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ON THE BALANCED PANTOGRAPH EQUATION OF MIXED TYPE

ПРО РІВНЯННЯ РІВНОВАЖНОГО ПАНТОГРАФА МІШАНОГО ТИПУ

We consider the balanced pantograph equation (BPE) $y'(x)+y(x) = \sum_{k=1}^{m} p_k y(a_k x)$, where $a_k, p_k > 0$ and $\sum_{k=1}^{m} p_k = 1$. It is known that if $K = \sum_{k=1}^{m} p_k \ln a_k \leq 0$ then, under mild technical conditions, the BPE does not have bounded solutions that are not constant, whereas for $K > 0$ these solutions exist. In the present paper, we deal with a BPE of *mixed type*, i.e., $a_1 < 1 < a_m$, and prove that, in this case, the BPE has a nonconstant solution y and that $y(x) \sim cx^{\sigma}$ as $x \to \infty$, where $c > 0$ and σ is the unique positive root of the characteristic equation $P(s) = 1 - \sum_{k=1}^{m} p_k a_k^{-s} = 0$. We also show that y is unique (up to a multiplicative constant) among the solutions of the BPE that decay to zero as $x \to \infty$.

Розглянуто рівняння збалансованого пантографа (РЗП) $y'(x)+y(x) = \sum_{k=1}^{m} p_k y(a_k x)$, де $a_k, p_k > 0$ і $\sum_{k=1}^{m} p_k = 1$. Відомо, що якщо $K = \sum_{k=1}^{m} p_k \ln a_k \leq 0$, то за м'яких технічних умов РЗП не має обмежених розв'язків, які не є сталими; водночас у випадку $K > 0$ такі розв'язки існують. У цій статті ми маємо справу з РЗП *мішаного типу*, тобто $a_1 < 1 < a_m$, і доводимо, що в цьому випадку РЗП має несталий розв'язок y і, крім того, $y(x) \sim cx^{\sigma}$ при $x \to \infty$, де $c > 0$, а σ — єдиний додатний корінь характеристичного рівняння $P(s) = 1 - \sum_{k=1}^{m} p_k a_k^{-s} = 0$. Також показано, що y є єдиним (з точністю до мультиплікативної константи) серед розв'язків РЗП, що спадають до нуля при $x \to \infty$.

1. Introduction. The classical *pantograph* equation is the functional differential equation of the form

$$y'(x) = ay(\lambda x) + by(x), \tag{1}$$

where a, b are constants and $\lambda > 0$ is a rescaling parameter. This equation has emerged in a striking number of applications including number theory, astrophysics and population dynamics. A more detailed account of applications can be found in [1]. The term “pantograph” dates back to the seminal 1971 paper by Ockendon and Taylor [13]. (The name “pantograph equation” was coined later by Iserles [9].). The initial value problem formed by equation (1) and the condition $y(0) = 1$ was studied in detail for various domains of parameters by Kato and McLeod [10]. They showed that the asymptotic behavior of solutions as $x \to \infty$ differs markedly depending on whether $0 < \lambda < 1$ or $1 < \lambda$ and also on the cases $\mathrm{Re}(b) < 0$, $\mathrm{Re}(b) > 0$ or $\mathrm{Re}(b) = 0$.

An obvious generalization of equation (1) is to include several scaling factors and enquire about the asymptotic behavior of solutions. We note that the analysis by Kato and McLeod (*op cit.*) for just one functional argument and two parameters is quite elaborate and that the addition of further terms escalates the analysis. Nonetheless, if some structure is imposed on the additional terms then the problem becomes more tractable. Some partial answers have been given by [1 – 3, 5]. In particular, Derfel [5] introduced the *balanced pantograph equation* (BPE)

$$y'(x) + y(x) = \sum_{k=1}^{m} p_k y(a_k x), \tag{2}$$

where p_k are positive numbers such that

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$$\sum_{k=1}^{m} p_k = 1 \tag{3}$$

and

$$0 < a_1 < a_2 < \ldots < a_m. \tag{4}$$

Without loss of generality we can always assume that $a_k \neq 1$. This generalization targets applications of the pantograph equation where the solution is a probability density function. The cell division equation studied by Hall and Wake [8] is a BPE with one functional argument. A feature of any BPE is that $y(x) = \text{const}$ is always a solution.

