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DYNAMICS OF A RANDOM HOPFIELD NEURAL LATTICE MODEL WITH ADAPTIVE SYNAPSES AND DELAYED HEBBIAN LEARNING

ДИНАМІКА ВИПАДКОВОЇ МОДЕЛІ НЕЙРОННОЇ ҐРАТКИ ХОПФІЛДА З АДАПТИВНИМИ СИНАПСАМИ ТА ЗАТРИМАНИМ ГЕББОВИМ НАВЧАННЯМ

A Dong – Hopfield neural lattice model with random external forcing and delayed response to the evolution of interconnection weights is developed and studied. The interconnection weights evolve according to the Hebbian learning rule with a decay term and contribute to changes in the states after a short delay. The lattice system is first reformulated as a coupled functional-ordinary differential equation system on an appropriate product space. Then the solution of the system is shown to exist and be unique. Furthermore it is shown that the system of equations generates a continuous random dynamical system. Finally, the existence of random attractors for the random dynamical system generated by the Dong – Hopfield model is established.

Розроблено та досліджено модель нейронної ґратки Донга – Хопфілда з випадковим зовнішнім впливом та затримкою реакції на еволюцію ваг взаємозв'язку. Вагові коефіцієнти взаємозв'язків розвиваються згідно з правилом навчання Гебба з затухаючим членом та впливають на зміну станів після короткої затримки. Ґратчаста система спочатку переформулюється у вигляді зв'язаної системи функціональних звичайних диференціальних рівнянь у відповідному добутку просторів. Далі показано, що розв'язок цієї системи існує та є єдиним, а також що система рівнянь породжує неперервну випадкову динамічну систему. Насамкінець встановлено існування випадкових атракторів для випадкової динамічної системи, що породжена моделлю Донга – Хопфілда.

1. Introduction. Delays are often included in neural models to account for the transmission time of signals between neurons. For example, Wang, Kloeden, and Yang [19] investigated a version of the Amari neural lattice model with delays, where a sigmoidal function characterised by a small positive parameter ε was used to approximate the Heaviside function in the vector field of the original Amari lattice model [1, 8, 9]. The existence and uniqueness of solutions to a delay sigmoidal lattice system as well as the existence of a global attractor were established in [19]. See also Kloeden and Villarragut [16] for a different neural field lattice model with delays and also Zhou [20].

Hopfield neural lattice models were introduced and investigated recently in [7, 11, 18], albeit without delays. In this work we will study a neural lattice version of the Dong – Hopfield model [6]. This model is particularly interesting, because it includes adaptive behavior of the interconnection weights in its dynamics and is thus a special case of a plastic self-organising velocity field in a conceptual model of a cognitive system introduced by Janson and Marsden [15] and analyzed mathematically by Janson and Kloeden [13, 14].

Specifically, here we will consider the following Dong – Hopfield model with delays and random forcing on 1-dimensional infinite lattice:

$$\gamma_i \frac{d}{dt} u_i(t) = -u_i + \sum_{|j-i|\le N} H_{i,j}\big(w_{i,j}(t-\tau_{i,j})\big) g_j(u_j(t)) + I_i(\theta_t(\omega)), \quad i \in \mathbb{Z}, \tag{1}$$

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$$\lambda_{i,j}\frac{d}{dt}w_{i,j}(t) = -w_{i,j} + g_i(u_i)g_j(u_j), \quad i,j \in \mathbb{Z}, \tag{2}$$

equipped with the initial conditions

$$u_i(0) = u_{o,i}, \qquad w_{i,j}(s) = \phi_{i,j}(s) \quad \text{for} \quad s \in [-h,0], \quad h = \sup_{i,j}\{\tau_{i,j}\} < \infty, \qquad i,j \in \mathbb{Z}. \tag{3}$$

Here, γ_i and $\lambda_{i,j}$ are positive time constants, u_i is the state of activity of the ith neuron, g_i is the activation function that takes u_i as the neuronal input, $w_{i,j}$ represents the correlation of the neuron activities $g_i(u_i)$ and $g_j(u_j)$, and $H_{i,j}$ is a bounded, continuous, and monotonic increasing function of the correlation $w_{i,j}$.

The term $I_i(\theta_t(\omega))$ describes the random external stimuli to the ith neuron represented by a measure-preserving dynamical system $\{\theta_t\}_{t\in\mathbb{R}}$. More precisely, the noise is represented in canonical form given by a metric dynamical system $\Theta = \{\theta_t\}_{t\in\mathbb{R}}$ acting on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying (see, e.g., [2])

(i) $\theta_0(\omega) = \omega$ for all $\omega \in \Omega$;

(ii) $\theta_{s+t}(\omega) = \theta_s \circ \theta_t(\omega)$ for all $\omega \in \Omega$ and any $s, t \in \mathbb{R}$;

(iii) $(t, \omega) \mapsto \theta_t(\omega)$ is measurable for all $\omega \in \Omega$ and any $s, t \in \mathbb{R}$;

(iv) $\theta_t\mathbb{P} = \mathbb{P}$ for every $t \in \mathbb{R}$.

Notice that the connection strength of a synapse represented by the function $H_{i,j}$ in equation (1) varies with respect to time, according to equation (2) that describes Hebbian learning with a decay term [17]. When the evolution of correlation functions is not fed into the updates of neuron activities instantaneously, a nonnegative constant $\tau_{i,j} \geq 0$ is introduced here to model the time delay in the process of updating the connection strength from the recent change in correlation $w_{i,j}$ of neuron activities. The term $\sum_{|j-i|\leq N} H_{i,j}(w_{i,j}(t-\tau_{i,j}))g_j(u_j(t))$ describes the interconnection structure of neurons through the connection matrix $H_{i,j}(w_{i,j}(t-\tau_{i,j}))$, in which $N > 0$ represents the size of interconnection neighbourhood.

The goal of this work is to study the long term dynamics of system (1), (2). This paper is organized as follows. First in Section 2 notation and preliminaries are provided, including reformulation of system (1), (2) and definition of the state space. In Section 3 the existence and uniqueness of solutions to (1), (2) is shown, and in Section 4 the existence of random attractors for the continuous random dynamical system generated by the solution to (1), (2) is established.

2. Preliminaries. In the Dong–Hopfield model (1), (2), for each $i, j \in \mathbb{Z}$, the mapping $H_{i,j} : \mathbb{R} \to \mathbb{R}$ is continuous, bounded, and monotonic increasing. For simplicity of exposition, throughout this paper $H_{i,j}$ will be assumed to have a special form $H_{i,j}(w_{i,j}) = f(\beta w_{i,j})$, where $\beta > 0$ is a positive constant and f is a piecewise linear function

$$f(x) = \begin{cases} 1, & x > 1, \\ x, & -1 \leq x \leq 1, \\ -1, & x < -1. \end{cases} \tag{4}$$

Clearly the function f is uniformly bounded, globally Lipschitz, and satisfies

$$f(0) = 0, \qquad |f(x)| \leq \min\{1, |x|\}, \qquad |f(x) - f(y)| \leq |x - y| \qquad \forall x, y \in \mathbb{R}. \tag{5}$$

Consequently, for every $i, j \in \mathbb{Z}$, the function $H_{i,j}$ has similar properties to (5) with the global Lipschitz constant β.

