DOI: 10.37863/umzh.v75i1.7390

UDC 517.9

A. Rontó (Masaryk Univ., Brno, Czech Republic),
M. Rontó[1] (Inst. Math., Univ. Miskolc, Hungary),
N. Rontoová (Brno Univ. Technology, Czech Republic)

ON THE JUMP CONTROL PROBLEM FOR BOUNDARY-VALUE PROBLEMS WITH STATE-DEPENDENT IMPULSES

ЗАДАЧА КЕРУВАННЯ СТРИБКОМ ДЛЯ КРАЙОВИХ ЗАДАЧ ІЗ ЗАЛЕЖНИМИ ВІД СТАНУ ІМПУЛЬСАМИ

We show how an appropriate parametrization technique and special successive approximations can help to control unknown jumps in the case of nonlinear boundary-value problems with state-dependent impulses. The practical application of the technique is shown on a numerical example.

Показано як відповідний метод параметризації та спеціальні послідовні наближення можуть допомогти керувати невідомими стрибками у випадку нелінійних крайових задач з імпульсами, що залежать від стану. Практичне застосування цієї техніки показано на числовому прикладі.

1. Introduction and problem setting. Boundary-value problems for differential equations with state-dependent jumps have recently attracted much attention (see [1] and references therein). According to the authors' best knowledge, the papers [2, 3] are the first where a numerical-analytic technique is described for the nonlinear boundary-value problems with state-dependent impulses, which allows one to combine the solvability analysis with the effective construction of approximate solutions. The work [2] deals with the nonlinear system of differential equations

$$u'(t) = f(t, u(t)), \qquad t \in [a, b], \tag{1}$$

where $-\infty < a < b < \infty$ and $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in a suitable domain. The differential system (1) is considered under the linear two-point boundary condition $Au(a) + Cu(b) = d$, where d is a constant vector and A, C are given square matrices, and under the state-dependent impulse condition

$$u(t+) - u(t-) = \gamma(u(t-)), \quad g(t, u(t-)) = 0. \tag{2}$$

The functions $\gamma : \mathbb{R}^n \to \mathbb{R}^n$ and $g : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ here are continuous in suitable domains and the time instants $t \in (a, b)$ appearing in (2) are *a priori* unknown. The jumps occurring according to (2) are called state-dependent because both the jump time and its magnitude depend on $u(t-)$ through the equation $g(t, u(t-)) = 0$, which determines whether the jump occurs at time t or not. In this way, different solutions of such a system may undergo jumps at different times. The jump of a solution occurs at the times where it meets the set

$$\{(t, x) : \ g(t, x) = 0\}, \tag{3}$$

[1] Corresponding author, e-mail: matronto@uni-miskolc.hu.

usually referred to as a barrier set. The study of [2] is focused on single-jump solutions, i. e., those which are allowed to meet the barrier only once on the given time interval. The work [3] deals with a more general multiimpulse boundary-value problem, where equation (1) is considered under the nonlinear two-point boundary condition

$$V(u(a), u(b)) = 0, \tag{4}$$

where $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and under the jump condition

$$u(t+) - u(t-) = \gamma_t(u(t-)), \quad g(t, u(t-)) = 0.$$

Solutions of (1), (4) under this jump condition are allowed in [3] to meet the barrier finitely many times.

In [2, 3], a technique is used, which had been at first suggested in [4] for the investigation of existence and approximate construction of solutions of a class of nonlinear boundary-value problems for ordinary differential equations without impulses. It belongs to the few approaches that offer constructive possibilities both for the investigation of the existence of solution and it approximate construction (see [5 – 13]).

As in [2, 3], we shall suppose in the sequel that potential solutions are left-continuous. A left-continuous vector function $u: \ [a, b] \to \mathbb{R}^n$ is called a single-jump solution of problem (1), (4), (2) if (4) holds and there exists a time instant $\tau \in (a, b)$ such that the restrictions $u|_{[a,\tau]},\ u|_{(\tau,b]}$ have continuous derivatives, u satisfies (1) on $[a, b] \setminus \{\tau\}$, the jump condition (2) is activated at $t = \tau$ and afterwards the graph of u does not meet the barrier set (3) at any other time within the given time interval.

In this paper, we consider a specific control problem which is, in some sense, inverse to problems mentioned above, by imposing the jump condition

$$u(\tau+) - u(\tau-) = \gamma, \quad g(\tau, u(\tau-)) = 0, \tag{5}$$

where neither the jump time τ nor its magnitude γ are specified beforehand. More precisely, we focus on functions u satisfying (1) for $t \in [a, b]$ except points τ where the jump (5) occurs and require that u satisfies the boundary condition (4) and the additional condition

$$u_i(a) = c_i, \quad i = 1, \ldots, j; \quad u_k(b) = c_k, \quad k = j + 1, \ldots, n, \tag{6}$$

where $c_k,\ k = 1, 2, \ldots, n,$ are given constants. The time instant τ and the magnitude of the jump γ in (5) are unknown and should be determined. We limit our consideration to solutions u which have jump (5) at a single point from (a, b). Thus, the problem here is to find a left-continuous function u, time instant τ and jump magnitude γ such that u satisfies the differential equation (1) on $[a, b] \setminus \{\tau\}$, has jump of magnitude γ at τ according to (5) and satisfies conditions (4), (6). We will refer to it as to problem (1), (4) – (6).

2. Notation and subsidiary statements . In what follows, 1_n and 0_n are, respectively, the unit and zero matrices of dimension n; $r(K)$ is the maximal, in modulus, eigenvalue of a matrix K. Similarly to [3], we put $|x| = \mathrm{col}(|x_1|, \ldots, |x_n|)$ for any $x = \mathrm{col}(x_1, \ldots, x_n)$ and understand inequalities between vectors and the operations "max" and "min" for vector functions componentwise.

If $\Omega \subset \mathbb{R}^n$, $f\colon [a,b]\times\Omega \to \mathbb{R}^n$ is a function, and K is an $n\times n$ matrix with nonnegative entries, the notation $f \in \mathrm{Lip}_K(\Omega)$ means that f satisfies the componenwise Lipschitz condition $|f(t,u_1) - f(t,u_2)| \le K|u_1 - u_2|$ for $t \in [a,b]$ and $u_1, u_2 \in \Omega$.

For any nonnegative vector $\varrho \in \mathbb{R}^n$, a componentwise ϱ-neighbourhood of a point $z \in \mathbb{R}^n$ is defined as $\mathcal{O}_\varrho(z) := \{\xi \in \mathbb{R}^n : |\xi - z| \le \varrho\}$. The ϱ-neighbourhood of a set $\Omega \subset \mathbb{R}^n$ is defined by the equality

$$\mathcal{O}_\varrho(\Omega) := \bigcup_{\xi\in\Omega} \mathcal{O}_\varrho(\xi). \tag{7}$$

Given two sets Ω_0 and Ω_1 in $\mathbb{R}^n$, we put

$$\mathcal{B}(\Omega_0,\Omega_1) := \{(1-\theta)z + \theta\eta\colon\ z\in\Omega_0,\ \eta\in\Omega_1,\ \theta\in[0,1]\}. \tag{8}$$

We need two auxiliary statements. Let $-\infty < t_0 < t_1 < \infty$.