There are three cases for BPE equations. The first is the "pure retarded" case, where $a_m < 1$. This case is of limited interest because the solution to initial value problems involving pure retarded equations is unique [14]: the constant solution is the only solution. The next is the "pure advanced" case where $a_1 > 1$. Here, we know that solutions are not unique: it is possible to get nonconstant solutions. Special cases of the pure advanced BPE for more than one argument have been studied in connection with asymmetrical cell division by [15, 16], where certain existence, uniqueness and monotonicity results are detailed. It is clear that we need at least one $a_k > 1$ to induce nonuniqueness. Here, we look at the third case where some of the arguments are retarded and some are advanced. Specifically, we focus on the *mixed case*, where $m > 1$ and

$$0 < a_1 < 1 < a_m. \tag{5}$$

The mixed case was studied in [1 – 3, 5]. It turns out that the asymptotic behavior of solutions depends critically on the quantity

$$K := \sum_{k=1}^{m} p_k \ln a_k. \tag{6}$$

Namely, it was proved by Derfel *op cit.* that if $K < 0$ (the *subcritical case*) then equation (2) does not have nonconstant solutions that are bounded; whereas, if $K > 0$ (the *supercritical case*) then such a solution always exists. The existence of nonconstant bounded solutions for the *critical case*, $K = 0$, remained open until 2008, when it was showed in [1] that no such solutions exist. In the present paper we will refine the results for the supercritical case. In particular, we will establish the decay rate of the second order terms in the asymptotic expansion of solutions as $x \to \infty$.

2. The characteristic equation. We will see in the sections to follow that the asymptotic behavior of solutions to the BPE depends crucially on solutions to the *characteristic equation*

$$P(s) = 0, \tag{7}$$

where $s \in \mathbb{C}$ and

$$P(s) = 1 - \sum_{k=1}^{m} \frac{p_k}{a_k^s}. \tag{8}$$

In this section, we look at a few results about solutions to this equation.

Determining solutions to equation (7) can prove formidable;however, the situation for the real zeros tractable as the next lemma indicates.

Lemma 2.1. *Let P be as defined in* (8)*, where p_k and α_k satisfy conditions* (3) *and* (5)*. Then P has precisely two real roots counting multiplicity. One root is always $s = 0$: if $P'(0) < 0$, then the other root must be negative*; *if $P'(0) > 0$, the other root must be positive. If $P'(0) = 0$, then P has a root of multiplicity* 2 *at $s = 0$.*

Proof. Let $\sigma \in \mathbb{R}$. It is clear that $P(0) = 0$ and that condition (5) implies $P(\sigma) \to -\infty$ as $\sigma \to \pm\infty$. Now,

$$P'(\sigma) = \sum_{k=1}^{m} \frac{p_k \ln a_k}{a_k^{\sigma}},$$

so that, for any $\sigma \in \mathbb{R}$,

$$P''(\sigma) = -\sum_{k=1}^{m} \frac{p_k \ln^2 a_k}{a_k^{\sigma}} < 0.$$

The graph of P is thus convex down and intersects the real axis at $\sigma = 0$. Since P must be negative for all $|\sigma|$ sufficiently large, this indicates that P has at most two distinct real roots. If $P'(0) < 0$, then $P(\sigma) > 0$ for some $\sigma < 0$ and therefore the other root must be negative. Similarly, if $P'(0) > 0$ the other root must be positive. Note that P' is monotonic strictly decreasing, so that in both these cases the zero must be of order 1. Finally, if $P'(0) = 0$, then P must have a zero of order 2 at 0.

The lemma is proved.

The challenge is to determine the complex solutions. There are two tractable results concerning complex zeros. First we introduce a definition. The scaling factors a_k, $k = 1, \ldots, m$, are called *multiplicatively commensurable* if there exists a number $\nu > 1$ and rational numbers $\zeta_1, \ldots, \zeta_m$ such that $a_k = \nu^{\zeta_k}$ for $k = 1, \ldots, m$. For reasons which will be explained below, this case will also the *arithmetic* or *lattice* case. Without loss of generality it can be assumed in this case that there is $q > 1$ and integers $n_1, \ldots, n_m$ (positive or negative) such that

$$a_k = q^{n_k} \tag{9}$$

for $k = 1, \ldots, m$.