In Hopfield neural networks, the activation function $g_i : \mathbb{R} \to \mathbb{R}$ is monotonic, in fact sigmoidal, with binary asymptotes. In this work, g_i's are assumed to be a common sigmoidal function characterized by a small positive parameter ε of the form

$$\sigma_\epsilon(x) = \frac{1}{1 + e^{-x/\epsilon}}, \qquad x \in \mathbb{R}, \quad 0 < \epsilon < 1. \tag{6}$$

Notice that σ_ϵ is uniformly bounded, monotonically increasing and differentiable with a uniformly bounded derivative,

$$|\sigma_\epsilon(x)| < 1, \qquad 0 < \frac{d}{dx}\sigma_\epsilon(x) \le \frac{1}{\epsilon} \quad \text{for all} \quad x \in \mathbb{R}.$$

It then follows immediately from the mean value theorem for derivatives that the sigmoidal function $x \mapsto \sigma_\epsilon(x)$ is globally Lipschitz with the Lipschitz constant $1/\epsilon$, i.e.,

$$\big|\sigma_\epsilon(x) - \sigma_\epsilon(y)\big| \le \frac{1}{\epsilon}|x - y|, \quad x, y \in \mathbb{R}. \tag{7}$$

Moreover, the following lemma holds.

Lemma 2.1. *The product of sigmoidal functions $(x, y) \mapsto \sigma_\epsilon(x)\sigma_\epsilon(y)$ is globally Lipschitz with the Lipschitz constant $1/\epsilon$.*

Proof. First, by using the inequality $|ab - cd| \le |c||b - d| + |b||a - c|$ for a, b, c, $d \in \mathbb{R}$, and the boundedness property of the sigmoidal function, we have, for any x, $\tilde{x}$, $y, \tilde{y} \in \mathbb{R}$, that

$$\begin{aligned}|\sigma_\epsilon(x)\sigma_\epsilon(y) - \sigma_\epsilon(\tilde{x})\sigma_\epsilon(\tilde{y})| &\le |\sigma_\epsilon(\tilde{x})|\,|\sigma_\epsilon(y) - \sigma_\epsilon(\tilde{y})| + |\sigma_\epsilon(y)||\sigma_\epsilon(x) - \sigma_\epsilon(\tilde{x})| \\ &\le |\sigma_\epsilon(y) - \sigma_\epsilon(\tilde{y})| + |\sigma_\epsilon(x) - \sigma_\varepsilon(\tilde{x})|.\end{aligned}$$

It then follows immediately from (7) that

$$\big|\sigma_\epsilon(x)\sigma_\epsilon(y) - \sigma_\epsilon(\tilde{x})\sigma_\epsilon(\tilde{y})\big| \le \frac{1}{\epsilon}\big(|x - \tilde{x}| + |y - \tilde{y}|\big).$$

2.1. Model reformulation. Using the special forms of $H_{i,j}$ defined by (4) and g_i defined by (6), the equations (1), (2) become

$$\gamma_i \frac{d}{dt}u_i(t) = -u_i + \sum_{|j-i|\le N} f(\beta w_{i,j}(t - \tau_{i,j}))\sigma_\epsilon(u_j(t)) + I(\theta_t(\omega)), \quad i \in \mathbb{Z}, \tag{8}$$

$$\lambda_{i,j}\frac{d}{dt}w_{i,j}(t) = -w_{i,j} + \sigma_\epsilon(u_i)\sigma_\epsilon(u_j), \quad (i, j) \in \mathbb{Z}^2. \tag{9}$$

Set $\boldsymbol{u} = (u_i)_{i\in\mathbb{Z}}$, $\boldsymbol{w} = (w_{i,j})_{(i,j)\in\mathbb{Z}^2}$, and define the operators Γ, Λ, and $\mathcal{S}^\epsilon$ component wise by

$$(\Gamma\boldsymbol{u})_i = \frac{u_i}{\gamma_i}, \qquad (\Lambda\boldsymbol{w})_{i,j} = \frac{w_{i,j}}{\lambda_{i,j}}, \qquad \mathcal{S}^\epsilon_{i,j}(\boldsymbol{u}) = \frac{\sigma_\epsilon(u_i)\sigma_\epsilon(u_j)}{\lambda_{i,j}}, \quad i, j \in \mathbb{Z}. \tag{10}$$

In addition, for $t \ge 0$, let

$$\boldsymbol{w}_t(s) = \boldsymbol{w}(t+s) = (w_{i,j}(t+s))_{i,j\in\mathbb{Z}}, \quad s \in [-h, 0],$$

be the segment function of $\boldsymbol{w}$, and let $\mathfrak{w} = \boldsymbol{w}_t$ be an element in an appropriate function space to be specified later. Define the operator F^ϵ component wise by

$$F_i^\epsilon(\boldsymbol{u}, \mathfrak{w}) = \frac{1}{\gamma_i} \sum_{|j-i|\leq N} f(\mathscr{b}\mathfrak{w}(-\tau_{i,j}))\sigma_\epsilon(u_j). \tag{11}$$

Now writing $\mathcal{I}(\theta_t(\omega)) = \left\{\dfrac{I_i(\theta_t(\omega))}{\gamma_i}\right\}_{i\in\mathbb{Z}}$, $\boldsymbol{u}_o = \{u_{o,i}\}_{i\in\mathbb{Z}}$, and $\boldsymbol{\phi}(\cdot) = \{\phi_{i,j}(\cdot)\}_{(i,j)\in\mathbb{Z}^2}$, the system (8), (9) with the initial condition (3) can be written as

$$\frac{d\boldsymbol{u}(t,\omega)}{dt} = -\Gamma\boldsymbol{u}(t) + F^\epsilon(\boldsymbol{u}(t), \boldsymbol{w}_t) + \mathcal{I}(\theta_t(\omega)) := \mathfrak{F}_1(\boldsymbol{u}(t), \boldsymbol{w}_t, \theta_t(\omega)), \tag{12}$$

$$\frac{d\boldsymbol{w}(t,\omega)}{dt} = -\Lambda\boldsymbol{w}(t) + \mathcal{S}^\epsilon(\boldsymbol{u}(t)) := \mathfrak{F}_2(\boldsymbol{u}(t), \boldsymbol{w}(t)), \tag{13}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_o, \qquad \boldsymbol{w}(s) = \boldsymbol{\phi}(s), \quad s \in [-h, 0], \tag{14}$$

where $\boldsymbol{w}(t) = \boldsymbol{w}_t(0)$.

***Remark* 2.1.** An alternative formulation of system (12), (13) is the vector form

$$\frac{d\boldsymbol{U}(t,\omega)}{dt} = -\mathrm{A}\boldsymbol{U} + \mathfrak{F}(\boldsymbol{U}_t, \theta_t(\omega)),$$

where

$$\boldsymbol{U}(t,\omega) = \begin{pmatrix} \boldsymbol{u}(t,\omega) \\ \boldsymbol{w}(t,\omega) \end{pmatrix}, \qquad \mathrm{A} = \begin{pmatrix} \Gamma & \\ & \Lambda \end{pmatrix}, \qquad \mathfrak{F}(\boldsymbol{U}_t, \theta_t(\omega)) = \begin{pmatrix} F^\epsilon(\boldsymbol{u}_t(0), \boldsymbol{w}_t) + \mathcal{I}(\theta_t(\omega)) \\ \mathcal{S}^\epsilon(\boldsymbol{u}_t(0)) \end{pmatrix}.$$

The aim of this work is to study existence and uniqueness of solutions and the asymptotic dynamics of the ordinary-delay differential equation system (12) – (14) in an appropriate space to be defined in the next subsection.