Lemma 1 ([10], Lemma 3.16)**.** *Let the functions $\{\alpha_m(\cdot, t_0, t_1) : m \ge 1\}$ be defined by the recurrence relation*

$$\alpha_{m+1}(t,t_0,t_1) = \left(1 - \frac{t-t_0}{t_1-t_0}\right)\int_{t_0}^{t} \alpha_m(s,t_0,t_1)ds + \frac{t-t_0}{t_1-t_0}\int_{t}^{t_1} \alpha_m(s,t_0,t_1)ds, \quad t\in[t_0,t_1], \tag{9}$$

where $m = 0,1,\ldots$ and $\alpha_0(t,t_0,t_1) = 1$. Then the estimate

$$\alpha_{m+1}(t,t_0,t_1) \le \frac{10}{9}\left(\frac{3(t_1-t_0)}{10}\right)^m \alpha_1(t,t_0,t_1), \quad t\in[t_0,t_1],$$

holds for any $m = 0,1,\ldots$.

For $m = 0$, formula (9) has the form

$$\alpha_1(t,t_0,t_1) = 2(t-t_0)\left(1 - \frac{t-t_0}{t_1-t_0}\right), \quad t\in[t_0,t_1]. \tag{10}$$

It follows immediately from (9), (10) that $\alpha_1(\cdot,t_0,t_1)$ is nonnegative and

$$\max_{t\in[t_0,t_1]} \alpha_1(t,t_0,t_1) = \frac{1}{2}(t_1-t_0), \qquad \min_{t\in[t_0,t_1]} \alpha_m(t,t_0,t_1) = 0$$

for all $m \ge 0$.

If $\Omega \subset \mathbb{R}^k$ is a compact set and $h\colon [a,b]\times\Omega \to \mathbb{R}^k$ is continuous, we put

$$\delta_{[t_0,t_1];\,\Omega}(f) := \max_{(t,x)\in[t_0,t_1]\times\Omega} (t,x) - \min_{(t,x)\in[t_0,t_1]\times\Omega} h(t,x). \tag{11}$$

We also set $\delta_{[a,b];\,\Omega}(h) = \delta_\Omega(h)$.

Lemma 2 ([14], Lemma 1)**.** *Let $\Lambda \subset \mathbb{R}^k$, $k \ge 1$, be a closed bounded set and $u\colon [t_0,t_1]\times\Lambda \to$ $\to \mathbb{R}^n$ be a continuous vector-function. Then, for an arbitrary $t \in [t_0,t_1]$, the following inequality holds*:

$$\left|\int_{t_0}^{t}\left(h(\tau,\lambda) - \frac{1}{t_1-t_0}\int_{t_0}^{t_1} h(s,\lambda)ds\right)d\tau\right| \le \frac{1}{2}\alpha_1(t,t_0,t_1)\,\delta_{[t_0,t_1];\,\Omega}(h).$$

3. Sets of parameters and auxiliary two model problems. Following [2], we are going to approximate a single-jump solution u of problem (1), (4)–(6) by suitable sequences of functions separately on the interval $[a,\tau]$ preceding the jump time τ and on the interval $[\tau,b]$ succeeding to it. Recall that the jump time and the value of the jump are unknown and are treated as parameters.

Let us certain compact convex sets Ω_a, $\Omega_{\tau-}$, Ω_b and Γ. In practice, it is convenient to choose these sets as n-dimensional parallelepipeds.

Introduce the variables

$$z\in\Omega_a,\quad \lambda\in\Omega_{\tau-},\quad \eta\in\Omega_b,\quad \gamma\in\Gamma,\quad \tau\in(a,b), \tag{12}$$

where γ and τ will represent the jump size and time, whereas z, η and λ will have the meaning of solution values at the points a, b and τ:

$$z\to u(a),\quad \eta\to u(b),\quad \lambda\to u(\tau). \tag{13}$$

According to the imposed condition (6), the vectors z and η in (13) should have the form

$$z=\operatorname{col}(c_1,c_2,\ldots,c_j,z_{j+1},z_{j+2},\ldots,z_n),\quad \eta=\operatorname{col}(\eta_1,\eta_2,\ldots,\eta_j,c_{j+1},c_{j+2},\ldots,c_n), \tag{14}$$

and therefore, in view of (14), the vectors z, η and λ actually contain only $3n$ scalar free variables

$$(z_{j+1},z_{j+2},\ldots,z_n),\quad (\eta_1,\eta_2,\ldots,\eta_j),\quad (\lambda_1,\lambda_2,\ldots,\lambda_n),\quad (\gamma_1,\gamma_2,\ldots,\gamma_n), \tag{15}$$

which are the parameters whose values we need to determine. Furthermore, let us put

$$\Omega_{\tau+}:=\{x+\gamma:\ x\in\Omega_{\tau-},\ \gamma\in\Gamma\}$$

and, according to (8), construct the sets

$$\Omega_{a,\tau-}:=\mathcal{B}(\Omega_a,\Omega_{\tau-}),\quad \Omega_{\tau+,b}:=\mathcal{B}(\Omega_{\tau+},\Omega_b).$$

The simplest way to choose the parameter sets, which also seems to be sufficient for most applications, is to take a compact convex set $\Omega_a\subset\mathbb{R}^n$ and put

$$\Omega_b=\{x+\gamma: x\in\Omega_a,\gamma\in\Gamma\},\quad \Omega_{a,\tau-}=\Omega_a,\quad \Omega_{\tau+}=\Omega_b.$$

The technique to be applied will require to define suitable neighbourhoods of sets where the values of parameters are looked for. For this purpose, according to (8), we construct neighbourhoods $\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})$ and $\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})$ of the sets $\Omega_{a,\tau-}$ and $\Omega_{\tau+,b}$ with ϱ_0 and ϱ_1 such that

$$\varrho_0\geq\frac{b-a}{4}\delta_{\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})}(f),\quad \varrho_1\geq\frac{b-a}{4}\delta_{\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})}(f). \tag{16}$$

We shall assume that ϱ_0 and ϱ_1 can be chosen so that inequalities (16) are satisfied. The relation between the introduced sets is the following:

$$(\Omega_a\cup\Omega_{\tau-})\subset\Omega_{a,\tau-}\subset\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-}),\quad (\Omega_{\tau+}\cup\Omega_b)\subset\Omega_{\tau+,b}\subset\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b}).$$

We will study single-jump solutions of the original jump control problem through two auxiliary two-point boundary-value problems

$$x'(t) = f(t, x(t)), \quad t \in [a, \tau], \quad x(a) = z, \quad x(\tau) = \lambda, \tag{17}$$

$$y'(t) = f(t, y(t)), \quad t \in [\tau, b], \quad y(\tau) = \lambda + \gamma, \quad y(b) = \eta, \tag{18}$$

where $\tau,\ z$ and η have form (14) and all the variables listed in (15) are treated as free parameters the range for which is described by (12). Note that, although both (17) and (18) are overdetermined problems (with n equations and $2n$ boundary conditions), one can see, that in fact, due to the nature of the conditions imposed, no complications arise when (17) and (18) are treated simultaneously. The families of simpler problems (17) and (18) can be efficiently used in the constructive analysis of the original impulsive problem (1), (4) – (6).

4. Successive approximations for problems with parameters. We will treat the auxiliary problems (17), (18) by using iterations constructed similarly to [2].

