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DYNAMICS OF ONE-DIMENSIONAL MAPS AND GURTIN – MACCAMY'S POPULATION MODEL. PART I. ASYMPTOTICALLY CONSTANT SOLUTIONS

ДИНАМІКА ОДНОВИМІРНИХ ВІДОБРАЖЕНЬ ТА ПОПУЛЯЦІЙНА МОДЕЛЬ ГУРТІНА – МАККЕМІ. ЧАСТИНА І. АСИМПТОТИЧНО СТАЛІ РОЗВ'ЯЗКИ

Motivated by the recent work by Ma and Magal [Proc. Amer. Math. Soc. (2021); https://doi.org/10.1090/proc/15629] on the global stability property of the Gurtin – MacCamy's population model, we consider a family of scalar nonlinear convolution equations with unimodal nonlinearities. In particular, we relate the Ivanov and Sharkovsky analysis of singularly perturbed delay differential equations in [https://doi.org/10.1007/978-3-642-61243-5_5] with the asymptotic behavior of solutions of the Gurtin – MacCamy's system. According the classification proposed in [https://doi.org/10.1007/978-3-642-61243-5_5], we can distinguish three fundamental kinds of continuous solutions of our equations, namely, solutions of the asymptotically constant type, relaxation type and turbulent type. We present various conditions assuring that all solutions belong to the first of these three classes. In the setting of unimodal convolution equations, these conditions suggest a generalized version of the famous Wright's conjecture.

На основі нещодавньої роботи Ма та Магала [Proc. Amer. Math. Soc. (2021); https://doi.org/10.1090/proc/15629] щодо властивості глобальної стабільності популяційної моделі Гуртіна – Маккемі розглянуто сім'ю скалярних нелінійних рівнянь згортки з унімодальними нелінійностями. Зокрема, аналіз сингулярно збурених диференціальних рівнянь із запізненням, запропонований Івановим та Шарковським в [https://doi.org/10.1007/978-3-642-61243-5_5], пов'язано з асимптотикою розв'язків системи Гуртіна – Маккемі. За класифікацією, запропонованою в [https://doi.org/10.1007/978-3-642-61243-5_5], можна виділити три основних типи неперервних розв'язків наших рівнянь, а саме: розв'язки асимптотично сталого типу, релаксаційного та турбулентного типів. Наведено різні умови, які гарантують, що всі розв'язки належать до першого з трьох згаданих класів. У постановці унімодальних рівнянь згортки ці умови пропонують узагальнену версію відомої гіпотези Райта.

1. Introduction: a unimodal convolution equation. In this paper, we study convergence properties of nonnegative continuous solution $b\colon \mathbb{R}_+ \to \mathbb{R}_+ := [0, +\infty)$ to the nonlinear convolution equation

$$b(t) = f\left(\sigma(t) + \int_0^t \beta(a) b(t-a)\, da\right), \tag{1}$$

where $\beta \in L^1(\mathbb{R}_+)$ is a nonnegative function normalized by $\int_0^\infty \beta(a)da = 1$ and the continuous function $\sigma\colon \mathbb{R}_+ \to \mathbb{R}_+$ is converging at $+\infty$ (without loss of generality, we can assume that $\sigma(+\infty) = 0$).

Equation (1) is equivalent to the nonlinear Volterra integral equation

$$B(t) = \sigma(t) + \int_0^t \beta(a) f(B(t-a))\, da, \tag{2}$$

where

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$$B(t) = \sigma(t) + \int_0^t \beta(a)b(t-a)da, \quad b(t) = f(B(t)).$$

All the past century, equation (2) was the object of intensive studies by many authors and by means of different approaches. We refer to the encyclopedia [11] for the historical notes and further references. The asymptotic behavior of solutions of (2) strongly depends on specific properties of the nonlinearity f and kernel β. Even if much can be said about dynamics in (2) for some particular classes of f, β, in general, the studies of (2) give onto difficult open problems (some of them will be discussed later). In this paper, in addition to the condition $\sigma(+\infty) = 0$ we assume the following hypothesis:

(UM) $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is a unimodal function satisfying the Lipschitz condition with some constant* $L > 0$ *and such that* $f(0) = f(+\infty) = 0$, $f_0 := f(u_0) = \max\{f(u), u \geq 0\}$, $f(u) > 0$ *for all* $u > 0$. *Next, equation* $f(u) = u$ *has a unique positive solution* $u = \kappa$, $f(u) - u > 0$ *for* $u \in (0, \kappa)$, *and if the support of* β *is not compact then the upper right-hand Dini derivative* $D^+f(0) = \limsup_{x \to 0^+}(f(x)/x)$ *satisfies* $D^+f(0) > 1$ (*see Fig.*1).

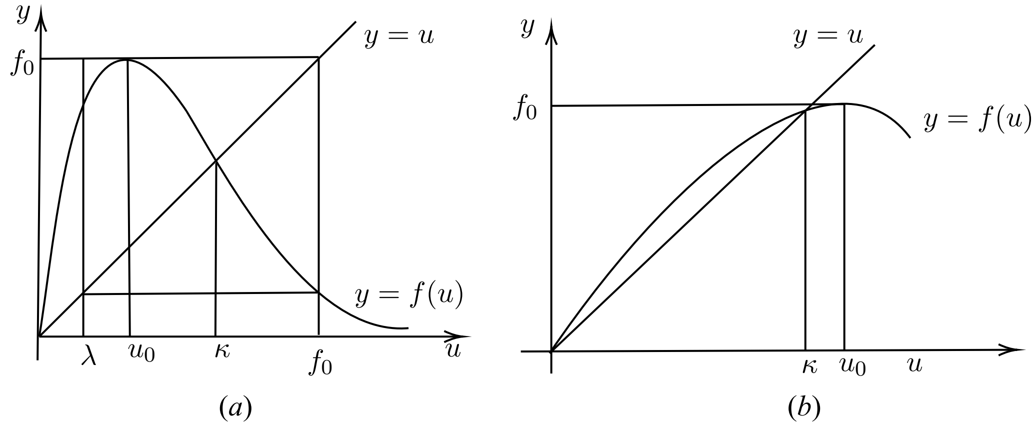

Fig. 1. Graphs of the unimodal nonlinearities $f : [0, \infty) \to [0, \infty)$: (*a*) $u_0 < \kappa$, (*b*) $u_0 > \kappa$.

In Section 5, we show how the unimodal equation (1) naturally appears in the theory of Gurtin–MacCamy's population model [11, 12, 20, 21, 31].