2.2. Sequence spaces and standing assumptions. Noticing that the interconnection term involves infinitely many pairs of (i, j), we follow Han and Kloeden [10] and consider a weighted space of bi-infinite real-valued sequences with integer indices $i \in \mathbb{Z}$. In particular, given a positive sequence of weights $(\rho_i)_{i\in\mathbb{Z}}$, we consider the separable Hilbert space

$$\ell_\rho^2 := \left\{\boldsymbol{u} = \{u_i\}_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} \rho_i u_i^2 < \infty\right\}$$

with the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_\rho := \sum_{i\in\mathbb{Z}} \rho_i u_i v_i \quad \text{for} \quad \boldsymbol{u} = \{u_i\}_{i\in\mathbb{Z}}, \qquad \boldsymbol{v} = \{v_i\}_{i\in\mathbb{Z}} \in \ell_\rho^2$$

and norm

$$\|\boldsymbol{u}\|_\rho := \sqrt{\sum_{i\in\mathbb{Z}} \rho_i u_i^2}.$$

The weights $\{\rho_i\}_{i\in\mathbb{Z}}$ are assumed to satisfy the following assumptions.

Assumption 2.1. *$\rho_i > 0$ for all $i \in \mathbb{Z}$ and $\rho_\Sigma := \sum_{i\in\mathbb{Z}} \rho_i < \infty$.*

Assumption 2.2. *There exists a positive constant κ such that*

$$\rho_{i\pm 1} \le \kappa\rho_i \quad \textit{for all} \quad i \in \mathbb{Z}.$$

The time constants $\gamma_i,\ \lambda_{i,j}$ are assumed to satisfy the following assumption.

Assumption 2.3. *$0 < m_\gamma \le \gamma_i \le M_\gamma$ for each $i \in \mathbb{Z}$.*

Assumption 2.4. *$0 < m_\lambda \le \lambda_{i,j} \le M_\lambda$ for each $(i,j) \in \mathbb{Z}^2$.*

The external forcing terms are assumed to satisfy the following assumption.

Assumption 2.5. *$I_i(\theta_t(\omega))$ is continuous for each $i \in \mathbb{Z}$ and satisfies*

$$\sum_{i\in\mathbb{Z}} \rho_i I_i^2(\theta_t(\omega)) \le I_\infty < \infty \quad \forall t \in \mathbb{R}, \quad \omega \in \Omega.$$

A close inspection of the lattice systems (1), (2) shows that for each $i \in \mathbb{Z}$, the state u_i in (1) is only affected by the weights $w_{i,j}$ with $|j-i| \le N$. An investigation of the coupled system can thus be restricted to components with indices in the set

$$\mathbb{I}_N := \big\{(i,j) \in \mathbb{Z}\times\mathbb{Z} \colon |j-i| \le N \text{ for each } i \in \mathbb{Z}\big\}.$$

The other $w_{i,j}$ components for $(i,j) \notin \mathbb{I}_N$ remain uniformly bounded and need not be considered further (see Lemma 4.2 below). Therefore, the connection weights can be considered as forming a "band diagonal matrix".

Let $\ell^2_{\rho,2N+1}$ denote the Hilbert sequence space with the norm

$$\|\boldsymbol{w}\|_{\rho,2N+1} := \sqrt{\sum_{i\in\mathbb{Z}} \rho_i \sum_{|j-i|\le N} w_{i,j}^2}, \qquad \boldsymbol{w} = (w_{i,j})_{(i,j)\in\mathbb{I}_N}.$$

The appropriate function space for the solutions of the delayed components of system (12), (13) is the Banach space $\mathcal{C}\big([-h,0],\ell^2_{\rho,2N+1}\big)$ of all continuous functions $\mathfrak{w}$ from $[-h,0]$ to $\ell^2_{\rho,2N+1}$ with the norm

$$\|\mathfrak{w}\|_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})} = \max_{s\in[-h,0]} \|\boldsymbol{w}(s)\|_{\rho,2N+1}.$$

For simplicity of notations, throughout the rest of this paper, denote by

$$\mathfrak{E} := \mathcal{C}\big([-h,0],\ell^2_{\rho,2N+1}\big), \qquad \mathfrak{X} := \ell^2_\rho \times \mathfrak{E}.$$

Then the appropriate function space to study the solutions of the ordinary-functional differential equation system (12), (13) is the Banach product space $\mathfrak{X}$ with the norm

$$\big\|(\boldsymbol{u},\mathfrak{w})\big\|_{\mathfrak{X}} := \big(\|\boldsymbol{u}\|_\rho^2 + \|\mathfrak{w}\|_{\mathfrak{E}}^2\big)^{1/2}.$$

3. Existence and uniqueness of solutions. In this section, we study the existence and uniqueness of solutions to the system (12)–(14) in the product space $\mathfrak{X}$, assuming that $(\boldsymbol{u}_o, \boldsymbol{\phi}(\cdot)) \in \mathfrak{X}$. To that end, we first verify that $(\mathfrak{F}_1, \mathfrak{F}_2)$ in well-defined on $\mathfrak{X}$. More precisely, we show that $\mathfrak{F}_1(\boldsymbol{u}, \mathfrak{w}, \theta_t(\omega))$ takes value in ℓ^2_ρ for every $(\boldsymbol{u}, \mathfrak{w}) \in \mathfrak{X}$, $t \in \mathbb{R}$, and $\omega \in \Omega$, and that $\mathfrak{F}_2(\boldsymbol{u}, \boldsymbol{w})$ takes value in $\ell^2_{\rho,N}$ for every $\boldsymbol{u} \in \ell^2_\rho$ and $\boldsymbol{w} \in \ell^2_{\rho,2N+1}$.

Clearly, due to Assumptions 2.3, 2.4, and 2.5

$$\|\Gamma \boldsymbol{u}\|_\rho \leq \frac{1}{m_\gamma}\|\boldsymbol{u}\|_\rho \quad \forall \boldsymbol{u} \in \ell^2_\rho,$$

$$\|\Lambda \boldsymbol{w}\|_{\rho,2N+1} \leq \frac{1}{m_\lambda}\|\boldsymbol{w}\|_{\rho,2N+1} \quad \forall \boldsymbol{w} \in \ell^2_{\rho,2N+1},$$

$$\big\|\mathcal{I}(\theta_t(\omega))\big\|_\rho \leq \frac{1}{m_\gamma}\sum_{i\in\mathbb{Z}} \rho_i I_i^2(\theta_t(\omega)) < \infty \quad \forall t \in \mathbb{R}, \quad \omega \in \Omega.$$

In addition, for every $\boldsymbol{u} \in \ell^2_\rho$,

$$\|\mathcal{S}^\epsilon \boldsymbol{u}\|^2_{\rho,2N+1} \leq \frac{1}{m_\lambda^2}\sum_{i\in\mathbb{Z}}\rho_i \sum_{|j-i|\leq N} \sigma^2_\epsilon(u_i)\sigma^2_\epsilon(u_j) < \frac{1}{m_\lambda^2}\sum_{i\in\mathbb{Z}}\rho_i \sum_{|j-i|\leq N} 1 \leq \frac{(2N+1)\rho_\Sigma}{m_\lambda^2},$$

i.e., $\mathcal{S}^\epsilon \boldsymbol{u}$ takes value in $\ell^2_{\rho,2N+1}$.