4.1. The interval before jump. To study the model two-point boundary-value problem (17) on the domain $[a, \tau] \times \mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})$ corresponding to the time interval where the jump does not yet occur, we define the sequence of functions $\{x_m(\cdot, \tau, z, \lambda) : m \geq 0\}$ involving the parameters $\tau \in (a, b)$, $z \in \Omega_a,\ \lambda \in \Omega_{\tau-}$ introduced according to (14) and (13), by putting

$$x_{m+1}(t, \tau, z, \lambda) = x_0(t, \tau, z, \lambda) + \int\limits_a^t f(s, x_m(s, \tau, z, \lambda))ds - \frac{t-a}{\tau-a}\int\limits_a^\tau f(\tau, x_m(s, \tau, z, \lambda))ds, \tag{19}$$

$$x_0(t, \tau, z, \lambda) = \left(1 - \frac{t-a}{\tau-a}\right)z + \frac{t-a}{\tau-a}\lambda \tag{20}$$

for $t \in [a, \tau],\ m = 0, 1, \ldots$. The following proposition is, in fact, [2] (Proposition 3.1).

Proposition 1. *Let $\tau \in (a, b),\ z \in \Omega_a$ and $\lambda \in \Omega_{\tau-}$ be fixed. Then*

$$x_m(a, \tau, z, \lambda) = z, x_m(\tau, \tau, z, \lambda) = \lambda$$

for any $m \geq 0$. Furthermore, if $\lim_{m\to+\infty} x_m(\cdot, \tau, z, \lambda) =: x(\cdot)$ exists uniformly on $[a, \tau]$, then $x(\cdot)$ is a solution of the problem

$$x'(t) = f(t, x(t)) + \frac{1}{\tau-a}\left(\lambda - z - \int\limits_a^\tau f(s, x(s))ds\right), \quad t \in [a, \tau], \tag{21}$$

$$x(a) = z, \quad x(\tau) = \lambda. \tag{22}$$

Theorem 1. *Let there exists a nonnegative vector ϱ_0 such that the inequality*

$$\varrho_0 \geq \frac{b-a}{4}\delta_{\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})}(f)$$

holds and $f : [a, b] \times \mathcal{O}_{\varrho_0}(\Omega_{a,\tau-}) \to \mathbb{R}^n$ satisfies the Lipschitz condition $f \in \mathrm{Lip}_K(\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-}))$ with a matrix K for which $r(K) < \dfrac{10}{3(b-a)}$. Then, for all fixed $\tau \in (a, b),\ z \in \Omega_a,\ \lambda \in \Omega_{\tau-}$:

1) *functions* (19) *are continuously differentiable on $[a, \tau]$ for $m \geq 0$;*
2) $\{x_m(t, \tau, z, \lambda) : t \in [a, \tau], m \geq 0\} \subset \mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})$;
3) $\{x_m(\cdot, \tau, z, \lambda) : m \geq 0\}$ *converges to a limit function $x_\infty(\cdot, \tau, z, \lambda)$ uniformly on $[a, \tau]$;*

4) $x_\infty(\cdot,\tau,z,\lambda)$ *is a solution of the boundary-value problem* (21), (22) *and this problem has no other solutions with values in* $\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})$;

5) *the estimate*

$$|x_\infty(t,\tau,z,\lambda) - x_m(t,\tau,z,\lambda)| \le \frac{5}{9}\alpha_1(t,a,\tau)\bar{K}^m\big(1_n - \bar{K}\big)^{-1}\delta_{\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})}(f) \tag{23}$$

holds for all $t \in [a,\tau]$ *and* $m \ge 1$, *where*

$$\bar{K} = \frac{3}{10}(b-a)K. \tag{24}$$

This statement, which ensures the uniform convergence of sequence (19), (20), is proved essentially in the same manner as [2] (Theorem 3.2).

4.2. The interval after jump. In order to consider the model two-point boundary-value problem (18) in the domain $[\tau,b]\times\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})$ describing the solution after the jump has occurred, introduce the sequence of functions $y_m(\cdot,\tau,\lambda,\gamma,\eta),\ t\in[\tau,b]$, by the relations

$$y_{m+1}(t,\tau,\lambda,\gamma,\eta) = \lambda + \gamma + \int\limits_\tau^t f(s, y_m(s,\tau,\lambda,\gamma,\eta))ds -$$

$$- \frac{t-\tau}{b-\tau}\int\limits_\tau^b f(s,y_m(s,\tau,\lambda,\gamma,\eta))ds + \frac{t-\tau}{b-\tau}(\eta-\lambda-\gamma) \tag{25}$$

for $t\in[\tau,b],\ m=0,1,2,\ldots,$ where

$$y_0(t,\tau,\lambda,\gamma,\eta) = \left(1-\frac{t-\tau}{b-\tau}\right)(\lambda+\gamma) + \frac{t-\tau}{b-\tau}\eta, \quad t\in[\tau,b]. \tag{26}$$

Formulae (25) and (26) involve parameters $\tau\in(a,b),\ \lambda\in\Omega_{\tau-},\ \gamma\in\Gamma,\ \eta\in\Omega_b$. The sequence of functions defined by (25) and (26) on $[\tau,b]$ is an analogue of that given by (19) and (20) on $[a,\tau]$. The following statement is easily proved similarly to Proposition 1.

Proposition 2. *Let* $\tau\in(a,b),\ \lambda\in\Omega_{\tau-},\ \gamma\in\Gamma$ *and* $\eta\in\Omega_b$ *be fixed. Then*

$$y_m(\tau,\tau,\lambda,\gamma,\eta) = \lambda+\gamma, \qquad y_m(b,\tau,\lambda,\gamma,\eta) = \eta$$

for any $m\ge 0$. *Furthermore, if* $\lim_{m\to+\infty} y_m(\cdot,\tau,\lambda,\gamma,\eta) := y(\cdot)$ *exists uniformly on* $[\tau,b]$, *then* $y(\cdot)$ *is a solution of the problem*

$$y'(t) = f(t,y(t)) + \frac{1}{b-\tau}\left(\eta-\lambda-\gamma-\int\limits_\tau^b f(s,y(s))ds\right), \quad t\in[\tau,b], \tag{27}$$

$$y(\tau) = \lambda+\gamma, \qquad y(b) = \eta. \tag{28}$$

Furthermore, Proposition 2 and an argument analogous to the proof of Theorem 1 allow us to obtain the following theorem.

Theorem 2. *Let there exists a nonnegative vector* ϱ_1 *such that*

$$\varrho_1 \geq \frac{b-a}{4}\delta_{\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})}(f),$$

the function f *is continuous on* $[a,b]\times\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})$ *and* $f\in \mathrm{Lip}_K(\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b}))$ *with a matrix* K *for which* (24) *holds. Then, for any fixed* $\tau\in(a,b),\ \lambda\in\Omega_{\tau-},\ \gamma\in\Gamma$ *and* $\eta\in D_\eta$:

1) *functions* (25) *are continuously differentiable on* $[\tau,b]$ *for any* $m\geq 0$;

2) $\{y_m(t,\tau,\lambda,\gamma,\eta)\colon t\in[\tau,b],\ m\geq 0\}\subset\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})$;

3) $\{y_m(\cdot,\tau,\lambda,\gamma,\eta)\colon m\geq 0\}$ *converges to a limit function* $y_\infty(\cdot,\tau,\lambda,\gamma,\eta)$ *uniformly on* $[\tau,b]$;

4) $y_\infty(\cdot,\tau,\lambda,\gamma,\eta)$ *is a solution of the boundary-value problem* (27), (28) *and this problem has no other solutions with values in* $\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})$;

5) *the estimate*

$$|y_\infty(t,\tau,\lambda,\gamma,\eta)-y_m(t,\tau,\lambda,\gamma,\eta)|\leq\frac{5}{9}\alpha_1(t,\tau,b)\bar{K}^m(1_n-\bar{K})^{-1}\delta_{\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})}(f) \qquad (29)$$

holds for all $t\in[\tau,b]$ *and* $m\geq 1$, *where* $\bar{K}$ *is given by* (24).