Let us start with one basic yet important example. Fix large $\mu \gg 1$ (so that $0 < \epsilon := 1/\mu$ is a small parameter) and consider one simplest form $\beta_\mu(a)$ for $\beta(a)$, called the weak delay kernel and defined as $\mu e^{-\mu(a-1)}$ for $a \geq 1 > 0$ and 0 otherwise. In a complementary fashion, suppose that $\sigma(t) = 0$ for $t \geq 1$. Then $b(t)$ solves (2) if and only if satisfies the singularly perturbed delay differential equation

$$\epsilon b'(t) = -b(t) + f(b(t-1)), \qquad t \geq 1, \quad b(s) = \sigma(s), \quad s \in [0, 1], \tag{3}$$

which was, in particular, analysed by A. F. Ivanov and A. N. Sharkovsky [8]. Taking limit in (3) as $1/\mu = \epsilon \to 0^+$ (so that $\beta_\mu(a)$ weakly* converges to the shifted Dirac delta $\delta(a-1)$), we recover the standard scalar difference equation with continuous argument and given initial function:

$$b(t) = f(b(t-1)), \qquad t \geq 1, \quad b(s) = \sigma(s), \quad s \in [0, 1]. \tag{4}$$

As it was shown in [8], the behavior of solutions to (4) can vary from very simple (asymptotically constant or asymptotically periodic) form to rather complicated (turbulent) type. One of questions suggested by A. F. Ivanov and A. N. Sharkovsky [8, p. 178, Section 2] is how much of the dynamics in (4) can be inherited by (3). Clearly, this question can be generalized and we can ask how much of the dynamics in (4) can be inherited by equation (1). One of goals of this note is to shed some light on this problem.

2. Uniform persistence and a criterion of the absolute global attractivity. The following result is well-known, for the reader's convenience and completeness of exposition, in Appendix A we present a short proof of it.

Proposition 1. *Assume that* **(UM)** *is satisfied. There exists a unique continuous solution* $b: \mathbb{R}_+ \to \mathbb{R}_+$ *of equation* (1) *which can be found by the method of successive approximations. If, in addition,* $\sigma(s) > 0$ *for some* $s \geq 0$, *then there exists* $t_0 > 0$ *such that* $b(t) > 0$ *for all* $t \geq t_0$.

In this section, we provide explicit asymptotic lower estimates at $+\infty$ for the solution $b(t)$ of equation (1). We use the notation $[\lambda, \Lambda]$ for the minimal attracting interval for the map $f: (0, f_0] \to (0, f_0]$. See Fig. 1(*a*), where $\Lambda = f_0$, $\lambda = f(f_0)$ (the existence of the attracting interval $[\lambda, \Lambda]$, called the persistence attractor, for the unimodal map f with $u_0 \leq \kappa$ is proven in [27, Proposition 5.12]) and Fig. 1(*b*), where $\Lambda = \lambda = \kappa$. More exactly, we prove the following result.

Theorem 1. *Suppose that the hypothesis* **(UM)** *is satisfied. Then independently on the choice of the initial function* $\sigma \not\equiv 0$, *we have*

$$m := \liminf_{t\to\infty} b(t) \geq \lambda, \qquad M := \limsup_{t\to\infty} b(t) \leq \Lambda, \quad \textit{and} \quad \kappa \in [m, M] \subseteq f^n\big([m, M]\big), \quad n \in \mathbb{N}.$$

Proof. Take a monotone sequence of positive real numbers $\{t_j\}_{j=1}^{\infty}$, where t_0 is defined in the statement of Proposition 1 and $t_j \to +\infty$, $b(t_j) \to m$. Define $b_j: [-t_j, +\infty) \to (0, +\infty)$ by $b_j(a) = b(t_j + a)$. The sequence $\{b_j\}$ is equicontinuous since b is uniformly continuous, and it is uniformly bounded since b is bounded. Thus, by the Ascoli – Arzelà theorem, without loss of generality, we can assume that $\{b_j\}$ converges to a continuous function $\psi: \mathbb{R} \to \mathbb{R}_+$, and convergence is uniform on compact subsets of $\mathbb{R}$. Note that $m \leq \psi(t) \leq M$. Indeed, take $a \in \mathbb{R}$ and $T \geq 0$. From the properties of $\{t_j\}$ we know that $t_j + a \geq T$ for all j large enough, which implies that

$$\inf_{t\geq T} b(t) \leq b_j(a) = b(t_j + a) \leq \sup_{t\geq T} b(t),$$

then, first taking $j \to \infty$ and then considering $T \to \infty$, we get that $m \leq \psi(a) \leq M$.

It follows from (1) that

$$b_j(s) = b(t_j + s) = f\left(\sigma(t_j + s) + \int_0^{t_j+s} \beta(a) b_j(s - a)\,da\right), \quad s \geq 0. \tag{5}$$

Suppose for a moment that $m = 0$, i.e., $b(t_j) \to 0$. At this stage of the proof, it is convenient to use equivalent form (2) of our equation. Observe that $\liminf_{t\to+\infty} B(t) = \lim_{j\to+\infty} B(t_j) = 0$ and $B(t) > 0$ for all $t \geq t_0$. Thus we can choose an increasing sequence $s_j \to +\infty$ in such a way that $B(s_j)$, $f(B(s_j))$ are decreasing and

$$f(B(s_j))/B(s_j) \to D^+ f(0), \qquad B(s_j) = \min\big\{B(s): s \in [t_0, s_j]\big\}.$$

Monotonicity of f on $[0, u_0]$ and boundedness of $B(t)$ then assure that, for all large j,

$$f(B(s_j)) = \min\{f(B(s)) : s \in [t_0, s_j]\},$$

so that

$$B(s_j) \geq \int_0^{s_j} \beta(s_j - a) f(B(a)) da \geq \int_{t_0}^{s_j} \beta(s_j - a) f(B(s_j)) da = f(B(s_j)) \int_0^{s_j - t_0} \beta(a) da.$$

Thus,

$$1 \geq \lim_{j \to +\infty} \frac{f(B(s_j))}{B(s_j)} \int_0^{s_j - t_0} \beta(a) da = D^+ f(0) > 1,$$

a contradiction proving that $m > 0$. Observe also that if $f'(0) = 1$ and the support of β is compact, we also get a contradiction: $B(s_j) \geq f(B(s_j)) > B(s_j) > 0$.

Next, considering $j \to \infty$ in equation (5), we obtain

$$\psi(s) = f\left(\int_0^{\infty} \beta(a) \psi(s - a) da\right), \qquad m = \psi(0) = f\left(\int_0^{\infty} \beta(a) \psi(-a) da\right).$$

Clearly,

$$m \leq \int_0^{\infty} \beta(a) \psi(-a) da \leq M$$

and, therefore, $m \in f\big([m, M]\big)$.