It remains to show that the operator F^ϵ takes value in ℓ^2_ρ for every $\boldsymbol{u} \in \ell^2_\rho$ and $\mathfrak{w} \in \mathfrak{C}$. In fact, by (5), (11), and Assumption 2.3 we have

$$\begin{aligned}|F_i^\epsilon(\boldsymbol{u}, \mathfrak{w})|^2 &\leq \frac{1}{m_\gamma^2}\left|\sum_{|j-i|\leq N} f(\mathfrak{b}\mathfrak{w}_{i,j}(-\tau_{i,j}))\sigma_\epsilon(u_j)\right|^2 \\ &\leq \frac{1}{m_\gamma^2}(2N+1)\sum_{|j-i|\leq N}\big|f(\mathfrak{b}\mathfrak{w}_{i,j}(-\tau_{i,j}))\big|^2 \\ &\leq \frac{\mathfrak{b}^2}{m_\gamma^2}(2N+1)\sum_{|j-i|\leq N}\big|\mathfrak{w}_{i,j}(-\tau_{i,j})\big|^2, \quad i \in \mathbb{Z}.\end{aligned}$$

It then follows directly that

$$\big\|F^\epsilon(\boldsymbol{u}, \mathfrak{w})\big\|^2_\rho \leq \frac{\mathfrak{b}^2}{m_\lambda^2}(2N+1)\sum_{i\in\mathbb{Z}}\rho_i\sum_{|j-i|\leq N}\big|\mathfrak{w}_{i,j}(-\tau_{i,j})\big|^2 \leq \frac{\mathfrak{b}^2}{m_\lambda^2}(2N+1)\|\mathfrak{w}\|^2_{\mathfrak{C}}.$$

We next show that the system (12), (13) has a unique solution on $\mathfrak{X}$.

Theorem 3.1. *Suppose that Assumptions* 2.1–2.5 *hold. Then, for every initial data* $(\boldsymbol{u}_o, \boldsymbol{\phi}(\cdot)) \in \mathfrak{X}$, *system* (12)–(14) *has a unique global solution* $(\boldsymbol{u}, \boldsymbol{w}_t) \in \mathfrak{X}$ *satisfying* $\boldsymbol{u}(0) = \boldsymbol{u}_o$ *and* $\boldsymbol{w}_0(s) = \boldsymbol{\phi}(s)$ *for* $s \in [-h, 0]$.

Proof. It suffices to show that the mappings $F^\epsilon : \mathfrak{X} \to \ell^2_\rho$ and $\mathcal{S}^\epsilon : \ell^2_\rho \to \ell^2_{\rho,2N+1}$ are globally Lipschitz. To this end, we first show that the mapping $F_i^\epsilon : \mathfrak{X} \to \mathbb{R}$ is globally Lipschitz. In fact, for any $\boldsymbol{u}, \tilde{\boldsymbol{u}} \in \ell^2_\rho$ and $\mathfrak{w}, \tilde{\mathfrak{w}} \in \mathfrak{C}$ by Assumption 2.3 we have

$$\begin{aligned}\big|F_i^\epsilon(\boldsymbol{u},\mathfrak{w})-F_i^\epsilon(\tilde{\boldsymbol{u}},\tilde{\mathfrak{w}})\big| &\le \frac{1}{m_\gamma}\sum_{|j-i|\le N}\big|f(\beta\mathfrak{w}_{i,j}(-\tau_{i,j}))\sigma_\epsilon(u_j)-f(\beta\tilde{\mathfrak{w}}_{i,j}(-\tau_{i,j}))\sigma_\epsilon(\tilde u_j)\big| \\ &\le \frac{1}{m_\lambda}\sum_{|j-i|\le N}\big|f(\beta\mathfrak{w}_{i,j}(-\tau_{i,j}))-f(\beta\tilde{\mathfrak{w}}_{i,j}(-\tau_{i,j}))\big||\sigma_\epsilon(\tilde u_j)| \\ &\quad+\frac{1}{m_\lambda}\sum_{|j-i|\le N}|f(\beta\mathfrak{w}_{i,j}(-\tau_{i,j}))||\sigma_\epsilon(u_j)-\sigma_\epsilon(\tilde u_j)|.\end{aligned}$$

Then, by the Lipschitz continuity of f and σ_ϵ and boundedness of f,

$$\big|F_i^\epsilon(\boldsymbol{u},\mathfrak{w})-F_i^\epsilon(\tilde{\boldsymbol{u}},\tilde{\mathfrak{w}})\big| \le \frac{\beta}{m_\gamma}\sum_{|j-i|\le N}\big|\mathfrak{w}_{i,j}(-\tau_{i,j})-\tilde{\mathfrak{w}}_{i,j}(-\tau_{i,j})\big|+\frac{1}{m_\gamma\epsilon}\sum_{|j-i|\le N}|u_j-\tilde u_j|,$$

from which it follows that

$$\big\|F^\epsilon(\boldsymbol{u},\mathfrak{w})-F^\epsilon(\tilde{\boldsymbol{u}},\tilde{\mathfrak{w}})\big\|_\rho^2 \le \frac{2\beta^2}{m_\gamma^2}S_1+\frac{2}{m_\gamma^2\epsilon^2}S_2, \tag{15}$$

where

$$\begin{aligned} S_1&=\sum_{i\in\mathbb{Z}}\rho_i\left(\sum_{|j-i|\le N}|\mathfrak{w}_{i,j}(-\tau_{i,j})-\tilde{\mathfrak{w}}_{i,j}(-\tau_{i,j})|\right)^2, \\ S_2&=\sum_{i\in\mathbb{Z}}\rho_i\left(\sum_{|j-i|\le N}|u_j-\tilde u_j|\right)^2.\end{aligned}$$

Notice that

$$S_1\le(2N+1)\sum_{i\in\mathbb{Z}}\rho_i\sum_{|j-i|\le N}\big|\mathfrak{w}_{i,j}(-\tau_{i,j})-\tilde{\mathfrak{w}}_{i,j}(-\tau_{i,j})\big|^2\le(2N+1)\big\|\mathfrak{w}-\tilde{\mathfrak{w}}\big\|_{\mathfrak{C}}^2. \tag{16}$$

In addition, by Assumption 2.2

$$\begin{aligned} S_2&\le(2N+1)\sum_{i\in\mathbb{Z}}\rho_i\sum_{|j-i|\le N}|u_j-\tilde u_j|^2 \\ &\le(2N+1)(1+2\kappa+\ldots+2\kappa^N)\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_\rho^2. \end{aligned} \tag{17}$$

Inserting (16) and (17) into (15) gives

$$\big\|F^\epsilon(\boldsymbol{u},\mathfrak{w})-F^\epsilon(\tilde{\boldsymbol{u}},\tilde{\mathfrak{w}})\big\|_\rho^2\le K\big(\|\mathfrak{w}-\tilde{\mathfrak{w}}\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})}+\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_\rho^2\big), \tag{18}$$

where

$$K:=\frac{2(2N+1)}{m_\gamma^2}\max\left\{\beta^2,\frac{1+2\kappa+\ldots+2\kappa^N}{\epsilon^2}\right\}. \tag{19}$$

Next, by Lemma 2.1 and Assumption 2.4,

$$\left|\mathcal{S}^{\epsilon}_{i,j}(\boldsymbol{u}) - \mathcal{S}^{\epsilon}_{i,j}(\tilde{\boldsymbol{u}})\right| \leq \frac{1}{m_\lambda \epsilon}\big(|u_i - \tilde{u}_i| + |u_j - \tilde{u}_j|\big), \quad i, j \in \mathbb{Z},$$

and thus

$$\left\|\mathcal{S}^{\epsilon}(\boldsymbol{u}) - \mathcal{S}^{\epsilon}(\tilde{\boldsymbol{u}})\right\|^2_{\rho,2N+1} \leq \frac{2}{\epsilon^2}\sum_{i\in\mathbb{Z}} \rho_i \sum_{|j-i|\leq N} \Big(|u_i - \tilde{u}_i|^2 + |u_j - \tilde{u}_j|^2\Big)$$

$$\leq \frac{2}{\epsilon^2}(2N+1)\|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|^2_\rho + \frac{2}{\epsilon^2} S_2,$$

where S_2 is the same as above. It then follows by (17) that

$$\left\|\mathcal{S}^{\epsilon}(\boldsymbol{u}) - \mathcal{S}^{\epsilon}(\tilde{\boldsymbol{u}})\right\|^2_{\rho,2N+1} \leq \frac{4}{\epsilon^2}(2N+1)(1+\kappa+\ldots+\kappa^N)\|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|^2_\rho. \tag{20}$$

Standard existence and uniqueness theorems such as in [3, 19] can then be applied to give the existence and uniqueness of solutions to system (12), (13).