Let us now define the sequence of functions $u_m(\cdot,\tau,z,\lambda,\gamma,\eta)\colon[a,b]\to\mathbb{R}^n,\ m\geq 0$, by setting

$$u_m(t,\tau,z,\lambda,\gamma,\eta):=\begin{cases}x_m(t,\tau,z,\lambda) & \text{if} \quad t\leq\tau,\\ y_m(\cdot,\tau,\lambda,\gamma,\eta) & \text{if} \quad t>\tau.\end{cases} \qquad (30)$$

By Propositions 1 and 2, these functions satisfy the conditions

$$u_m(a,\tau,z,\lambda,\gamma,\eta)=z, \qquad u_m(b,\tau,z,\lambda,\gamma,\eta)=\eta,$$

$$u_m(\tau,\tau,z,\lambda,\gamma,\eta)=\lambda, \qquad u_m(\tau+,\tau,z,\lambda,\gamma,\eta)=\lambda+\gamma$$

for all $m\geq 0$. Similarly to (30), we can define the function

$$u_\infty(t,\tau,z,\lambda,\gamma,\eta):=\begin{cases}x_\infty(t,\tau,z,\lambda) & \text{if} \quad t\leq\tau,\\ y_\infty(t,\tau,\lambda,\gamma,\eta) & \text{if} \quad t>\tau.\end{cases} \qquad (31)$$

Theorems 1 and 2 guarantee that, under the conditions assumed, the function $u_\infty\colon[a,b]\times(a,b)\times\times\Omega_a\times\Omega_{\tau-}\times\Gamma\times\Omega_b\to\mathbb{R}^n$ is well defined and has range in $\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})\cup\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})$. Functions (30) and (31) can be used to describe the solutions of the original control problem (1), (4)–(6).

5. Determining equations for parameter values. For arbitrary $\tau\in(a,b),\ z\in\Omega_a,\ \lambda\in\in\Omega_{\tau-},\gamma\in\Gamma$ and $\eta\in\Omega_b$ let us put

$$\Delta_0(\tau,z,\lambda):=\lambda-z-\int_a^\tau f(s,x_\infty(s,\tau,z,\lambda))ds, \qquad (32)$$

$$\Delta_1(\tau,\lambda,\gamma,\eta):=\eta-\lambda-\gamma-\int_\tau^b f(s,y_\infty(s,\tau,\lambda,\gamma,\eta))ds. \qquad (33)$$

It follows from Theorems 1 and 2 that formulae (32), (33) define mappings $\Delta_0\colon(a,b)\times\Omega_a\times\Omega_{\tau-}\to\to\mathbb{R}^n$ and $\Delta_1\colon(a,b)\times\Omega_{\tau-}\times\Gamma\times\Omega_b\to\mathbb{R}^n$. The assertions of Theorems 1 and 2 can be reformulated in terms of the functions Δ_0 and Δ_1, which allows one to describe the relation of function (31) to the original problem (1), (4)–(6).

Theorem 3. *The following assertions are true.*

1. *Under assumptions of Theorem* 1, *the function* $x_\infty(\cdot,\tau,z,\lambda)$ *is a solution of the differential equation*

$$x'(t) = f(t,x(t)) + \frac{1}{\tau - a}\Delta_0(\tau,z,\lambda), \quad t \in [a,\tau], \tag{34}$$

satisfying the two-point boundary conditions (22). *The boundary-value problem* (34), (22) *has no other solutions with range in* $\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})$.

2. *Under assumptions of Theorem* 2, *the function* $y_\infty(\cdot,\tau,\lambda,\gamma,\eta)$ *is a solution of the differential equation*

$$y'(t) = f(t,y(t)) + \frac{1}{b - \tau}\Delta_1(\tau,\lambda,\gamma,\eta), \quad t \in [\tau,b], \tag{35}$$

satisfying the two-point boundary conditions (28). *The boundary-value problem* (35), (28) *has no other solutions with range in* $\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})$.

This statement is proved by analogy to [2] using the Lipschitz conditions for f. It implies that, instead of functional terms in equations (21) and (27), one can consider constant forcing terms. Expressions (32) and (33) appearing in (34), (35) can be regarded as optimal values of the forcing terms.

Theorem 4. *Let* $\tau \in (a,b),\ z \in \Omega_a,\ \lambda \in \Omega_{\tau-},\ \gamma \in \Gamma,\ \eta \in \Omega_b$ *and* $\mu \in \mathbb{R}^n$ *be fixed.*

1. *Let there exist* ϱ_0 *and a matrix* K *such that assumptions of Theorem* 1 *hold. Then a solution* $x(\cdot)$ *of the differential equation*

$$x'(t) = f(t,x(t)) + \frac{1}{\tau - a}\mu, \quad t \in [a,\tau], \tag{36}$$

has values in $\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})$ *and satisfies the two-point boundary conditions* (22) *if and only if* $\mu = \Delta_0(\tau,z,\lambda)$.

2. *Let there exist* ϱ_1 *and a matrix* K *such that assumptions of Theorem* 2 *hold. Then a solution* $y(\cdot)$ *of the differential equation*

$$y'(t) = f(t,y(t)) + \frac{1}{b - \tau}\mu, \quad t \in [\tau,b], \tag{37}$$

has values in $\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})$ *and satisfies the two-point boundary conditions* (28) *if and only if*

$$\mu = \Delta_1(\tau,\lambda,\gamma,\eta). \tag{38}$$

Proof. Let us prove, e.g., the assertion concerning equation (37). By virtue of Theorem 2, the function $y_\infty(\cdot,\tau,\lambda,\gamma,\eta)$ is well defined and, therefore, equality (33) defines the function $\Delta_1 : (a,b) \times \Omega_{\tau-} \times \Gamma \times \Omega_b \to \mathbb{R}^n$. Assume that value of μ in (37) is given by equality (38). Then (37) coincides with equation (35). According to Theorem 3, the function $y_\infty(\cdot,\tau,\lambda,\gamma,\eta)$ is a solution of problem (37), (28) and its graph lies in the set $[\tau,b] \times \mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})$. Theorem 3 also guarantees that (36) with μ given by (38) does not have any other solutions with properties (28) and graphs lying in $[a,b] \times \mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})$.