In the same way, we can find a monotone sequence of positive real numbers $\{T_j\}_{j=1}^{\infty}$ such that $T_j \to +\infty$, $b(T_j) \to M$ and $b(T_j + a)$ converges to a continuous function $\zeta : \mathbb{R} \to [m, M]$ uniformly on compact subsets of $\mathbb{R}$. Then the limiting function $\zeta(t) = \lim_{j \to +\infty} b(t + T_j)$ satisfies the renewal equation

$$\zeta(t) = f\left(\int_0^{\infty} \beta(a) \zeta(t - a) da\right), \quad t \in \mathbb{R}, \tag{6}$$

and

$$M = \zeta(0) = f\left(\int_0^{\infty} \beta(a) \zeta(-a) da\right), \qquad m \leq \int_0^{\infty} \beta(a) \zeta(-a) da \leq M.$$

Therefore, $M \in f\big([m, M]\big)$ allowing to conclude that $[m, M] \subseteq f\big([m, M]\big)$. The latter inclusion assures the existence of a fixed point of f on $[m, M]$. Since $m > 0$, this point coincides with κ. Thus, $m \leq \kappa \leq M$. By iterating the above inclusion, we also get

$$[m, M] \subseteq f^n\big([m, M]\big), \quad n \in \mathbb{N}.$$

Since $[m, M]$ is attracted by the set $[\lambda, \Lambda]$, we conclude that $[m, M] \subset [\lambda, \Lambda]$ so that $\lambda \leq m$.

The theorem is proved.

***Remark* 1.** While proving the inequality $m > 0$, we followed argumentation in [4, Lemma 3.1]. Conversely, our proof indicates how the result of [4, Lemma 3.1] can be improved: in the cited lemma, a) monotonicity of $f(u)$ can be assumed only in a small right neighbourhood of $u = 0$; b) any restriction on the upper bound of solution $b(t)$ can be omitted whenever $\limsup_{u\to\infty} f(u)/u > 1$ and $f(u) > 0$ for $u > 0$.

***Remark* 2.** A seemingly open question is whether the conclusion of Theorem 1 remains valid in the case when $f'(0) = 1$ and the support of β is not compact.

The renewal equation (6) has exactly two nonnegative constant solutions, $\zeta = 0$ and $\zeta = \kappa$, the following statement is obvious:

Lemma 1. *Assume that* **(UM)** *is satisfied. If* $\zeta : \mathbb{R} \to \mathbb{R}_+$ *satisfies* (6), *then, for each fixed* $s \in \mathbb{R}$, *function* $\zeta(t + s)$, $t \geq 0$, *is solution of equation* (1) *with* $\sigma_s(t) = \int_t^\infty \beta(a)\zeta(t + s - a)da$. *If* $\zeta(t) = \kappa$ *is the unique solution of* (6) *having positive* $\inf_{t\in\mathbb{R}} \zeta(t)$, *then each nonzero solution of equation* (1) *converges to* κ *at* $+\infty$.

Assume now that f is continuously differentiable. By linearizing the renewal equation (6) at the equilibria, we obtain the linear convolution equations

$$\zeta(t) = \alpha \int_0^\infty \beta(a)\zeta(t - a)da, \qquad t \in \mathbb{R}, \quad \alpha \in \{f'(\kappa), f'(0)\}. \tag{7}$$

The following so-called characteristic equations at 0 and κ play a key role (see, e.g., [1, 4, 9]) in the studies of equations (6):

$$1 = \alpha \int_0^{+\infty} e^{-\lambda a}\beta(a)da, \quad \alpha \in \{f'(\kappa), f'(0)\}. \tag{8}$$

Clearly, equation (8) determines exponents λ of solutions $\zeta(t) = e^{\lambda t}$ of each linearization. Therefore, the existence of roots λ for equation (8) having $\Re\lambda > 0$ is an indicator of instability of the respective steady state. Equation (8) also appears in a natural way while solving linear inhomogeneous version of (7) by means of the Laplace transform. Actually, the integral expression in (8) is the Laplace transform $\mathcal{L}(\beta)(\lambda)$ of $\beta \in L^1(\mathbb{R}_+)$. Thus, it is well defined on the half-plane $\{\Re\lambda \geq 0\}$ where $\mathcal{L}(\beta)(\lambda) \to 0$ as $|\lambda| \to \infty$ in virtue of the Riemann – Lebesgue lemma. As a consequence, equation (8) has at most a finite set of solutions with $\Re\lambda \geq 0$.

When $f'(0) > 1$, the characteristic equation at 0 has a unique positive eigenvalue. This reflects the instability property of the solution $\zeta \equiv 0$. Next, $f'(\kappa) \leq 1$ in view of the hypothesis **(UM)**. If $f'(\kappa) = 0$ then the set of eigenvalues is vacuous. Suppose for a moment that the characteristic equation at κ has an eigenvalue λ_j in the closed right half-plane, $\Re\lambda_j \geq 0$. Then necessarily $|f'(\kappa)| \geq 1$ in view of the relations

$$|f'(\kappa)|^{-1} = \left| \int_0^{+\infty} e^{-\lambda_j a}\beta(a)da \right| \leq \int_0^{+\infty} e^{-\Re\lambda_j a}\beta(a)da \leq 1.$$

Furthermore, it easy to see that if $f'(\kappa) = 1$ then $\lambda = 0$ is the unique (simple) eigenvalue in the closed right half-plane, and if $f'(\kappa) = -1$ then the characteristic equation does not have eigenvalues λ_j with $\Re\lambda_j \geq 0$. This shows that the equilibrium $\zeta = \kappa$ might have good stability properties when $|f'(\kappa)| \leq 1$. Actually this is true under additional conditions imposed on the nonlinearity f.

***Corollary* 1.** *In addition to the hypothesis* **(UM)**, *assume that f satisfies one of the following conditions*:

(a) *the composition map $f \circ f : [0, f_0] \to [0, f_0]$ has exactly two fixed points*, 0 *and* κ;

(b) $|f'(\kappa)| \le 1$, *f is C^3-smooth and possesses the negative Schwarz derivative*

$$(Sf)(u) = \frac{f'''(u)}{f'(u)} - \frac{3}{2}\left(\frac{f''(u)}{f'(u)}\right)^2, \quad u \in [0, f_0].$$

Then $b(\infty) = \kappa$ independently on the choice of σ from the corresponding functional class.

Proof. Each of the above conditions guarantees that $\lambda = \Lambda = \kappa$, see [25, Theorems 4.1 and 5.3].

Note that the convergence conditions of Corollary 1 does not depend on the specific form of kernel (thus we can call it absolute convergence conditions in analogy to absolute stability conditions in the theory of delayed differential equations). On the other hand, in the next section we show that the statement of Corollary 1 is not necessarily optimal while considering particular kernels: even if $f'(\kappa) < -1$, the characteristic equation (8) can have either an empty set of eigenvalues or all of them can have negative real parts.