Lemma 2.2. *Suppose that P has a zero at $s = \sigma + i\tau$, where $\sigma, \tau \in \mathbb{R}$.*

1. *If $P(\sigma) = 0$ and $\tau \neq 0$, then a_k are multiplicatively commensurate. In this case there are an infinite number of zero on the line $\mathrm{Re}(s) = \sigma$.*

2. *Suppose that P has two distinct real zeros. Then σ cannot be between these zeros.*

Proof. Equating real components, the equation $P(s) = 0$ gives

$$\sum_{k=0}^{m} \frac{p_k \cos(\tau a_k)}{a_k^{\sigma}} = 1. \tag{10}$$

1. Suppose $P(\sigma) = 0$. If this relation also holds for some $\tau \neq 0$, then equation (10) implies $\cos(\tau a_k) = 1$ for $k = 1, \ldots, m$. This observation means that $\tau a_k = 2\pi m_k$ for some nonzero integer m_k. Since $\tau \neq 0$ must be the same for all these arguments, we see that a_k must be multiplicatively commensurate. In this case, using (9), we have $\tau \ln a_k = \tau n_k \ln q$ so that $\tau_m = \tau + 2\pi m / \ln q$, where m is any integer, also produces a zero.

2. If σ is between the two real zeros of P, then $P(\sigma) > 0$, equation (10) cannot be satisfied for any values of τ.

The lemma is proved.

3. A probabilistic approach. In this section, we relate the mixed BPE problem to one in probability theory. Specifically, we relate the decay of solutions to BPE (2) with results from the Kesten – Grincevičius – Goldie theorem about tails of special random series (perpetuities). This theorem is presented in the next section. For this study we will be concerned primarily with the mixed BPE for the supercritical case $K > 0$. We will call this **Problem A** for succinctness.

A bridge to a probabilistic interpretation of the BPE is provided by results for the *archetypal equation* (AE)

$$y(x) = \iint_{\mathbb{R}^2} y(a(x-b))\,\mu(da,db), \tag{11}$$

where $x \in \mathbb{R}$ and $\mu(da,db)$ is a probability measure (cf. [2, 3, 5]). The integral in (11) has the meaning of the expectation $\mathbb{E}$ with respect to a random vector (α,β) with distribution $P\{(\alpha,\beta) \in da \times db\} = \mu(da,db)$. Thus, equation (11) can be written in the equivalent and compact form

$$y(x) = \mathbb{E}(y(\alpha(x-\beta))). \tag{12}$$

It was observed by Derfel [5] (see also [2, 3]) that equation (11) is a rich source of important functional and functional differential equations with rescaling, specified by a suitable choice of the probability measure μ. In particular, the BPE (2) is a special case of the AE (11) with $\mathbb{P}(\alpha = a_j) = p_j$ and β conditional on $\alpha = a_j$ having the unit exponential distribution with density $g(x) = e^{-x}$ on $[0,\infty)$. These choices produce the AE

$$y(x) = p_1 \int_0^\infty y(a_1(x-t))e^{-t}\,dt + \ldots + p_m \int_0^\infty y(a_m(x-t))e^{-t}\,dt,$$

and it can be shown (cf. [2, 5]) that this equation is equivalent to (2).

We have the following result about the existence of bounded continuous solutions.

Theorem 3.1 [3, 5]**.** *Let $\alpha > 0$ and suppose that, for all $c \in \mathbb{R}$,*

$$\mathbb{P}(a(c-b) = c) < 1 \tag{13}$$

and

$$\iint_{\mathbb{R}^2} \ln\sup(|\beta|,1)\,\mu(da,db) = \mathbb{E}(\ln\sup(|\beta|,1)) = \mathbb{E}\bigl(\ln^+|\beta|\bigr) < \infty. \tag{14}$$

Let

$$K := \iint_{\mathbb{R}^2} \ln a\,\mu(da,db) = \mathbb{E}(\ln\alpha).$$

If $K \le 0$, then equation (11) *does not have a bounded continuous solution apart from constant solutions. If $K > 0$, then equation* (11) *has a nonconstant bounded and continuous solution y. This solution is the probability distribution function F_ψ of the convergent random series*

$$\psi = \beta_1 + a_1^{-1}\beta_2 + a_1^{-1}a_2^{-1}\beta_3 + \ldots + a_1^{-1}a_2^{-1}\ldots a_{n-1}^{-1}\beta_n + \ldots = \beta_1 + \sum_{n=2}^\infty \beta_n \prod_{j=1}^{n-1} a_j^{n-1},$$

i.e.,

$$y(x) = F_\psi(x) = \mathbb{P}(\psi < x). \tag{15}$$

Here, (α_n, β_n) *is a sequence of independent identically distributed random points with the same distribution* μ *as* (α, β)*. Moreover,*

$$\psi \stackrel{d}{=} \alpha^{-1}\psi + \beta, \tag{16}$$

where $\stackrel{d}{=}$ *means equality in distribution.*

Equation (16) is an example of a *stochastic fixed point equation*

$$X \stackrel{d}{=} AX + B, \tag{17}$$

where the random variable X and the random pair (A, B) are independent. For the AE (11), let

$$X = \psi, \qquad A = \alpha^{-1}, \qquad B = \beta, \qquad \text{and} \qquad K = \mathbb{E}\{\ln \alpha\} = -\mathbb{E}\{\ln A\}. \tag{18}$$