The theorem is proved.

4. Existence of random attractors. This section is devoted to the existence of random pullback attractors for system (12), (13). The reader is referred to, e.g., [2, 5] for the concept of random dynamical system and random attractors. Here we only provide the minimum amount of preliminaries.

Let $\Theta = \{\theta_t\}_{t\in\mathbb{R}}$ be a metric dynamical system acting on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\psi :$ $\mathbb{R}^+ \times \Omega \times \mathfrak{X} \to \mathfrak{X}$ satisfy

(i) initial condition: $\psi(0, \omega, x) = x$ for all $\omega \in \Omega$ and $x \in \mathfrak{X}$,

(ii) cocycle property: $\psi(s + t, \omega, x) = \psi(s, \theta_t(\omega), \psi(t, \omega, x))$ for all $s, t \in \mathbb{R}^+$, $\omega \in \Omega$ and $x \in \mathfrak{X}$,

(iii) measurability: $(t, \omega, x) \mapsto \psi(t, \omega, x)$ is measurable,

(iv) continuity: $x \mapsto \psi(t, \omega, x)$ is continuous for all $(t, \omega) \in \mathbb{R} \times \Omega$.

The pair (Θ, ψ) is called a random dynamical system (RDS).

Let $\mathcal{D}$ be a collection of random subsets of $\mathfrak{X}$. Then $\mathcal{D}$ is said to be *inclusion-closed* if $\mathcal{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\tilde{\mathcal{D}} = \{\tilde{D}(\omega)\}_{\omega\in\Omega}$ with $\tilde{D}(\omega) \subset D(\omega)$ for all $\omega \in \Omega$ imply that $\tilde{\mathcal{D}} \in \mathcal{D}$. Such a family of random sets in $\mathfrak{X}$ is called a *universe*. Given a universe $\mathcal{D}$ of tempered random sets in a Banach space $\mathfrak{X}$, a random set $\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is called a *random attractor* for a given random dynamical system (Θ, ψ) if: (i) $\mathcal{A}$ is a compact random set in $\mathfrak{X}$; (ii) $\mathcal{A}$ is invariant in the sense that $\psi(t, \omega, A(\omega)) = A(\theta_t\omega)$ for all $t \geq 0$ and $\omega \in \Omega$; (iii) $\mathcal{A}$ pullback attracts every $D \in \mathcal{D}$ in the sense that $\lim_{t\to\infty} \mathrm{dist}_{\mathfrak{X}}(\psi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), A(\omega)) = 0$ for $\omega \in \Omega$. Here $\mathcal{D}$ is called *basin* of attraction of $\mathcal{A}$, which are typically bounded tempered subsets. Throughout this work, they are assumed to be random tempered bounded subsets.

The following proposition on the existence of random attractor will be applied below.

Proposition 4.1. *Let $\mathcal{D}$ be a universe of random sets in $\mathfrak{X}$, and (Θ, ψ) be a continuous random dynamical system on $\mathfrak{X}$ over $(\Omega, \mathcal{F}, \mathbb{P})$ with noise $\{\theta_t\}_{t\in\mathbb{R}}$. Suppose that $\mathcal{K} = \{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is a closed pullback absorbing set for (Θ, ψ) and ψ is $\mathcal{D}$-pullback asymptotically compact with respect to $\mathcal{D}$. Then (Θ, ψ) has a unique $\mathcal{D}$-random attractor $\mathcal{A} = \{A(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ with component set given by*

$$A(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \psi(t, \theta_{-t}(\omega), K(\theta_{-t}(\omega)))}, \quad \omega \in \Omega.$$

4.1. Generation of a dynamical system. In this subsection, we show that the solution to the system (12), (13) generates a random dynamical system. To that end, we first show that the solution mappings are continuous in initial conditions. Let $(\boldsymbol{u}, \boldsymbol{w})(t; \boldsymbol{u}_o, \boldsymbol{\phi})$ and $(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{w}})(t; \tilde{\boldsymbol{u}}_o, \tilde{\boldsymbol{\phi}})$ be two solutions of system (12), (13) with initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_o$, $\boldsymbol{w}(s) = \boldsymbol{\phi}(s)$ for $s \in [-h, 0]$ and $\tilde{\boldsymbol{u}}(0) = \tilde{\boldsymbol{u}}_o$ and $\tilde{\boldsymbol{w}}(s) = \tilde{\boldsymbol{\phi}}(s)$ for $s \in [-h, 0]$, respectively. Set

$$\Delta \boldsymbol{u}(t) := \boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t), \qquad \Delta \boldsymbol{w}(t) := \boldsymbol{w}(t) - \tilde{\boldsymbol{w}}(t), \quad t \geq 0.$$

Then from (12), (13) we have

$$\frac{d\Delta \boldsymbol{u}(t)}{dt} = -\Gamma \Delta \boldsymbol{u}(t) + F^\epsilon(\boldsymbol{u}(t), \boldsymbol{w}_t) - F^\epsilon(\tilde{\boldsymbol{u}}(t), \tilde{\boldsymbol{w}}_t), \tag{21}$$

$$\frac{d\Delta \boldsymbol{w}(t)}{dt} = -\Lambda \Delta \boldsymbol{w}(t) + \mathcal{S}^\epsilon(\boldsymbol{u}(t)) - \mathcal{S}^\epsilon(\tilde{\boldsymbol{u}}(t)). \tag{22}$$

Taking the inner product $\langle \cdot, \cdot \rangle_\rho$ of equation (21) with $\Delta \boldsymbol{u}$, and noticing that $\langle \Gamma \boldsymbol{u}, \boldsymbol{u} \rangle_\rho \geq \dfrac{1}{M_\gamma} \|\boldsymbol{u}\|_\rho^2$ for every $\boldsymbol{u} \in \ell_\rho^2$ we have

$$\frac{d\|\Delta \boldsymbol{u}(t)\|_\rho^2}{dt} \leq -\frac{2}{M_\gamma} \|\Delta \boldsymbol{u}\|_\rho^2 + 2\langle F^\epsilon(\boldsymbol{u}(t), \boldsymbol{w}_t) - F^\epsilon(\tilde{\boldsymbol{u}}(t), \tilde{\boldsymbol{w}}_t), \Delta \boldsymbol{u}(t) \rangle_\rho.$$

It then follows by the Lipschitz property of F^ϵ in (18) that

$$\frac{d\big\|\Delta \boldsymbol{u}(t)\big\|_\rho^2}{dt} \leq K \|\Delta \boldsymbol{u}(t)\|_\rho^2 + K \|\boldsymbol{w}_t - \tilde{\boldsymbol{w}}_t\|^2_{\mathcal{C}([-h,0], \ell^2_{\rho, 2N+1})},$$

which can be integrated to get

$$\big\|\Delta \boldsymbol{u}(t)\big\|_\rho^2 \leq \|\boldsymbol{u}_o - \tilde{\boldsymbol{u}}_o\|_\rho^2 + K \int\limits_0^t \Big(\|\Delta \boldsymbol{u}(s)\|_\rho^2 + \|\Delta \boldsymbol{w}_s\|^2_{\mathcal{C}([-h,0], \ell^2_{\rho, 2N+1})} \Big) ds. \tag{23}$$