3. One particular case: Gamma distribution delay kernels. In this section, fixing parameters $\mu > 0$, $h \ge 0$, $n = 0, 1, 2, \ldots$, we take the following important class of normalized Gamma distribution kernels

$$\beta_n(a, h) = \begin{cases} \dfrac{\mu^{n+1}(a-h)^n}{n!} e^{-\mu(a-h)}, & \text{if} \quad a \ge h, \\ 0, & \text{if} \quad a < h. \end{cases}$$

In particular, these kernels were considered in [12, 20–22, 24]. Two special cases, when $h = 0, n = 0$ and when $h = 0, n = 1$, are called weak delay kernel and strong delay kernel, respectively [24]. With β_n the renewal equation (6) can be written as

$$\zeta(t) = f\left(\int_0^{+\infty} \frac{\mu^{n+1} a^n}{n!} e^{-\mu a} \zeta(t - a - h) da\right), \quad t \in \mathbb{R}. \tag{9}$$

Note that after setting

$$\xi(t) = \int_0^{+\infty} \frac{\mu^{n+1} a^n}{n!} e^{-\mu a} \zeta(t - a) da, \quad t \in \mathbb{R}, \tag{10}$$

we recover $\zeta(t)$ as $\zeta(t) = f(\xi(t-h))$.

In the case $n = 0$, there exists a very close relation between the convolution equation (9) and scalar delay differential equation of the Mackey–Glass type (see, e.g., [22, Section 1.3]). Below we study more general situation when $n \in \mathbb{N}$. Let D denote the differentiation operator: $Df(t) = f'(t)$.

Lemma 2. *$\zeta : \mathbb{R} \to \mathbb{R}_+$ is a continuous solution of equation* (9) *if and only if $\xi : \mathbb{R} \to \mathbb{R}_+$ is the bounded classical solution of the $(n+1)$th order scalar delay differential equation*

$$(D + \mu)^{n+1} \xi(t) = \mu^{n+1} f(\xi(t - h)), \quad t \in \mathbb{R}. \tag{11}$$

Proof. Suppose that ξ is a bounded solution of (11). One way to obtain the integral representation (10) is the use of the Laplace transform method and exploiting the fact that the inverse Laplace transform of $\mu^{n+1}/(\lambda + \mu)^{n+1}$ is precisely β_n. However, it is more convenient to write equation (11) as the system

$$\begin{aligned}
(D+\mu)\xi_{n+1} &= \xi_n, \\
&\dots\dots\dots\dots\dots\dots \\
(D+\mu)\xi_2 &= \xi_1, \\
(D+\mu)\xi_1 &= \mu^{n+1} f(\xi_{n+1}(t-h)),
\end{aligned} \tag{12}$$

where $\xi_{n+1}(t) := \xi(t)$. Note that boundedness of $\xi_{n+1}(t)$ guarantees that each component $\xi_j(t)$ is also bounded on $\mathbb{R}$. The linear part (when we take $f = 0$) of (12) can be written as $z' = (-\mu I + J)z$ where $z = (\xi_{n+1}, \dots, \xi_1)$ and $-\mu I + J$ denotes the Jordan block with the eigenvalue $-\mu < 0$, I is the identity matrix. Since the system $z' = (-\mu I + J)z$ is exponentially stable, each bounded solution of (12) is given by

$$z(t) = \int\limits_{-\infty}^{t} e^{(-\mu I+J)(t-s)} F(s)ds = \int\limits_{-\infty}^{t} e^{-\mu(t-s)} \sum_{k=0}^{n} J^k \frac{(t-s)^k}{k!} F(s)ds,$$

where $F(s) = \big(0, 0, \dots, 0, \mu^{n+1} f(\xi_{n+1}(s-h))\big)^T$ is a column vector. Clearly, formula (10) coincides with the equation for the first component of $z(t)$ in the above integral representation. Consequently, each bounded solution of (11) satisfies (10). A straightforward verification also shows that (10) yields a solution of (11).

The lemma is proved.

***Remark* 3.** As it is shown in [26], system (12) can be used to model the control of protein synthesis in the cell. In [26, Chapter 4, Section 2] and [26, Chapter 5, Section 6] this system was studied in the case of the positive feedback loop controlling the translation of DNA to mRNA (that corresponds to the situation when $f'(\kappa) \geq 0$ in **(UM)**).

Hence, the problem of finding bounded solutions for (9) can be reduced to the problem of description of the complete orbits for system (12). In general, the dynamics in (12) can be quite complicated (including the option of 'turbulent' behavior of solutions). In next result, we consider the simplest case when $h = 0$ and $n = 0$ or $n = 1$.

Theorem 2. *In addition to the hypothesis* **(UM)**, *assume that* (1) $h = n = 0$ *or* (2) $h = 0$, $n = 1$ *and* f *is* C^1*-smooth with* $f'(\kappa) < 1$. *Then* $\zeta(t) \equiv \kappa$ *is the unique positive solution of equation* (9) *satisfying condition* $\inf_{t\in\mathbb{R}} \zeta(t) > 0$.

Proof. Take first $n = 0$, then

$$\xi'(t) = -\mu\xi(t) + \mu f(\xi(t)), \quad t \in \mathbb{R}.$$

Since the unique bounded solution of this equation with the positive $\inf_{t\in\mathbb{R}} \xi(t)$ is its constant solution $\xi(t) \equiv \kappa$, we conclude that also $\zeta(t) \equiv \kappa$.

Take now $n = 1$, then after rescaling the time variable ($t \to \mu t$), we get

$$\xi''(t) + 2\xi'(t) + \xi(t) - f(\xi(t)) = 0.$$

This equation can be written as the system

$$x' = y, \qquad y' = -2y - x + f(x).$$

Clearly, it has exactly two equilibria, $(0,0)$ and $(\kappa,0)$. Applying the Bendixson – Dulac theorem [6] with the Dulac function $D \equiv 1$, we find that this system does not have periodic solutions. Thus, in view of the Poincaré – Bendixson theorem each bounded solution $(x(t),y(t))$ of the system satisfying $\inf_{t\in\mathbb{R}} x(t) > 0$ should be either the equilibrium $(\kappa,0)$ or the homoclinic solution to this equilibrium. Since the characteristic equation at $(\kappa,0)$ is $(\lambda+1)^2 = f'(\kappa) < 1$, this equilibrium is locally exponentially stable and therefore does not possess homoclinic solutions.

The theorem is proved.

***Remark* 4.** It is worth to mention that if $f'(\kappa)$ is a sufficiently large negative number then the main conclusion of Theorem 2 does not hold in the higher dimensions $n = 2, 3, \ldots$. This situation will be analyzed in the second part of our studies.