Conversely, assume that, for a certain value of μ, problem (37), (28) has a solution y with range in $\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})$. Then, by (28),

$$y(t) = \lambda + \gamma + \int\limits_{\tau}^{t} f(s, y(s))ds + \frac{t-\tau}{b-\tau}\mu, \quad t \in [\tau, b], \tag{39}$$

and, therefore,

$$\eta = \lambda + \gamma + \int\limits_{\tau}^{b} f(s, y_\infty(s, \tau, \lambda, \gamma, \eta))ds + \frac{b-\tau}{b-\tau}\mu,$$

because (28) implies that $y(b) = \eta$. The last equality means that

$$\mu = \eta - \lambda - \gamma - \int\limits_{\tau}^{b} f(s, y(s))ds. \tag{40}$$

Substituting (40) into (39), we obtain

$$y(t) = \lambda + \gamma + \int\limits_{\tau}^{t} f(s, y(s))ds + \frac{t-\tau}{b-\tau}\left(\eta - \lambda - \gamma - \int\limits_{\tau}^{b} f(s, y(s))ds\right) =$$

$$= \int\limits_{\tau}^{t} f(s, y(s))ds - \frac{t-\tau}{b-\tau}\int\limits_{\tau}^{b} f(s, y(s))ds + \lambda + \gamma + \frac{t-\tau}{b-\tau}(\eta - \lambda - \gamma)$$

for $t \in [\tau, b]$, whence, by differentiation, we get

$$y'(t) = f(t, y(t)) - \frac{1}{b-\tau}\int\limits_{\tau}^{b} f(s, y(s))ds + \frac{1}{b-\tau}(\eta - \lambda - \gamma), \quad t \in [\tau, b]. \tag{41}$$

Relation (41) coincides with equation (27), which means that y is a solution of problem (27), (28). However, by virtue of Theorem 2, the function $y_\infty(\cdot, \tau, \lambda, \gamma, \eta)$ is the only solution of (27), (28) having range in $\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})$, and therefore the function y should coincide with $y_\infty(\cdot, \tau, \lambda, \gamma, \eta)$. Replacing the function y by $y_\infty(\cdot, \tau, \lambda, \gamma, \eta)$ in (40) and recalling (33), we obtain (38). The assertion for equation (36) is proved by analogy.

Theorem 4 is proved.

Equations (36) and (37) are ordinary differential equations and the difference between (36), (37) and (1) consists in the presence of a constant forcing term. This circumstance allows one to establish a relation between function (31) the jump control problem (1), (4) – (6).

Theorem 5. *Let there exist vectors* ϱ_0 *and* ϱ_1 *and matrices* K_0 *and* K_1 *such that* $f \in \in \mathrm{Lip}_{K_0}(\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})) \cup \mathrm{Lip}_{K_1}(\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b}))$ *and conditions of Theorems* 1 *and* 2 *hold. Then the following assertions hold*:

1. *If the equations*

$$\Delta_0(\tau, z, \lambda) = 0, \quad \Delta_1(\tau, \lambda, \gamma, \eta) = 0, \quad g(\tau, \lambda) = 0, \quad V(z, \eta) = 0 \tag{42}$$

hold for certain values $(\tau, z, \lambda, \gamma, \eta) \in (a, b) \times \Omega_a \times \Omega_{\tau-} \times \Gamma \times \Omega_b$ *and, in addition,*

$$g(t, y_\infty(t, \tau, \lambda, \gamma, \eta)) \neq 0 \quad \textit{for any } t \in (\tau, b], \tag{43}$$

then the function $u_\infty(\cdot, \tau, z, \lambda, \gamma, \eta)$ *given by* (31) *is a solution of the original control problem* (1), (4) – (6) *with exactly one jump of magnitude* γ *at the time* τ.

2. *If* $u(\cdot)$ *is a solution of control problem* (1), (4) – (6) *with exactly one jump of magnitude* γ *at a time instant* τ *such that*

$$\begin{gathered} \{u(t) : t \in [a, \tau]\} \subset \mathcal{O}_{\varrho_0}(\Omega_{a,\tau-}), \qquad \{u(t) : t \in [\tau, b]\} \subset \mathcal{O}_{\varrho_1}(\Omega_{\tau+,b}), \\ u(a) \in \Omega_a, \quad u(\tau) \in \Omega_{\tau-}, \quad \gamma \in \Gamma, u(b) \in \Omega_b, \end{gathered} \tag{44}$$

then $(\tau, u(a), u(\tau), \gamma, u(b))$ *necessarily satisfy the determining system* (42). *If, moreover, this solution has no other jumps, then* (43) *also holds for the values indicated.*

Proof. 1. Let τ, z, η, λ, and γ be as in (12) and let $u = u_\infty(\cdot, \tau, z, \lambda, \gamma, \eta)$ be defined by (31). Theorem 3 ensures that the restrictions $x := u|_{[a,\tau]}$ and $y := u|_{[\tau,b]}$ satisfy (34) and (35). Assume that $(\tau, z, \lambda, \gamma, \eta)$ satisfy (42). The first two equations of system (42) then imply that u satisfies (1) on $[a, b] \setminus \{\tau\}$. On the other hand, x and y satisfy, respectively, (22) and (28), and therefore, by (31),

$$u(a) = z, \quad u(\tau-) = u(\tau) = \lambda, \quad u(\tau+) = \gamma + \lambda, \quad u(b) = \eta. \tag{45}$$

It is clear from (45) that

$$u(\tau+) - u(\tau-) = \gamma, \tag{46}$$

whereas the third equation in (42) yields $g(\tau, u(\tau-)) = 0$. This means that u satisfies the jump condition (5) at the time τ. In view of assumption (43), this is the only time instant where the jump occurs. Finally, due to the last equation in (42), u satisfies the two-point condition (4). Thus, u is a single-jump solution of (1), (4) – (6).

2. Let u be a single-jump solution of (1), (4) – (6) satisfying inclusions (44). Then there is a unique $\tau \in (a, b)$ such that (46) holds with a certain value of γ. Putting $x := u|_{[a,\tau]}$ and $y := u|_{[\tau,b]}$, we find that x and y have continuous derivatives, satisfy respectively (36) and (37) with $\mu = 0$ and have the properties

$$x(a) = u(a), \qquad x(\tau) = u(\tau), \qquad y(\tau) = u(\tau) + \gamma, \qquad y(b) = u(b).$$

In other words, x and y are solutions of the respective problems (36), (22) and (37), (28) with $\mu = 0$ and

$$z = u(a), \qquad \lambda = u(\tau), \qquad \eta = u(b). \tag{47}$$

Applying Theorem 4, we obtain that $(\tau, z, \lambda, \gamma, \eta)$ with z, λ, η given by (47) necessarily satisfy the first two equations in (42). In view of Theorem 3, it follows that $x = x_\infty(\cdot, \tau, z, \lambda)$ and $y = y_\infty(\cdot, \tau, \lambda, \gamma, \eta)$, which means that u has form (31), whence the required assertions follow.

Theorem 5 is proved.

Note that system (42) consists of $3n + 1$ algebraic or transcendental scalar equations for $3n + 1$ scalar variables τ, z_{j+1}, z_{j+2}, $\ldots$, z_n, η_1, η_2, $\ldots$, η_j, γ_1, γ_2, $\ldots$, γ_n. Under conditions of Theorem 5, system (42) allows one to determine all possible solutions u of problem (1), (4) – (6) having exactly one jump and possessing properties (44). Conditions (44) mean that the graph of u is located in a neighbourhood of the set where the Lipschitz condition holds.

6. Construction of approximate solutions. By Theorem 5, equations (42) determine the parameter values corresponding to solutions of problem (1), (4)–(6). The solvability of the determining system (42) can be established by analogy to [7, 8, 11] using the approximate determining system

$$\lambda = z + \int_a^\tau f(s, x_m(s, \tau, z, \lambda))ds, \qquad \eta = \lambda + \gamma \int_\tau^b f(s, y_m(s, \tau, \lambda, \gamma, \eta))ds \tag{48}$$

with the additional condition

$$g(t, y_m(t, \tau, \lambda, \gamma, \eta)) \neq 0 \quad \text{for any} \quad t \in (\tau, b]. \tag{49}$$

In contrast to equations (42), (43), the explicitly unknown limit function is replaced in equations (48), (49) by the mth iteration for a fixed m. As a consequence, equations (48) and (49) can be constructed in finitely many steps.

