a measurable way w.r.t. (y, z) . We consider the product

$$\mu_{y_1, y_2, z_1, z_2} = \nu_{y_1, z_2} \otimes \nu_{y_2, z_2}$$

and apply Lemma B.1 with this family and $\mathbb{S}_1 = \mathbb{S} \times \mathbb{S} \times \mathbb{S} \times \mathbb{S}$ and $\mathbb{S}_2 = \mathbf{D}([0, \tau], \mathbb{S} \times \mathbb{S})$. Let Ξ_2 be the corresponding measurable mapping from Lemma B.1.

Finally, consider law λ_z of the process Y , restricted to $[\tau, T]$ (or $[\tau, \infty)$ for $T = \infty$) and conditioned by $Y(\tau) = z$. We define μ_{z_1, z_2} as

the product of $\lambda_{z_1}, \lambda_{z_2}$ if $z_1 \neq z_2$;

the measure which corresponds to two identical components distributed as λ_z if $z_1 = z_2 = z$.

Let Ξ_3 be the corresponding measurable mapping from Lemma B.1.

Now we define the mapping $\Xi: \mathbb{S} \times \mathbb{S} \times [0, 1] \times [0, 1] \times [0, 1] \rightarrow \mathbf{D}(J_T, \mathbb{S} \times \mathbb{S})$ as follows:

(i) Given $y_1, y_2 \in \mathbb{S}, u \in [0, 1]$, denote $z = (z_1, z_2) = \Xi_1(y_1, y_2, u)$.

(ii) Given in addition $u_2 \in [0, 1]$, define the part of the trajectory of the required element of $\mathbf{D}(J_T, \mathbb{S} \times \mathbb{S})$ on $[0, \tau]$ as $\Xi_2(y_1, y_2, z_1, z_2, u_2)$.

Given in addition $u_3 \in [0, 1]$, define the part of the trajectory of the required element of $\mathbf{D}(J_T, \mathbb{S} \times \mathbb{S})$ on $[\tau, T]$ (or $[\tau, \infty)$) as $\Xi_3(z_1, z_2, u_3)$.

By the construction, the process $Z = \Xi(y_1, y_2, U_1, U_2, U_3)$ with independent U_1, U_2, U_3 uniformly distributed on $[0, 1]$ satisfies the required properties. On the other hand, it is easy to construct a measurable mapping $[0, 1] \rightarrow [0, 1]^3$ which maps uniform distribution on $[0, 1]$ into the uniform distribution on $[0, 1]^3$. Taking composition of this mapping with Ξ we get the required mapping Υ .

Lemma B.2 is proved.

O. Kulyk was supported by the Polish National Science Center (grant 2019/33/B/ST1/029230). A. Pilipenko and S. Röelly were partially supported by the Alexander von Humboldt Foundation between the University of Potsdam and the Institute of Mathematics of the National Academy of Sciences of Ukraine (Research Group Linkage cooperation *Singular diffusions: analytic and stochastic approaches*).

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Received 18.12.22