Hence, with either weak or strong delay kernel, the hypothesis **(UM)** guarantees that each solution $b(t)$ of equation (1) satisfies $b(+\infty) = \kappa$ independently on the choice of σ and f from the corresponding functional classes. This agrees with the fact that the corresponding characteristic equations

$$1 = f'(\kappa)\int_0^{+\infty} \mu e^{-(\mu+\lambda)a}da, \qquad 1 = f'(\kappa)\int_0^{+\infty} \mu^2 a e^{-(\mu+\lambda)a}da$$

has only eigenvalues λ_j with $\Re\lambda_j \le 0$ if $f'(\kappa) \in (0,1]$ and has an empty set of eigenvalues if $f'(\kappa) \le 0$. It is worth to mention that the characteristic function

$$\chi_n(z) = (\lambda+\mu)^{n+1} - \mu^{n+1} f'(\kappa)$$

for the linearization of differential equation (11) with $h = 0$ at $\xi = \kappa$ has better analytic properties since it is defined for all complex λ and the respective eigenvalues λ_j has nonpositive real parts if $f'(\kappa) \le 1$, $h = 0$, $n = 0, 1$. In virtue of this difference between two types of characteristic equations sometimes we can obtain more information about the asymptotic behavior of bounded solutions from the differential forms (11) or (12) than from the integrated form (9).

Next, in the case when $h > 0$, $n = 0$ and **(UM)** is assumed, equation (11) is called the Mackey – Glass type equation

$$\xi'(t) = -\mu\xi(t) + \mu f(\xi(t-h)), \quad t \in \mathbb{R}, \tag{13}$$

and it the was object of intensive studies [30]. In particular, it was hypothesized in [16] (see also [26, p. 116] and [17]) that under the assumption **(UM)** and the condition of negativity of the Schwarz derivative $Sf(x)$, the positive equilibrium of equation (13) attracts all positive orbits once it is locally asymptotically stable. The famous Wright's conjecture can be viewed as particular case of this general hypothesis and the recent affirmative solution of the Wright conjecture in [29] gives an additional support for this more general formulation of the equivalence between local and global stabilities. In this way, assuming **(UM)** and the inequality $Sf(x) < 0$, we can expect that the only positive global solution of (9) separated from 0 is its equilibrium $\xi(t) = \kappa$ if and only if the characteristic equation

$$\lambda + \mu = \mu f'(\kappa) e^{-\lambda h} \tag{14}$$

does not have roots λ_j with positive real part $\Re\lambda_j > 0$. The next result seems to be a rather good approximation to this criterion, cf. [16].

Theorem 3. *In addition to the hypothesis* **(UM)**, *assume that $n = 0$ and C^3-smooth f possesses the negative Schwarz derivative. Then if either $f'(\kappa) \in [0,1]$ or $f'(\kappa) < 0$ and the following* *inequality holds*:

$$e^{-h\mu} > -f'(\kappa)\ln\frac{(f'(\kappa))^2 - f'(\kappa)}{(f'(\kappa))^2+1}, \tag{15}$$

then $\zeta(t) \equiv \kappa$ *is the unique positive solution of equation* (9) *having* $\inf_{t\in\mathbb{R}}\zeta(t) > 0$.

Proof. In view of Corollary 1, it suffices to consider the case $f'(\kappa) < -1$. By [7, Theorem 2.9], inequality (15) implies that the equilibrium $\xi(t) = \kappa$ of equation (13) is locally exponentially stable. From [17, Corollary 2.3] we also obtain that each positive solution $\xi(t)$ of (13) satisfies $\xi(+\infty) = \kappa$. This implies the conclusion of the theorem.

***Remark* 5.** In the case when the lower bound λ for the minimal attracting interval $[\lambda, \Lambda]$ (cf. Theorem 1) is larger than $\min f^{-1}(\kappa)$ (precisely this position is shown on Fig. 1(*a*)) and equation (14) has eigenvalues with positive real parts then equation (13) has at least one slowly oscillating periodic solution due to the famous result by K. P. Hadeler and J. Tomiuk in [13]. It can also have 'turbulent' solution: concise survey written by H.-O. Walther [30] can be recommended as a source of further references.

***Remark* 6.** Suppose that $\mu \gg 1$ and set $\epsilon = 1/\mu > 0$. Then we obtain from (13) equation (3) with small positive parameter ϵ. Therefore, all results of A. F. Ivanov and A. N. Sharkovsky from [8] concerning the global solutions of (3) have a straightforward interpretation in the framework of equation (9) with $n = 0, h = 1$ and large $\mu > 0$. See also fundamental work [23] by J. Mallet-Paret and R. D. Nussbaum and Section 7 in [8].

The above discussion and the mentioned conjecture from [15, 16] suggest one additional open problem: *In addition to* **(UM)**, *suppose that* C^3*-smooth* f *possesses the negative Schwarzian. Prove* (*or find a counterexample to the next statement*) *that* $\zeta(t) = \kappa$ *is the unique bounded solution of equation* (6) *whenever the characteristic equation* (8) *with* $\alpha = f'(\kappa)$ *does not have characteristic values with nonnegative real part* (*by the Paley – Wiener theorem* [11], *the latter amounts to integrability of the resolvent of* β *on* $\mathbb{R}_+$. See also Nyquist's criterion in [11, Corollary 6.5].

In the next section, we provide one more argument supporting the veracity of the proposed hypothesis.

4. Nonincreasing kernels. As we have seen in the previous section, it is convenient to reduce the renewal equation (6) by means of the change of variables

$$\xi(t) = \int_0^\infty \beta(a)\zeta(t-a)da, \qquad \zeta(t) = f(\xi(t)), \quad t \in \mathbb{R},$$

to the next, more standard, form of the convolution equation

$$\xi(t) = \int_0^\infty \beta(a)f(\xi(t-a))da = \int_{-\infty}^t \beta(t-a)f(\xi(a))da, \quad t \in \mathbb{R}. \tag{16}$$

In this section, we consider the particular situation when kernel $\beta(a)$ is a nonincreasing function. Our next assertion shows that then each solution $\xi(t)$ of (16) satisfies the following nonlinear differential equation with unbounded delay:

$$\xi'(t) = \beta(0)f(\xi(t)) + \int_0^{+\infty} f(\xi(t-a))d\beta(a). \tag{17}$$

Lemma 3. *Suppose that* $f:\mathbb{R}_+\to\mathbb{R}_+$ *is a unimodal continuous function and* $\beta\ge 0$ *is a nonincreasing normalized kernel. Then* $\beta(t)=o(1/t)$, $t\to+\infty$, *and each solution of equation* (16) *is a continuously differentiable function satisfying* (17).

Proof. Set $\gamma=\limsup_{t\to+\infty} t\beta(t)$. If $\gamma\in(0,+\infty]$, then there exist some $K>0$ and an increasing sequence $t_j\to+\infty$ such that $t_j\beta(t_j)\ge K$. This implies that $\beta(t)\ge K/t_j$ for $t\in[0,t_j]$ and, consequently, for each positive integer q, it holds that

$$\int\limits_{t_q}^{+\infty}\beta(a)da\ge\lim_{n\to+\infty}K\sum_{j=q+1}^{n}\frac{t_j-t_{j-1}}{t_j}\ge\lim_{n\to+\infty}K\frac{1}{t_n}\sum_{j=q+1}^{n}(t_j-t_{j-1})=K,$$

This means that $\int_0^{+\infty}\beta(a)da=+\infty$, a contradiction proving that $\lim_{t\to+\infty}t\beta(t)=0$.