Then AE (11) reduces to the BPE (2) if we take A as the discrete random variable taking values α_k^{-1} with probabilities p_k, i.e., $\mathbb{P}(A = \alpha_k^{-1}) = p_k$, and $B = \beta$ conditioned on A having a unit exponential distribution with density $g(x) = e^{-x}$ on $[0, \infty)$. It is clear that conditions (13) and (14) are satisfied and that $K = \sum_{k=1}^{m} p_k \ln a_k$ in agreement with the earlier definition (6). We will call (15) the *canonical solution* of (12).

4. Main results.

Theorem 4.1. *Let* a_k *be non multiplicatively commensurable.*

1. *There exists a solution* y *to Problem A such that*

$$y(x) \sim cx^{-\sigma_1} \quad \text{as} \quad x \to \infty, \tag{19}$$

where c *is a nonzero constant and* σ_1 *is the positive root of* $P(\sigma) = 0$.

2. *Every nonconstant solution* y *to Problem A that tends to zero as* $x \to \infty$ *satisfies* (19).

3. *Every nonconstant solution* y *to Problem A that has a limit* L *satisfies*

$$y(x) \sim L + cx^{-\sigma_1} \quad \text{as} \quad x \to \infty, \tag{20}$$

where at least one of the constants L *or* c *is a nonzero.*

4. *Every nonconstant solution* y *to Problem A that has a limit as* $x \to \infty$ *coincides, up to an affine transformation, with the canonical solution of* (2).

***Corollary* 4.1.** *Let* p_k *and* a_k *be such that* $K > 0$ *and suppose that* a_k *are multiplicatively commensurable so that, without loss of generality, we can assume* a_k *satisfy* (9)*. Then and the statements of Theorem* 4.1 *remain valid with the difference that relations* (19) *and* (20) *are replaced by*

$$y(uq^n) \sim D\left(\frac{\ln u}{\ln q}\right)(uq^n)^{-\sigma_1}$$

and

$$y(uq^n) \sim L + D\left(\frac{\ln u}{\ln q}\right)(uq^n)^{-\sigma_1},$$

respectively, as $n \to \infty$*, where* u *is any positive real number and* D *is a* 1*-periodic function.*

Remark. The assumption that the BPE is of mixed type is essential for the validity of the results. It is known that pure advanced equations ($a_k > 1$ for all k) may have solutions that decay exponentially (cf. [10, 15, 16]).

The proof of Theorem 4.1 is based on the Kesten – Grincevičius – Goldie theorem [6, 7, 11], which we present here in the form given in [4]. First, however, note that a random variable A is called *arithmetic* if it is supported on some set $h\mathbb{Z}$ where $h > 0$, i.e., on a set of points that belong to some arithmetic progression.

Theorem 4.2 (Kesten – Grincevičius – Goldie). *Assume that the following conditions hold*:

1) $A > 0$ *and the law of* $\log A$ *is nonarithmetic;*

2) *there exists* $\sigma > 0$ *such that*

$$\mathbb{E}[A^\sigma] = 1, \qquad \mathbb{E}[B^\sigma] < \infty, \qquad \textit{and} \qquad \mathbb{E}[A^\sigma \log^+ A] < \infty;$$

3) *for every* $x \in \mathbb{R}$,

$$\mathbb{P}(Ax + B = x) < 1.$$

Then equation (17) *has a solution* X *that is independent of* (A, B) *and there exist constants* c_+, c_- *such that* $c_+ + c_- > 0$, *and*

$$\mathbb{P}(X > x) \sim c_+ x^{-\sigma} \qquad \textit{and} \qquad \mathbb{P}(X < -x) \sim c_- x^{-\sigma} \quad \textit{as} \quad x \to \infty. \tag{21}$$

If the support of X *is* (a, ∞) *for some* $a > 0$, *then* $c_+ > 0$ *and* $c_- = 0$.

***Proof of Theorem* 4.1.** The BPE (2) is a special case of the AE (11) and we know from Theorem 3.1 that there exists a nonconstant bounded and continuous solution, *viz.*, the canonical solution (15). It remains to show that asymptotic behavior is as claimed and this is where Theorem 4.2 is crucial.