If $(\hat{\tau}, \hat{z}, \hat{\lambda}, \hat{\gamma}, \hat{\eta}) \in (a, b) \times \Omega_a \times \Omega_{\tau-} \times \Gamma \times \Omega_b$ is a root of system (48) such that (49) holds, then, due to estimates (23) and (29) of Theorems 1 and 2, the function

$$\hat{u}(t) := \begin{cases} x_m(t, \hat{\tau}, \hat{z}, \hat{\lambda}) & \text{if} \quad t \in [a, \hat{\tau}], \\ y_m(t, \hat{\tau}, \hat{\lambda}, \hat{\gamma}, \hat{\eta}) & \text{if} \quad t \in [\hat{\tau}, b], \end{cases} \tag{50}$$

is natural to be regarded as the mth approximation to a solution of problem (1), (4)–(6) with a single jump of magnitude $\hat{\gamma}$ at the time $\hat{\tau}$.

The most technically difficult part of the above approach is the construction of the functions $x_m(\cdot, \tau, z, \lambda)$ and $y_m(\cdot, \tau, \lambda, \gamma, \eta)$ in (19) and (25). If the explicit integration in (19) and (25) is impossible or exceedingly complicated, one can use alternative versions of these formulas which, at the expense of a certain loss in accuracy, provide successive approximations more suitable for practical computations. Polynomial interpolation and interval division can be used here (see [12, 15–18]) as well as the "frozen" parameters scheme simplifying the construction of determining equations by reusing the results of computations from the preceding step [2–4]. The choice of the sets Ω_a, $\Omega_{\tau-}$, Γ and Ω_b which, of course, depends on a particular problem, can be facilitated by solving the zeroth approximate determining system (i. e., (48) with $m = 0$), which, together with the corresponding piecewise linear zeroth approximation, usually gives us a preliminary picture of where the graph of the solution is probably located and which regions should be selected when solving equations (48) numerically.

7. A numerical example. Let us demonstrate the approach on an example from [2] reformulated as a control problem with unknown jump magnitude. Consider the system of differential equations

$$\begin{aligned} u_1'(t) &= (u_2(t))^2 - \frac{t}{5}u_1(t) + \frac{t^3}{100} - \frac{t^2}{25}, \\ u_2'(t) &= \frac{t^2}{10}u_2(t) + \frac{t}{8}u_1(t) - \frac{21t^3}{800} + \frac{t}{16} + \frac{1}{5}, \quad t \in \left[0, \frac{1}{2}\right], \end{aligned} \tag{51}$$

with the two-point boundary condition

$$\begin{pmatrix} \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} -0.1212 \\ 0.0019 \end{pmatrix} \tag{52}$$

and the state-dependent jump occurring in accordance with the rule

$$u_1(\tau + 0) - u_1(\tau - 0) = \gamma_1, \qquad u_2(\tau + 0) - u_2(\tau - 0) = \gamma_2 \tag{53}$$

for τ such that

$$\left(u_1(\tau) + \frac{1}{2}\right)^2 + u_2(\tau) = \frac{1}{25}. \tag{54}$$

The jump magnitudes γ_1 and γ_2 in condition (53) are considered in [2] as fixed and equal to the values

$$\gamma_1 = \frac{1}{2}, \qquad \gamma_2 = -\frac{1}{10}. \tag{55}$$

The study of problem (51), (52) in [2] is focused on the single-jump case, where it is needed to determine a left-continuous vector function $u\colon \left[0, \frac{1}{2}\right] \to \mathbb{R}^2$, $u = \operatorname{col}(u_1, u_2)$, whose graph intersects the barrier set

$$\left\{(x_1, x_2) \in \mathbb{R}^2 \colon \left(x_1 + \frac{1}{2}\right)^2 + x_2 - \frac{1}{25} = 0\right\}$$

exactly once. That is, one looks for u satisfying condition (52) and such that there exists a unique time instant $\tau \in \left(0, \frac{1}{2}\right)$ at which (54) holds, the jump magnitude at τ is equal to the given value $\gamma = \operatorname{col}\left(\frac{1}{2}, -\frac{1}{10}\right)$ and u satisfies the differential equations (51) for $t \in \left[0, \frac{1}{2}\right] \setminus \{\tau\}$.

It was shown in [2] that the corresponding determining system has two solutions, which determine two single-jump solutions of problem (51)–(54); let us denote them by u^{I} and u^{II} and the corresponding jump times by τ^{I} and τ^{II}.

On the fourth step of iteration, we have obtained in [2] the numerical values of parameters

$$\begin{gathered} z_1 = -8.437478618, \quad z_2 = -3.968739309, \quad \eta_1 = 0.0076000002, \quad \eta_2 = 0.0076000002, \\ \lambda_1 = -1.493945318, \quad \lambda_2 = -1.493945318, \quad \tau = 0.377366355 \end{gathered} \tag{56}$$

determining the solution u^{I} and the values

$$\begin{gathered} z_1 = -0.492769235, \quad z_2 = 0.003615383, \quad \eta_1 = 0.007600000, \quad \eta_2 = 0.010065542, \\ \lambda_1 = -0.491120590, \quad \lambda_2 = 0.0039921156, \quad \tau = 0.181450846 \end{gathered}$$

for u^{II}. The residuals obtained after the substitution of the fourth approximations of u^{I} and u^{II} into the differential system (51) are of order 10^{-7} and 10^{-10}, respectively.

The meaning of variables in the expressions above is explained by the following scheme:

$$z_1 \to u_1(0), \quad z_2 \to u_2(0), \quad \eta_1 \to u_1(1/2), \quad \eta_2 \to u_2(1/2),$$
$$\lambda_1 \to u_1(\tau), \quad \lambda_2 \to u_2(\tau). \tag{57}$$

Let us now put this problem into another setting, where the jump magnitude in not fixed beforehand but should be determined so that an additional two-point condition of type (6) is satisfied. In this way, we shall regard both the time instant τ and the jump magnitude $\gamma = \mathrm{col}(\gamma_1, \gamma_2)$ in (53) as unknown parameters the values of which are to be found. For the convenience of comparison with [2], let us choose the additional two-point condition (6) so that it corresponds to the solution u^{I} determined by the parameter values (56), namely, impose the condition

$$u_1(0) = -8.437478618, \qquad u_2(1/2) = -4.498968764, \tag{58}$$

where the numbers in the right-hand side are numerical values of u^{I} at 0 and $1/2$. We are thus dealing with the control boundary-value problem (51)–(53), (58), where one is looking for u, τ, and γ.

Due to the way condition (58) is posed, it is obvious that this control problem should have a solution u_*^{I} equal to u^{I} with $\tau = \tau^{\mathrm{I}}$ and γ given by (55). Carrying out Maple computations according to the approach described in the above sections, we find that the results indeed essentially coincide with (56) for u^{I}:

$$z_2 = -3.968739309, \quad \eta_1 = 0.007600000, \quad \lambda_1 = -1.493945318, \quad \lambda_2 = -1.493945318,$$
$$\tau = 0.377366355, \quad \gamma_1 = 0.5, \quad \gamma_2 = -0.1. \tag{59}$$

The absence of z_1 and η_2 here is due to the imposed condition (58), which, according to (57), fixes those variables to the given values.