Consider now the equation

$$\xi(t)=\int\limits_0^{\infty}\beta(a)f(\xi(t-a))da=\int\limits_{-\infty}^{t}\beta(t-a)f(\xi(a))da,\quad t\in\mathbb{R}.$$

Set

$$F_t(a)=\int\limits_0^{a}f(\xi(t-s))ds=\int\limits_{t-a}^{t}f(\xi(s))ds,$$

then $0\le F_t(a)\le af_0,\ a\ge 0$, and, for each $T>0$, it holds that

$$\int\limits_0^{T}\beta(a)f(\xi(t-a))da=\int\limits_0^{T}\beta(a)dF_t(a)=\beta(T)F_t(T)-\int\limits_0^{T}F_t(a)d\beta(a),$$

in view of the integration by parts formula for the Riemann – Stieltjes integral. By taking limit as $T\to+\infty$, we find that

$$\xi(t)=\int\limits_0^{+\infty}\beta(a)f(\xi(t-a))da=-\int\limits_0^{+\infty}F_t(a)d\beta(a),\quad t\in\mathbb{R}.$$

This shows that $\xi(t)$ is differentiable on $\mathbb{R}$ and, for all $t\in\mathbb{R}$,

$$\xi'(t)=-\int\limits_0^{+\infty}(f(\xi(t))-f(\xi(t-a)))d\beta(a)=\beta(0)f(\xi(t))+\int\limits_0^{+\infty}f(\xi(t-a))d\beta(a).$$

The lemma is proved.

Let observe that nonincreasing kernels are meaningful from the biological point of view, e.g., see the seminal work [3] by K. Cooke and J. Yorke, where the simple kernel

$$\beta(a)=\begin{cases}1/h, & a\in[0,h],\\ 0, & \text{otherwise},\end{cases}$$

with some $h > 0$ was considered. With such a kernel equations (16) and (17) take the form

$$\xi(t) = \frac{1}{h}\int\limits_0^h f(\xi(t-a))da = \frac{1}{h}\int\limits_{t-h}^t f(\xi(a))da, \quad t \in \mathbb{R}, \tag{18}$$

and

$$\xi'(t) = \frac{1}{h}\big(f(\xi(t)) - f(\xi(t-h))\big), \quad t \in \mathbb{R}. \tag{19}$$

Reciprocally, if $\xi(t),\ t \geq 0,$ is a solution to the latter equation, then the derivative of the function $\xi(t) - \frac{1}{h}\int_{t-h}^t f(\xi(a))da$ is equal to 0. Thus, $\xi(t) = \frac{1}{h}\int_{t-h}^t f(\xi(a))da + C,\ t \geq 0$, for some $C \in \mathbb{R}$. In this way, to choose the solution of equation (18) between solutions of (19) defined for $t \geq 0$, we have to take initial value functions $\phi(s), s \in [-h, 0]$, satisfying the restriction $\phi(0) = \frac{1}{h}\int_{-h}^0 f(\phi(a))da$.

Under assumptions (**UM**) and continuous differentiability of f [3, Theorem 2] guarantees that each nonzero solution $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ of (18) converges to κ at $+\infty$ (i.e., $\xi(t) = \kappa$ is the only nonzero solution $\xi : \mathbb{R} \to \mathbb{R}_+$ of equation (18) uniformly separated from 0). In view of the problem proposed at the end of Section 3, this result agrees with the following statement.

Lemma 4. *Suppose that $\beta(a)$ is nonincreasing. Then the characteristic equation* (8) *with $\alpha = f'(\kappa) \leq 0$ does not have characteristic values λ with nonnegative real part.*

Proof. Due to observations given below formula (8), we can assume that $f'(\kappa) \leq -1$ and $\lambda \neq 0$. If $\lambda = i\omega$ for some $\omega > 0$, then

$$0 = \int\limits_0^{+\infty} \beta(a)\sin(\omega a)da = \sum_{j=0}^{+\infty}\int\limits_0^{2\pi/\omega} \beta(a + 2j\pi/\omega)\sin(\omega a)da \geq 0,$$

so that the last inequality is actually an equality and $\beta(a)$ is a constant function on each of the intervals $2\pi/\omega[j, j+1]$. But then

$$0 > 1/f'(\kappa) = \int\limits_0^{+\infty} \beta(a)\cos(\omega a)da = \sum_{j=0}^{+\infty}\int\limits_0^{2\pi/\omega} \beta(a + 2j\pi/\omega)\cos(\omega a)da = 0,$$

a contradiction. If $\Re\lambda > 0$, then we have

$$\lambda = -f'(\kappa)\int\limits_0^{+\infty} \beta(a)de^{-\lambda a} = f'(\kappa)\left(\beta(0) + \int\limits_0^{+\infty} e^{-\lambda a}d\beta(a)\right) =: f'(\kappa)\Lambda.$$

On the other hand, $\Re f'(\kappa)\Lambda < 0$ because of $\beta(0) > 0,\ f'(\kappa) < 0,\ \left|\int_{\mathbb{R}_+} e^{-\lambda a}d\beta(a)\right| < \beta(0)$, so that again we obtain a contradiction.

The lemma is proved.

The work [3] and Lemma 4 together with discussion at the end of Section 3 suggests the asymptotic constancy of solutions to (1) with nonincreasing kernel $\beta(a)$. This fact indeed holds as a consequence of the remarkable convergence theorem established by S.-O. Londen in [18, Theorem 1] (see also [19] and [2, 11] for further extensions). Below we present a rather short proof of the mentioned Londen result adapted to our dynamical style framework of unimodal convolution equations.

Proposition 2. *Assume that* **(UM)** *is satisfied and* $\beta \geq 0$ *is a nonincreasing normalized kernel. Then each nonzero solution* $b(t)$ *of equation* (1) *satisfies* $b(+\infty) = \kappa$.

Proof. It suffices to consider $u_0 < \kappa$. Let $m \leq \kappa \leq M$ be defined as in Theorem 1. We claim that then $\max_{[m,M]} f(u) = f(M)$. As a consequence of this relation, since f is decreasing in some open neighbourhood of κ, we conclude that $m = M = \kappa$.