Let A and B be as defined in equation (18). The assumption that α_k are not multiplicatively commensurable implies that $\log A = \log \alpha^{-1} = -\log \alpha$ is nonarithmetic. The equation $\mathbb{E}[A^\sigma] = 1$, is called the *Cramér – Lundberg condition* in ruin theory (cf. [4, p. 48]) and here this equation is the same as the characteristic equation (7). *Problem A* concerns the supercritical case and we know that $K > 0$. Lemma 2.1 implies that there is a unique positive solution σ_1 to this equation; consequently, there is $\sigma_1 > 0$ such that

$$\mathbb{E}[A^{\sigma_1}] = 1. \tag{22}$$

Now, for any $\sigma > 0$,

$$\mathbb{E}[|B|^\sigma] = \int x^\sigma e^{-x}\, dx < \infty \tag{23}$$

and

$$\mathbb{E}\big[A^\sigma \log_+ A\big] = \sum_{j=1}^{\ell} p_j a_j^{-\sigma} \log(\alpha_j)^{-1} = -\sum_{j=1}^{\ell} p_j \alpha_j^{-\sigma} \log a_j < \infty, \tag{24}$$

where the summation is over all j such that $a_j^{-1} > 1$. Equations (22), (23) and (24) show that condition 2 in Theorem 4.2 is satisfied, and condition 3 coincides with (13).

Theorem 4.2 shows that the canonical solution (15) satisfies (21) with $X = \psi$ and $\sigma = \sigma_1$. Evidently, $y_1(x) = 1$ is a solution of (2), and, for $y_2(x) = F_\psi(x) = \mathbb{P}(\psi < x)$, the function $y_1(x) - y_2(x) = 1 - F_\psi(x) = \mathbb{P}(\psi > x) = \bar{F}_\psi(x)$ is also a solution of (2), i.e., the *tail* of ψ is a solution. Theorem 4.2 implies that $\bar{F}_\psi(x)$ must also satisfy (21). Statements 3 and 4 in Theorem 4.1 follow from the general uniqueness theorem for the archetypal equation (cf. [3, Theorem 4.3]).

The theorem is proved.

***Proof of Corollary* 4.1.** The proof of Corollary 4.1 follows from the work of Grincevičius [7], who studied the case when $\ln A$ is supported on the set $h\mathbb{Z},\ h \geq 0$ (see also [4, p. 50, Remark]). Note that if (9) is satisfied, then $h = \ln q$. Under the assumptions of Theorem 4.1 he proved that for all but countably many $v \in \mathbb{R}$ there exists a periodic function $C(v)$ of period h such that

$$\mathbb{P}\Big(X > e^{v+nh}\Big) \sim C(v)e^{-\sigma_1(v+nh)}$$

as $n \to \infty$. Moreover, if $\mathbb{P}(B \geq 1) < 1$ and $\mathbb{P}(B = 0) < 1$, then the limit in equation (20) exists and is positive for every v. Let $e^h = q$ and $e^v = u$. Then (20) can be written in the form

$$\mathbb{P}(X > uq^n) \sim D\left(\frac{\ln u}{\ln q}\right)(uq^n)^{-\sigma_1} \tag{25}$$

as $n \to \infty$, where D is 1-periodic function.

Remarks. 1. It is known that if D is continuous then (25) can be replaced by

$$\mathbb{P}(X > x) \sim D\left(\frac{\ln x}{\ln q}\right)x^{-\sigma_1}$$

as $x \to \infty$ (see [12, Theorem 1]).

2. Theorem 4.1 can be easily extended by the same methods to the BPE with shifts

$$y'(x) + y(x) = \sum_{k=1}^{m} p_k y(a_k x + b_k), \tag{26}$$

purely functional equation

$$y(x) = \sum_{k=1}^{m} p_k y(a_k x + b_k), \tag{27}$$

and also some BPE of high order

$$Q(d/dx)y(x) = \sum_{k=1}^{m} p_k y(a_k x), \tag{28}$$

where $Q(s) = \prod_{j=1}^{n}(1 + s/\lambda_j),\ \lambda_j$ are *real* numbers and $\lambda_j \neq 0$. More precisely, each of the above equations (26)–(28) has two solutions $y(x)$ and $y_1(x) = 1 - y(x)$, such that

$$y_1(x) \sim c_+ x^{-\sigma_1}, \qquad y(-x) \sim c_- x^{-\sigma_1}$$

as $x \to \infty$ and $c_+ + c_- > 0$.

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