To detect other possible solutions, we define suitable auxiliary sets as in [2]. For the sets Ω_a and $\Omega_{\tau-}$, where one looks for the values $u(a)$ and $u(\tau)$, we take

$$O_{\varrho_0}(\Omega_{a,\tau-}) = \Omega_{\tau-} = \{(x_1, x_2) \colon -8.44 \le x_1 \le 0.15,\ -4.0 \le x_2 \le 0.15\}. \tag{60}$$

The corresponding set $\Omega_{a,\tau-}$ then coincides with (60). If we put, e.g., $\varrho_0 = \mathrm{col}(2.46,\ 0.2)$ then, according to (7), the ϱ_0-neighbourhood $\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-}) = \mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})$ of the set $\Omega_{a,\tau-}$ has the form

$$\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-}) = \{(x_1, x_2) \colon -10.9 \le x_1 \le 2.61,\ -4.2 \le x_2 \le 0.35\}. \tag{61}$$

Introduce the set for the control parameter γ

$$\Gamma := \{(\gamma_1, \gamma_2) \colon 0.25 \le \gamma_1 \le 0.65,\ -0.25 \le \gamma_2 \le -0.05\}.$$

Direct computations show that $f = \mathrm{col}(f_1, f_2) \colon (t, u_1, u_2) \mapsto \mathrm{col}(u_2^2 - tu_1/5 + t^3/100 - t^2/25, t^2 u_2/10 + tu_1/8 - 21t^3/800 + t/6 + 1/5)$ appearing in (51) belongs to $\mathrm{Lip}_{K_0}(\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-}))$ with the matrix

$$K_0 = \begin{pmatrix} \dfrac{1}{10} & \dfrac{42}{5} \\ \dfrac{1}{16} & \dfrac{1}{40} \end{pmatrix}.$$

Since $r(K_0) \approx 0.788$, we see that $\bar{K}_0 = \dfrac{3}{20} K_0$ has $r(\bar{K}_0) = 0.1182 < 1$. Moreover, computing the value of $\delta_{\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})}(f)$ according to (11) and (61), we obtain

$$\delta_{\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})}(f) = \begin{pmatrix} 18.991 \\ 0.958125 \end{pmatrix},$$

and, therefore,

$$\varrho_0 = \begin{pmatrix} 2.46 \\ 0.2 \end{pmatrix} \geq \frac{b-a}{4}\delta_{\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})}(f) = \frac{1}{8}\delta_{\mathcal{O}_{\varrho_0}(\Omega_{a,\tau-})}(f) = \begin{pmatrix} 2.373875 \\ 0.119765625 \end{pmatrix}.$$

The conditions of Theorem 1 are thus satisfied and, consequently, the sequence of functions (19) is convergent.

Let us put

$$\Omega_{\frac{1}{2}} = \Omega_{\tau+} = \{(y_1, y_2)\colon -7.94 \leq y_1 \leq 0.7,\ -4.15 \leq y_2 \leq 0.05\}$$

and choose the vector $\varrho_1 = \mathrm{col}(2.63, 0.15)$. Then the ϱ_1-neighbourhood $O_{\varrho_1}(\Omega_{\tau+,\frac{1}{2}}) = \mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})$ of the set $\Omega_{\tau+,\frac{1}{2}}$ has the form

$$\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b}) = \{(y_1, y_2)\colon -10.57 \leq y_1 \leq 3.33,\ -4.3 \leq y_2 \leq 0.2\}. \tag{62}$$

The computation shows that $f \in \mathrm{Lip}_{K_1}(\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b}))$ with the matrix

$$K_1 = \begin{pmatrix} \frac{1}{10} & \frac{43}{5} \\ \frac{1}{16} & \frac{1}{40} \end{pmatrix}.$$

Since $r(K_1) \approx 0.7966,$ it follows that for $\bar{K}_1 = \dfrac{3}{20}K$ we have $r(\bar{K}_1) \approx 0.11949 < 1.$ Moreover, according to (11) and (62),

$$\delta_{\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})}(f) = \begin{pmatrix} 19.88 \\ 0.98125 \end{pmatrix}$$

and

$$\varrho_1 = \begin{pmatrix} 2.63 \\ 0.15 \end{pmatrix} \geq \frac{b-a}{4}\delta_{\mathcal{O}_{\varrho_1}(\Omega_{\tau+,b})}(f) \approx \begin{pmatrix} 2.485 \\ 0.1227 \end{pmatrix}.$$

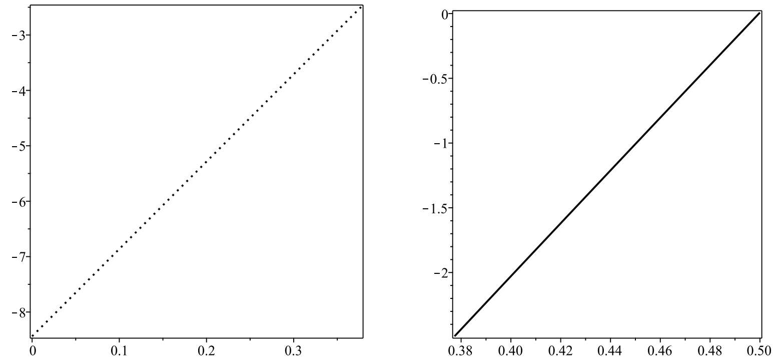
Thus, the assumptions of Theorems 2 hold, which guarantees the convergence of the function sequence (25).

Table 1. Numerical values of the phase parameters and jump magnitude for u_*^{II}

m	λ_1	λ_2	γ_1	γ_2	η_1	z_2
0	-2.49394187	-3.93580419	-0.01571395	-0.57215580	0.00760000	-3.96873931
1	-2.49394274	-3.93580764	0.00174158	-0.57213096	0.00760000	-3.96873931
2	-2.49394530	-3.93558179	0.00167162	-0.57212080	0.00760001	-3.96873930
3	-2.49394531	-3.93581792	0.00168085	-0.57212074	0.00760001	-3.96873931
4	-2.49394531	-3.93581793	0.00168083	-0.57212073	0.00759999	-3.96873931

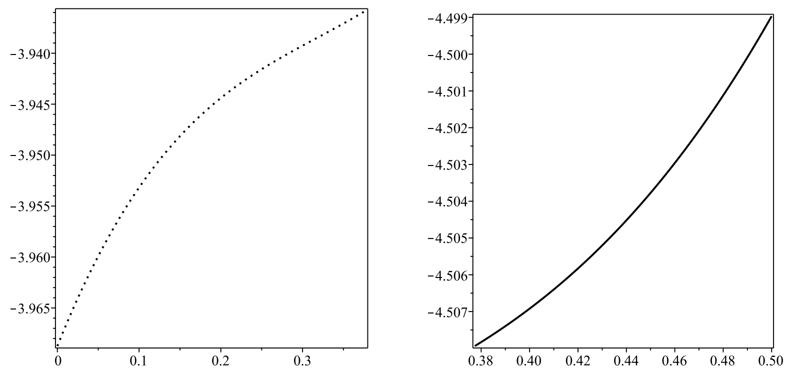
Table 2. Numerical values of the jump time for u_*^{II}

m	τ
0	0.37647668
1	0.37736941
2	0.37736590
3	0.37736635
4	0.37736635



(a) (b)

Fig. 1. First and second component of the solution on the pre-jump interval.