Next, taking the inner product of (22) with $\Delta \boldsymbol{w}$ in the space $\ell^2_{\rho, 2N+1}$ and using the Lipschitz property of $\mathcal{S}^\epsilon$ in (20) gives

$$\begin{aligned} \frac{d\|\Delta \boldsymbol{w}(t)\|^2_{\rho, 2N+1}}{dt} &= -2\Lambda \|\Delta \boldsymbol{w}(t)\|^2_{\rho, 2N+1} + 2\langle \mathcal{S}^\epsilon(\boldsymbol{u}(t)) - \mathcal{S}^\epsilon(\tilde{\boldsymbol{u}}(t)), \Delta \boldsymbol{w}(t) \rangle_{\rho, 2N+1} \\ &\leq B\big(\|\Delta \boldsymbol{u}(t)\|_\rho^2 + \|\Delta \boldsymbol{w}(t)\|^2_{\rho, 2N+1} \big) \end{aligned} \tag{24}$$

for some $B > 0$. Integrating the differential inequality (24) from 0 to t then gives

$$\begin{aligned} \big\|\Delta \boldsymbol{w}(t)\big\|^2_{\rho, 2N+1} &\leq \big\|\Delta \boldsymbol{w}(0)\big\|^2_{\rho, 2N+1} + B \int\limits_0^t \big(\|\Delta \boldsymbol{u}(s)\|_\rho^2 + \|\Delta \boldsymbol{w}(s)\|^2_{\rho, 2N+1} \big) ds \\ &\leq \|\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}\|^2_{\mathcal{C}([-h,0], \ell^2_{\rho, 2N+1})} + B \int\limits_0^t \Big(\|\Delta \boldsymbol{u}(s)\|_\rho^2 + \|\Delta \boldsymbol{w}_s\|^2_{\mathcal{C}([-h,0], \ell^2_{\rho, 2N+1})} \Big) ds, \end{aligned}$$

since $\big\|\Delta\boldsymbol{w}(t)\big\|^2_{\rho,2N+1} \le \|\Delta\boldsymbol{w}_t\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})}$ for $t \ge 0$.

Thus, for $t \ge h$ and any $r \in [0,h]$,

$$\big\|\Delta\boldsymbol{w}(t-r)\big\|^2_{\rho,2N+1} \le \|\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})} + B\int\limits_0^{t-r}\Big(\|\Delta\boldsymbol{u}(s)\|^2_\rho + \|\Delta\boldsymbol{w}_s\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})}\Big)ds$$

$$\le \|\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})} + B\int\limits_0^{t}\Big(\|\Delta\boldsymbol{u}(s)\|^2_\rho + \|\Delta\boldsymbol{w}_s\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})}\Big)ds,$$

since the integrals are monotonically increasing. This implies that

$$\|\Delta\boldsymbol{w}_t\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})} \le \|\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})} + B\int\limits_0^{t}\Big(\|\Delta\boldsymbol{u}(s)\|^2_\rho + \|\Delta\boldsymbol{w}_s\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})}\Big)ds$$

for $t \ge h$. The same inequality also holds for $0 \le t \le h$. In fact, notice that

$$\Delta\boldsymbol{w}_t(r) := \begin{cases}\Delta\boldsymbol{w}(t+r), & 0 \le t+r,\\ \boldsymbol{\phi}(r)-\tilde{\boldsymbol{\phi}}(r), & -h \le t+r \le 0,\end{cases} \qquad \text{for} \quad -h \le r \le 0,$$

and thus, for every $t \in [0,h]$, we have

$$\begin{aligned}\|\Delta\boldsymbol{w}_t\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})} &\le \max\Bigg\{\|\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})},\\ &\qquad \big\|\Delta\boldsymbol{w}(0)\big\|^2_{\rho,2N+1} + B\int\limits_0^{t}\Big(\big\|\Delta\boldsymbol{u}(s)\big\|^2_\rho + \|\Delta\boldsymbol{w}_s\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})}\Big)ds\Bigg\}\\ &\le \|\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}}\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})} + B\int\limits_0^{t}\Big(\big\|\Delta\boldsymbol{u}(s)\big\|^2_\rho + \|\Delta\boldsymbol{w}_s\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})}\Big)ds.\end{aligned} \tag{25}$$

Define $X(t) := \|\Delta\boldsymbol{u}(t)\|^2_\rho + \|\Delta\boldsymbol{w}_t\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})}$. Adding the inequalities (23) and (25) gives

$$X(t) \le X(0) + (K+B)\int\limits_0^t X(s)ds,$$

and it follows immediately from the Grownwall inequality that

$$X(t) \le X(0)e^{(K+B)T}, \quad 0 \le t \le T.$$

This is the desired continuity in initial conditions.

Notice that the cocycle property follows from the global existence and uniqueness of solutions to the initial value problem, we conclude the solution to system (12), (13) generates a continuous random dynamical system (θ, ψ) with ψ defined by

$$\psi(t, \omega, \boldsymbol{\phi}) = \big(\boldsymbol{u}(\cdot, \omega, \boldsymbol{\phi}), \boldsymbol{w}_t(\cdot, \omega, \boldsymbol{\phi})\big). \tag{26}$$

4.2. Existence of a random absorbing set. In this subsection, we construct a random absorbing set for the RDS defined by (26). To that end we first establish a priori estimate of $\boldsymbol{u}$ in the following lemma.

Lemma 4.1. *For any $R > 0$ and $\|\boldsymbol{u}_o\|_\rho^2 \le R$, there exists $T(R) > 0$ such that*

$$\|\boldsymbol{u}(t)\|_\rho^2 \le 1 + 2(2N+1)^2 \rho_{\Sigma} + 2m_\gamma M_\gamma I_\infty^2 =: R_\infty \quad \forall\, t \ge T(R).$$

Proof. First, for each $i \in \mathbb{Z}$, multiply the ith component of equation (8) by $2u_i(t)$ to obtain

$$\frac{d}{dt}u_i^2(t) = -\frac{2}{\gamma_i}u_i^2(t) + \frac{2}{\gamma_i}u_i(t)\sum_{|j-i|\le N} f(\beta w_{i,j}(t-\tau_{i,j}))\sigma_\epsilon(u_j(t)) + 2u_i(t)I_i(\theta_t(\omega))$$

$$\le -\frac{2}{\gamma_i}u_i^2(t) + \frac{1}{\gamma_i}\left(\frac{1}{2}u_i^2(t) + 2\left(\sum_{|j-i|\le N} f(\beta w_{i,j}(t-\tau_{i,j}))\sigma_\epsilon(u_j(t))\right)^2\right)$$

$$+ \frac{1}{2\gamma_i}u_i^2(t) + 2\gamma_i I_i^2(\theta_t(\omega)) = -\frac{1}{\gamma_i}u_i^2(t) + \frac{2}{\gamma_i}(N+1)^2 + 2\gamma_i I_i^2(\theta_t(\omega)).$$

Then by Assumptions 2.3 and 2.5 we have

$$\frac{d}{dt}u_i^2(t) \le -\frac{1}{m_\gamma}u_i^2(t) + \frac{2}{m_\gamma}(2N+1)^2 + 2M_\gamma I_\infty^2. \tag{27}$$