Indeed, suppose that $m < M$ and $\max_{[m,M]} f(u) = f(u_1)$ for some $u_1 \in [m, M)$. Choose increasing sequences $s_j < t_j$ such that

$$t_j - s_j \to \alpha \in (0, +\infty], \qquad b(t_j) \to u_1,\ b(s_j) = \frac{u_1 + M}{2}, \qquad b(t_j) < b(t) \leq b(s_j), \quad t \in [s_j, t_j),$$

and $b(t + t_j) \to \xi(t)$, $t \in \mathbb{R}$, uniformly on compact sets. The existence of such sequences is obvious (cf. the proof of Theorem 1). Then $\xi(0) = u_1$, $\xi(t) \geq u_1$ for $t \in [-\alpha, 0]$ and $m \leq \xi(t) \leq M$ for all $t \in \mathbb{R}$. Since $\xi : \mathbb{R} \to (0, +\infty)$ also satisfies (17), we find that

$$\xi'(t) \geq \beta(0) f(\xi(t)) + \int_0^{+\infty} f(u_1) d\beta(a) = \beta(0)\big(f(\xi(t)) - f(u_1)\big), \quad t \in \mathbb{R}.$$

Then a standard comparison result for the scalar ordinary differential inequalities (see [28, Theorem 9.6]) assures that $\xi(t) \leq \zeta(t) := u_1$, $t \leq 0$, where $\zeta(t) = u_1$ solves the initial value problem $\zeta(0) = u_1$ for the scalar differential equation

$$\zeta'(t) = \beta(0)\big(f(\zeta(t)) - f(u_1)\big), \quad t \in \mathbb{R}. \tag{20}$$

But this means that $\xi(t) = u_1$ for $t \in [-\alpha, 0]$ and therefore $\alpha = +\infty$. Consequently,

$$u_1 = \xi(0) = \int_0^{+\infty} \beta(a) f(\xi(-a)) da = \int_0^{+\infty} \beta(a) f(u_1) da = f(u_1) = \max_{[m,M]} f(u).$$

By Theorem 1, this is possible only if $u_1 = \kappa = m = M$, a contradiction that finalises the proof of the theorem.

5. Applications to Gurtin – MacCamy's population dynamics model. In this section, we show how our conclusions concerning asymptotic behavior of solutions to (1) can be used for description of long-time behavior of solutions to the following Gurtin – MacCamy's age-structured model [11, 12, 20, 21, 31] (also known as the McKendrick – von Foerster equation subject to a nonlinear boundary condition [5, 10])

$$\begin{aligned} &(\partial_t + \partial_a) u(t,a) = -\mu u(t,a), \qquad t > 0, \quad a > 0, \\ &u(t,0) = f\left(\int_0^{\infty} \gamma(a) u(t,a)\, da\right), \quad t > 0, \qquad u(0,\cdot) = u_0 \in L^1_+(\mathbb{R}_+), \end{aligned} \tag{21}$$

where $\mu > 0$, function f satisfies **(UM)** and the kernel $\beta(a) = \gamma(a)e^{-\mu a}$ is normalized by $\int_0^{\infty} \beta(a) da = 1$. Observe that system (21) has a unique positive equilibrium $\overline{u}(a) = \kappa e^{-\mu a}$, and its solution can be obtained by the method of characteristics:

$$u(t,a) = \begin{cases} e^{-\mu t}u_0(a-t), & \text{if} \quad a \geq t, \\ e^{-\mu a}b(t-a), & \text{if} \quad a \leq t, \end{cases}$$

where $b\colon \mathbb{R}_+ \to \mathbb{R}_+$ is the unique continuous solutions of equation (1) with

$$\sigma(t) := \int_t^\infty \gamma(a)e^{-\mu t}u_0(a-t)da.$$

Thus, we have the following theorem.

Theorem 4. *In addition to* **(UM)**, *suppose that one of the following conditions holds*:

1) $a \mapsto \gamma(a)e^{-\mu a}$ *is a nonincreasing function*;

2) $\gamma(a) = \mu e^{\mu h}$ *for* $a \geq h \geq 0$ *and* $\gamma(a) = 0$ *otherwise. In addition*, f *is* C^3*-smooth*, *has the negative Schwarzian derivative and inequality* (15) *is satisfied*;

3) $\gamma(a) \equiv \mu^2 a$ *and* f *is* C^1*-smooth with* $f'(\kappa) < 1$;

4) f *is* C^3*-smooth with* $|f'(\kappa)| \leq 1$ *and has negative Schwarzian derivative*;

5) *composed map* $f \circ f : [0, f_0] \to [0, f_0]$ *has exactly two fixed points*, 0 *and* κ.

Then the equilibrium $\overline{u}$ *is a global attractor for system* (21). *More precisely*, *if* $\sigma(t) \neq 0$ *then* $u(t,\cdot) \to \overline{u}$ *in* $L^1(\mathbb{R}_+)$.

Proof. Note that

$$\int_0^\infty |u(t,a) - \overline{u}(a)|\, da = \int_0^t |e^{-\mu a}b(t-a) - \kappa e^{-\mu a}|\, da + \int_t^\infty |e^{-\mu t}u_0(a-t) - \kappa e^{-\mu a}|\, da$$

$$= e^{-\mu t}\int_0^t e^{\mu a}|b(a) - \kappa|\, da + e^{-\mu t}\int_0^\infty |u_0(a) - \kappa e^{-\mu a}|\, da.$$

The last term clearly goes to 0 as $t \to \infty$. On the other hand, each of 5 conditions of the theorem guarantees that $b(t) \to \kappa$ as $t \to \infty$, so that an obvious application of the L’Hôpital rule implies that the first term also vanishes at $+\infty$.

Theorem 1 also provides explicit bounds for the compact global attractor [20] in (21):

Theorem 5. *Suppose that* **(UM)** *is satisfied and* λ, Λ *are as in Theorem* 1. *Then*, *independently on the choice of* $\sigma \not\equiv 0$, *we have that*

$$\lambda e^{-\mu a} \leq \liminf_{t\to\infty} u(t,a) \leq \limsup_{t\to\infty} u(t,a) \leq \Lambda e^{-\mu a} \quad a \geq 0.$$

Note that Theorems 4 and 5 complement and generalize Theorems 5.2, 5.4, and 6.5 in [20].

Appendix A. Existence, uniqueness and positivity of solutions for the Volterra unimodal convolution equation. It is clear that we can extend f on the interval $(-\infty, 0]$ as a constant function with value 0, without changing solvability property of equation (1). The step-by-step recursive method can be used to prove the existence of the global continuous solutions to equations (3) or (4). In general case, the existence and uniqueness of solution to (1) follows from the Banach fixed-point theorem (e.g., in its generalized version given in [14, Theorem 49.3]). For each $T > 0$, let consider the operator $\mathcal{A}$: $C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ given by the right-hand side of equation (1).

Lemma 5. *Let* **(UM)** *be satisfied. There exists a unique continuous solution* $b\colon \mathbb{R}_+ \to \mathbb{R}$ *of equation* (1) *which is the limit of the sequence* $b_n(t)$ *defined by the recurrence*

$$b_{n+1}(t) = \mathcal{A}b_n(t), \qquad b_0 \equiv 0, \quad n = 0, 1, 2, \ldots .$$

Moreover, $0 \le b(t) \le f_0 := \max_{s\ge 0} f(s)$ *for all* $t \ge 0$.