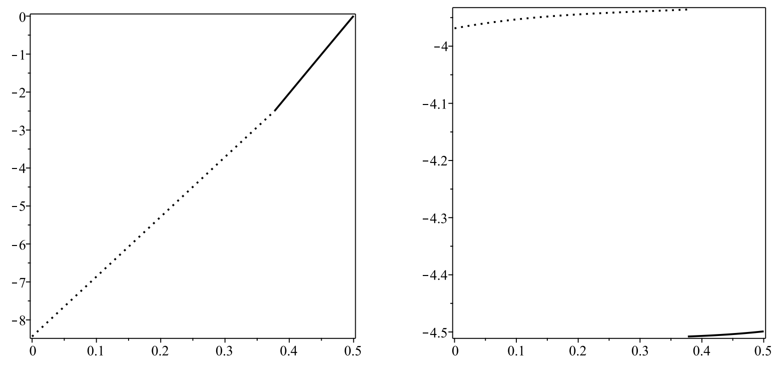


(a) (b)

Fig. 2. First and second component of the solution on the after-jump interval.

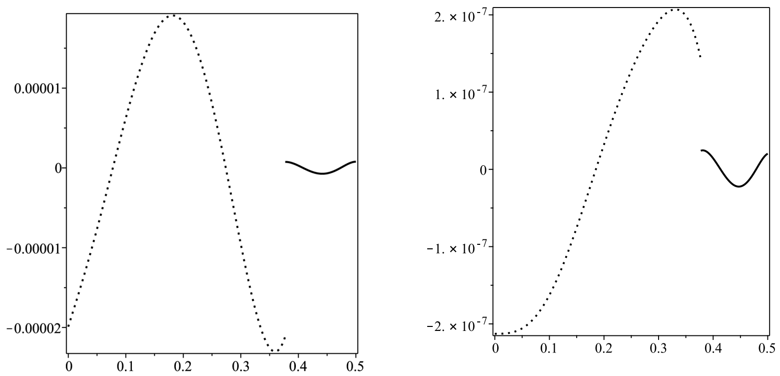
According to Theorem 5, the number of solutions of the determining system (42) coincides with the number of solutions of the given jump control problem. Computation shows that, along with the solution (59) corresponding to u_*^{I}, the approximate determining system of algebraic equations (48), (49) has another solution, which indicates the presence of another solution u_*^{II} of the jump control problem (51) – (54), (58).

Carrying out Maple computations using (19), (25) and (48) – (50), we numerically find approximate values of parameters that determine u_*^{II}. Tables 1 and 2 show the corresponding values of parameters, jump magnitude and jump time obtained at several steps of iteration. We see that the jump of u_*^{II} occurs at the time $\tau \approx 0.37736635$ and has magnitude $\gamma \approx \mathrm{col}(0.00168083, -0.57212073)$.



(*a*) (*b*)

Fig. 3. First and second component of the solution on the interval $[0, 0.5]$.



(*a*) (*b*)

Fig. 4. First and second component of the residual of the approximate solution on the interval $[0, 0.5]$.

Figures 1 and 2 show the components of the fourth approximations of u_*^{II} on the pre-jump and after-jump intervals $[0, \tau]$ and $[\tau, 1/2]$. Figure 3 shows the graph of the fourth approximations to u_*^{II} on the entire interval $[0, 1/2]$. The residual functions obtained after the substitution of the fourth approximation of u_*^{II} into the differential system (51) are shown on Fig. 4.

References

1. I. Rachůnková, J. Tomeček, *State-dependent impulses: boundary value problems on compact interval*, vol. 6, Atlantis Briefs Different. Equat., Atlantic Press, Paris (2015).
2. I. Rachůnková, L. Rachůnek, A. Rontó, M. Rontó, *A constructive approach to boundary value problems with state-dependent impulses*, Appl. Math. and Comput., **274**, 726 – 744 (2016).
3. A. Rontó, I. Rachůnková, M. Rontó, L. Rachůnek, *Investigation of solutions of state-dependent multi-impulsive boundary value problems*, Georgian Math. J., **24**, № 2, 287 – 312 (2017).
4. A. Rontó, M. Rontó, J. Varha, *A new approach to non-local boundary value problems for ordinary differential systems*, Appl. Math. and Comput., **250**, 689 – 700 (2015).
5. M. Ronto, A. M. Samoilenko, *Numerical-analytic methods in the theory of boundary-value problems*, World Sci. Publ. Co., River Edge, NJ (2000).

6. A. Rontó, M. Rontó, *On nonseparated three-point boundary value problems for linear functional differential equations*, Abstr. and Appl. Anal., Article ID 326052, 1 – 22 (2011).
7. A. Rontó, M. Rontó, *Existence results for three-point boundary value problems for systems of linear functional differential equations*, Carpathian J. Math., **28**, № 1, 163–182 (2012).
8. A. Rontó, M. Rontó, N. Shchobak, *Constructive analysis of periodic solutions with interval halving*, Bound. Value Probl., **2013**, № 57, 1 – 34 (2013).
9. A. Rontó, M. Rontó, N. Shchobak, *On finding solutions of two-point boundary value problems for a class of non-linear functional differential systems* (2012).
10. A. Rontó, M. Rontó, *Successive approximation techniques in non-linear boundary value problems for ordinary differential equations*, Handbook of Differential Equations: Ordinary Differential Equations, vol. IV, Handb. Different. Equat., Elsevier/North-Holland, Amsterdam (2008), p. 441 – 592.
11. A. Rontó, M. Rontó, *On constructive investigation of a class of non-linear boundary value problems for functional differential equations*, Carpathian J. Math., **29**, № 1, 91 – 108 (2013).
12. A. Rontó, M. Rontó, G. Holubová, P. Nečesal, *Numerical-analytic technique for investigation of solutions of some nonlinear equations with Dirichlet conditions*, Bound. Value Probl., № 58, 1 – 20 (2011).
13. B. Půža, A. Rontó, M. Rontó, N. Shchobak, *On solutions of nonlinear boundary-value problems the components of which vanish at certain points*, Ukrainian Math. J., **70**, № 1, 101 – 123 (2018).
14. A. Rontó, M. Rontó, N. Shchobak, *On periodic solutions of systems of linear differential equations with argument deviations*, J. Math. Sci., **263**, № 2, 282 – 298 (2022).
15. A. Rontó, M. Rontó, N. Shchobak, *Parametrisation for boundary value problems with transcendental non-linearities using polynomial interpolation*, Electron. J. Qual. Theory Different. Equat., Paper № 59, 1 – 22 (2018).
16. A. Rontó, M. Rontó, *Periodic successive approximations and interval halving*, Miskolc Math. Notes, **13**, № 2, 459 – 482 (2012).
17. A. Rontó, M. Rontó, N. Shchobak, *Notes on interval halving procedure for periodic and two-point problems*, Bound. Value Probl., **2014**, № 164, 1 – 20 (2014).
18. A. Rontó, M. Rontó, J. Varha, *On non-linear boundary value problems and parametrisation at multiple nodes*, Electron. J. Qual. Theory Different. Equat., Paper № 80, 1 – 18 (2016).

Received 23.11.22