Now multiplying (27) through by ρ_i and summing over $i \in \mathbb{Z}$ gives

$$\frac{d}{dt}\|\boldsymbol{u}(t)\|_\rho^2 \le -\frac{1}{m_\gamma}\|\boldsymbol{u}(t)\|_\rho^2 + \frac{2}{m_\gamma}(2N+1)^2\rho_{\Sigma} + 2M_\gamma I_\infty^2,$$

which is then integrated from 0 to t to obtain

$$\|\boldsymbol{u}(t)\|_\rho^2 \le \|\boldsymbol{u}_o\|_\rho^2 e^{-t/m_\gamma} + (2(2N+1)^2\rho_{\Sigma} + 2m_\gamma M_\gamma I_\infty^2)(1 - e^{-t/m_\gamma}).$$

Thus, for every $\|\boldsymbol{u}_o\|_\rho^2 \le R$, there exists a $T(R) > 0$ such that

$$\|\boldsymbol{u}(t)\|_\rho^2 \le 1 + 2(2N+1)^2\rho_\Sigma + 2m_\gamma M_\gamma I_\infty^2, \qquad t \ge T(R).$$

The lemma is proved.

We next construct an estimate for $\|\boldsymbol{w}_t\|^2_{\mathcal{C}([-h,0],\ell^2_{\rho,2N+1})}$. Notice that the differential equation (2) does not contain delay terms so it can be handled as an ordinary differential equation.

Lemma 4.2. *For every $\boldsymbol{u}_o \in \ell_\rho^2$ and $\boldsymbol{w}(0) \in \ell^2_{\rho,2N+1}$ there exists a T_0 independent of i, j such that*

$$w_{i,j}^2(t; \boldsymbol{u}_o, \boldsymbol{w}(0)) \le 1 + \frac{1}{m_\lambda}, \quad t \ge T_0,$$

for all $i \in \mathbb{Z}$ with $|j - i| \le N$. Moreover,

$$\|\boldsymbol{w}(t)\|^2_{\rho,2N+1} \le (2N+1)\rho_{\Sigma}\left(1 + \frac{1}{m_\lambda}\right), \quad t \ge T_0.$$

Proof. Let $i \in \mathbb{Z}$ with $|j-i| \le N$ be given. Multiplying the equation (2) by $2w_{i,j}(t)$ gives

$$\frac{d}{dt} w_{i,j}^2(t) = \frac{2}{\lambda_{i,j}}\, w_{i,j}(t)\sigma_\epsilon(u_i(t))\sigma_\epsilon(u_j) - \frac{2}{\lambda_{i,j}} w_{i,j}^2(t)$$
$$\le \frac{w_{i,j}^2}{\lambda_{i,j}} + \frac{\sigma_\epsilon^2(u_i)\sigma_\epsilon^2(u_j)}{\lambda_{i,j}} - \frac{2}{\lambda_{i,j}} w_{i,j}^2(t)$$
$$\le \frac{1}{m_\lambda} - \frac{1}{M_\lambda} w_{i,j}^2(t).$$

Integrating the above differential inequality from 0 to t results in

$$w_{i,j}^2(t) \le w_{i,j}^2(0)e^{-t/M_\lambda} + \frac{1}{m_\lambda}(1 - e^{-t/M_\lambda}) \le 1 + \frac{1}{m_\lambda}, \quad t \ge T_0,$$

which implies that

$$\|\boldsymbol{w}(t)\|_{\rho,2N+1}^2 = \sum_{i\in\mathbb{Z}} \rho_i \sum_{|j-i|\le N} w_{i,j}^2(t) \le (2N+1)\rho_\Sigma \left(1 + \frac{1}{m_\lambda}\right), \quad t \ge T_0.$$

It follows from the proof of Lemma 4.2 that the closed interval $\left[0, 1 + \dfrac{1}{m_\lambda}\right]$ is a positively invariant absorbing set for each $w_{i,j}^2(t)$ with the same rate of attraction for all $i \in \mathbb{Z}$ with $|j-i| \le N$.

In terms of the delay segments $\boldsymbol{w}_t \in \mathcal{C}\big([-h,0], \ell_{\rho,2N+1}^2\big)$ this gives

$$\|\boldsymbol{w}_t\|_{\mathcal{C}([-h,0],\ell_{\rho,2N+1}^2)}^2 \le (2N+1)\rho_\Sigma \left(1 + \frac{1}{m_\lambda}\right) =: R_\infty^*, \quad t \ge T(R) + h,$$

when $\|\boldsymbol{\phi}\|_{\mathcal{C}([-h,0],\ell_{\rho,2N+1}^2)}^2 \le R$.

The lemma is proved.

It follows directly from Lemmas 4.1 and 4.2 that the closed and bounded subset

$$\mathcal{K} := \left\{(\boldsymbol{u}, \mathfrak{w}) \in \mathfrak{X} \,:\, \|\boldsymbol{u}\|_\rho^2 \le R_\infty, \|\mathfrak{w}|_{\mathcal{C}([-h,0],\ell_{\rho,2N+1}^2)}^2 \le R_\infty^*\right\} \tag{28}$$

is positive invariant and (pullback and forwards) absorbing for lattice system (12), (13) on the state space $\ell_\rho^2 \times \mathcal{C}\big([-h,0], \ell_{\rho,2N+1}^2\big)$, and the following theorem holds.

Theorem 4.1. *The random dynamical system ψ has a random absorbing set $\mathcal{K}$ in $\mathfrak{D}$ such that, for every $\mathcal{D} = \{D(\omega)\}_{\omega\in\Omega} \in \mathfrak{D}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $T_D(\omega) > 0$ such that*

$$\psi(t, \theta_{-t}(\omega), D(\theta_{-t}(\omega))) \subset \mathcal{K} \quad \forall t \ge T_D(\omega).$$

4.3. Asymptotic compactness. The next step is to show the asymptotic compactness, which can be obtained by showing the solutions have light tails in the space $\mathfrak{X}$ as stated in the following lemma.

Lemma 4.3. *Suppose that Assumptions* 2.1 – 2.5 *hold. Then, for every $\omega \in \Omega$, and $\varepsilon > 0$ there exist $T = T(\omega, \mathcal{K}, \varepsilon) > 0$ and $\iota = \iota(\omega, \mathcal{K}, \varepsilon) > 0$ such that the solution $\big(\boldsymbol{u}(t,\omega,\Psi_o), \boldsymbol{w}_t(\cdot,\omega,\Psi_o)\big)$ of system* (12), (13) *with initial data $\Psi_o = (\boldsymbol{u}_o, \boldsymbol{\phi}(\cdot)) \in K(\theta_{-t}\omega)$ satisfies*

$$\sum_{|i|>\iota} \rho_i \left(u_i^2(t; \theta_{-t}(\omega), \Psi_o) + \sup_{-h\le s\le 0} \sum_{|j-i|\le N} w_{i,j}^2(t+s, \theta_{-t}(\omega), \Psi_o) \right) < \varepsilon, \quad t \ge T.$$

Proof. Consider a smooth increasing function $\xi : \mathbb{R}_+ \to \mathbb{R}$ that satisfies

$$\xi(x) = 0, \quad 0 \le x \le 1,$$
$$0 \le \xi(x) \le 1, \quad 1 \le x \le 2,$$
$$\xi(x) = 1, \quad x \ge 2.$$