Proof. Since

$$\big|(\mathcal{A}b_1)(t) - (\mathcal{A}b_2)(t)\big| \le L\left|\int_0^t \beta(a)(b_1(t-a) - b_2(t-a))da\right|, \quad t \in [0,T],$$

and the positive linear operator $\mathcal{B}\colon C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ defined by

$$(\mathcal{B}x)(t) = L\int_0^t \beta(a)x(t-a)da$$

has zero spectral radius, equation (1) has a unique solution $b(t)$ which can be found as the limit of successive approximations $b_n(t)$ converging uniformly on compact subsets of $\mathbb{R}_+$. The upper and lower estimates for $b(t)$ follows from the properties of f.

In fact, the solution $b(t)$ of equation (1) is uniformly continuous on $\mathbb{R}_+$ due to following well-known fact.

Lemma 6. *If* $x \in L^\infty(\mathbb{R}_+)$ *and* $\beta \in L^1(\mathbb{R}_+)$, *then the convolution* $(\mathcal{B}x)(t)$ *is uniformly continuous on* $\mathbb{R}_+$.

***Corollary* 2.** *Let* **(UM)** *be satisfied. Then solution of equation* (1) *is uniformly continuous on* $\mathbb{R}_+$.

Another well-known property of the solution $b(t)$ is its eventual positivity (see, e.g., [27, Corollary B.6] or [20, Proposition 3.6]). Here we present an alternative proof of this fact based on the following lemma (where for two set $A, B \subset \mathbb{R}$ of real numbers we define their sum as $A+B = \{a+b\colon a \in A, b \in B\}$).

Lemma 7. *Let* $\sigma, \beta\colon \mathbb{R}_+ \to \mathbb{R}_+$ *be nonnegative continuous functions with compact nonempty supports* $\mathrm{supp}(\beta)$, $\mathrm{supp}(\sigma)$. *Then*

$$\mathrm{supp}(\beta * \sigma) = \mathrm{supp}(\beta) + \mathrm{supp}(\sigma).$$

Proof. First, let us show that $\mathrm{supp}(\beta * \sigma) \subseteq \mathrm{supp}(\beta) + \mathrm{supp}(\sigma)$. Indeed, if $t_0 \in \mathrm{supp}(\beta * \sigma)$ then there exists a sequence $t_j \to t_0$ such that

$$\int_0^{t_j} \beta(a)\sigma(t_j - a)da > 0.$$

Consequently, for some $a \in (0, t_j)$, it holds that $a \in \mathrm{supp}(\beta), t_j - a \in \mathrm{supp}(\sigma)$. Therefore, $t_j = a + (t_j - a) \in \mathrm{supp}(\beta) + \mathrm{supp}(\sigma)$. Since the latter sum is a compact set, we conclude that $t_0 \in \mathrm{supp}(\beta) + \mathrm{supp}(\sigma)$.

Next, we prove the opposite inclusion $\mathrm{supp}(\beta * \sigma) \supseteq \mathrm{supp}(\beta) + \mathrm{supp}(\sigma)$. So suppose that $x \in \mathrm{supp}(\beta)$, $y \in \mathrm{supp}(\sigma)$. Then there exist sequences $x_j \to x, y_j \to y$ such that $\beta(x_j) > 0, \sigma(y_j) > 0$. Thus,

$$\int_0^{x_j+y_j} \beta(a)\sigma(x_j + y_j - a)da > 0$$

since the integrand is positive at the point $a = x_j$. Therefore, $x_j+y_j \in \operatorname{supp}(\beta*\sigma)$. Since $\operatorname{supp}(\beta*\sigma)$ is a compact set, we conclude that $x + y \in \operatorname{supp}(\beta * \sigma)$.

The lemma is proved.

Proposition 3. *Assume that* **(UM)** *is satisfied and* $\sigma(s) > 0$ *for some* $s \geq 0$. *Then there exists* $t_0 > 0$ *such that* $b(t) > 0$ *for all* $t \geq t_0$.

Proof. Since b is bounded by f_0, it holds that, for some positive M,

$$0 \leq \sigma(t) + \int_0^t \beta(a)b(t-a)da \leq \sigma(t) + f_0 \leq M, \quad t \geq 0.$$

So, we can choose $k > 0$ small enough such that $f(u) \geq ku$ for all $u \in [0, M]$. Let also some continuous and bounded measurable, respectively, functions $\sigma_1(t)$ and $\beta_1(t)$ be sufficiently close to $\sigma(t)$, $\beta(t)$ in their spaces, have compact supports and satisfy the inequalities

$$0 \leq \sigma_1(t) \leq \sigma(t), \qquad 0 \leq \beta_1(t) \leq \beta(t), \quad t \geq 0.$$

Then from equation (1) we obtain that

$$b(t) \geq k\left(\sigma_1(t) + \int_0^t \beta_1(a)b(t-a)da\right) \quad \forall t \geq 0.$$

Now, consider the operator $A\colon C_b(\mathbb{R}_+) \to C_b(\mathbb{R}_+)$ defined by $Ab = k(\beta_1 * b)$ (here $C_b(\mathbb{R}_+)$ is the Banach space of all continuous bounded functions with sup-norm). Then $A^2b = k^2((\beta_1 * \beta_1) * b) = k^2(\beta_2 * b)$, where $\beta_2 := \beta_1 * \beta_1$ is a nonnegative continuous function with nonempty compact support containing some nonempty open interval (p, q) such that $\beta_2(t) > 0$ for $t \in (p, q)$ (see Lemma 6).

If we set $\sigma_2 := k\sigma_1$, then the latter inequality takes the form $b(t) \geq \sigma_2(t) + (Ab)(t)$. By iterating, we obtain

$$b(t) \geq \sum_{n=0}^{\infty} A^n\sigma_2(t) \geq \sum_{n=0}^{\infty} (A^2)^n\sigma_2(t) \geq 0, \quad t \geq 0. \tag{A.1}$$

Clearly, $\sum_{n=0}^{\infty} A^n\sigma_2$ is well defined since $\|A\sigma\|_\infty \leq k\|\sigma\|_\infty$ so that $\|A\| \leq k < 1$.

By applying Lemma 7 to inequality (A.1), for each integer n and for $z = \inf\operatorname{supp}(\sigma_2)$ we have that

$$\operatorname{supp}(b) \supset \operatorname{supp}((A^2)^n\sigma_2) = \sum_{j=1}^{n} \operatorname{supp}(\beta_2) + \operatorname{supp}(\sigma_2) \supset n(p, q) + z = (z + np, z + nq).$$

Therefore, $\operatorname{supp}(b) \supset \bigcup_{n\geq 1}(z + np, z + nq) \supset (z + mp, +\infty)$, where $m = [p/(q - p)] + 2$. Finally, for $t > z + mp + q$, it holds that

$$b(t) \geq \sigma_2(t) + (A\sigma_2)(t) + (A^2b)(t) \geq \int_0^t \beta_2(a)b(t-a)da > 0.$$

The proposition is proved.

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