Given a positive integer k that is sufficiently large (to be specified later), define

$$\xi_k(|i|) = \xi\big(|i|/k\big), \quad i \in \mathbb{Z}.$$

Multiplying the equation (8) by $2\rho_i\xi_k(|i|)u_i(t)/\gamma_i$ and summing over $i \in \mathbb{Z}$ gives

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i u_i^2 = -2\sum_{i\in\mathbb{Z}}\frac{\xi_k(|i|)\rho_i u_i^2}{\gamma_i} + S_3 + S_4, \tag{29}$$

where

$$S_3 = 2\sum_{i\in\mathbb{Z}}\frac{\xi_k(|i|)\rho_i u_i}{\gamma_i}\sum_{|j-i|\le N} f(\mathfrak{b} w_{i,j}(t-\tau_{i,j}))\sigma_\epsilon(u_j(t)),$$

$$S_4 = 2\sum_{i\in\mathbb{Z}}\frac{\xi_k(|i|)\rho_i u_i}{\gamma_i} I_i(\theta_t(\omega)).$$

The terms on the right-hand side of (29) satisfy, respectively,

$$2\sum_{i\in\mathbb{Z}}\frac{\xi_k(|i|)\rho_i u_i^2}{\gamma_i} \ge \frac{2}{M_\gamma}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i u_i^2, \tag{30}$$

$$\begin{aligned} S_3 &\le \frac{1}{m_\gamma}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i\left(\frac{m_\gamma}{2M_\gamma}u_i^2 + 2\frac{M_\gamma}{m_\gamma}\Big(\sum_{|j-i|\le N} f(\mathfrak{b} w_{i,j}(t-\tau_{i,j}))\sigma_\epsilon(u_j(t))\Big)^2\right) \\ &\le \frac{1}{M_\gamma}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i u_i^2 + \frac{2M_\gamma}{m_\gamma^2}(2N+1)^2\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i, \end{aligned} \tag{31}$$

$$\begin{aligned} S_4 &\le \frac{1}{m_\gamma}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i\left(\frac{m_\gamma}{2M_\gamma}u_i^2 + 2\frac{M_\gamma}{m_\gamma}I_i^2(\theta_t(\omega))\right) \\ &\le \frac{1}{M_\gamma}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i u_i^2 + \frac{2M_\gamma}{m_\gamma^2}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i I_i^2(\theta_t(\omega)). \end{aligned} \tag{32}$$

Inserting (30)–(32) into (29) gives

$$\begin{aligned} &\frac{d}{dt}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i u_i^2 \\ &\qquad \le -\frac{1}{M_\gamma}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i u_i^2 + \frac{2M_\gamma}{m_\gamma^2}\left((2N+1)^2\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i + \sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i I_i^2(\theta_t(\omega))\right) \end{aligned}$$

$$\leq -\frac{1}{M_\gamma}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i u_i^2 + \frac{2M_\gamma}{m_\gamma^2}\left((2N+1)^2\sum_{|i|\geq k}\rho_i + \sum_{|i|\geq k}\rho_i I_i^2(\theta_t(\omega))\right). \tag{33}$$

Next, as in the proof of Lemma 4.2 but multiplying the equation (2) by $2\xi_k(|i|)\rho_i w_{i,j}$ for all $|j-i|\leq N$, and summing over $i\in\mathbb{Z}$ gives

$$\frac{d}{dt}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i\sum_{|j-i|\leq N} w_{i,j}^2(t) \leq -\frac{1}{M_\lambda}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i\sum_{|j-i|\leq N} w_{i,j}^2(t) + \frac{2N+1}{m_\lambda}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i. \tag{34}$$

Integrating the differential inequalities (33) and (34), respectively, gives

$$\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i u_i^2(t) \leq \sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i u_{i,o}^2 e^{-\frac{t}{M_\lambda}}$$

$$+\frac{2M_\gamma}{m_\gamma^2}\left((2N+1)^2\sum_{|i|\geq k}\rho_i + \sum_{|i|\geq k}\rho_i I_i^2(\theta_t(\omega))\right)\Big(1-e^{-\frac{t}{M_\lambda}}\Big),$$

$$\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i\sum_{|j-i|\leq N} w_{i,j}^2(t)$$

$$\leq \sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i\sum_{|j-i|\leq N} w_{i,j}^2(0)e^{-\frac{t}{M_\lambda}} + \frac{2N+1}{m_\lambda}\sum_{i\in\mathbb{Z}}\xi_k(|i|)\rho_i\Big(1-e^{-\frac{t}{M_\lambda}}\Big).$$

Hence, for every $\varepsilon>0$, $\omega\in\Omega$, there exist $\iota=\iota(\varepsilon,\omega,\mathcal{K})>0$ and $T=T(\varepsilon,\omega,\mathcal{K})>0$ independently of an specific initial conditions in $\mathcal{K}$ such that

$$\sum_{|i|\geq\iota}\rho_i u_i^2(t)\leq\varepsilon, \qquad \sum_{|i|\geq N}\rho_i\sum_{|j-i|\leq N} w_{i,j}^2(t)\leq\varepsilon, \quad t\geq T.$$

It follows from the latter inequality that

$$\sum_{|i|\geq\iota}\rho_i\sum_{|j-i|\leq N}\sup_{-h\leq s\leq 0} w_{i,j}^2(t+s)\leq\varepsilon, \quad t\geq T+h,$$

which implies the desired assertion of this lemma.

The lemma is proved.

4.4. Existence of a random attractor.

Theorem 4.2. *The random dynamical system (Θ,ψ) has a unique $\mathfrak{D}$-random attractor in $\mathcal{K}$.*

Proof. To apply Proposition 4.1 it remains to show that the random dynamical system (Θ,ψ) is $\mathfrak{D}$-pullback asymptotic compact in $\mathcal{K}$ defined by (28). To that end we need to show that any sequence $\big\{\psi(t_n,\theta_{-t_n}(\omega),\boldsymbol{u}_o^n,\boldsymbol{\phi}^n)\big\}_{n=1}^{\infty}$ with $(\boldsymbol{u}_o^n,\boldsymbol{\phi}^n)\in D(\theta_{-t_n}(\omega))$ has a (strongly) convergent subsequence in $K(\omega)$ when $t_n\to\infty$, where $\mathcal{D}=\{D(\omega)\}_{\omega\in\Omega}\in\mathfrak{D}$.

Write the terms of the sequence as $\boldsymbol{u}^n=\boldsymbol{u}(t_n,\theta_{-t_n}(\omega))\in\ell^2$ and $\mathfrak{w}^n=\mathfrak{w}(t_n,\theta_{-t_n}(\omega))\in\mathcal{C}\big([-h,0],\ell^2_{\rho,2N+1}\big)$. Denote $\mathfrak{w}^n(s)=\mathfrak{w}(t_n,\theta_{-t_n}(\omega))(s)\in\ell^2_{\rho,2N+1}$ for a fixed (but arbitrary) $s\in[-h,0]$.

Following Bates et al. [3], $\boldsymbol{u}^n$ and $\mathfrak{w}^n(s)$ are contained in closed and bounded balls in their respective Hilbert spaces, so have weakly convergent subsequences. The individual components of the terms in these subsequences converge in $\mathbb{R}$. Using this for a finite number of terms and the asymptotic tails result above for the rest shows that these weakly convergent subsequences actually converges strongly in their respective Hilbert space. The strong convergence in the space $\mathcal{C}\big([-h,0],\ell^2_{\rho,2N+1}\big)$ then follows from the uniform Lipschitz property of the $\boldsymbol{w}_t$ solutions. See Caraballo et al. [4] and Han & Kloeden [10, Chapter, 6 Section 3] for more details.

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