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M. A. Ali (Jiangsu Key Laboratory for NSLSCS, School Math. Sci., Nanjing Normal Univ., China),
M. Fečkan[1] (Comenius Univ. Bratislava and Math. Inst., Slovak Acad. Sci., Bratislava, Slovakia),
A. Mateen (COMSATS Univ. Islamabad Sahiwal Campus, Pakistan)

STUDY OF QUANTUM OSTROWSKI'S-TYPE INEQUALITIES FOR DIFFERENTIABLE CONVEX FUNCTIONS[2]

ДОСЛІДЖЕННЯ КВАНТОВИХ НЕРІВНОСТЕЙ ТИПУ ОСТРОВСЬКОГО ДЛЯ ДИФЕРЕНЦІЙОВНИХ ОПУКЛИХ ФУНКЦІЙ

We prove some new q-Ostrowski's-type inequalities for differentiable and bounded functions. Moreover, we present the relationship between the newly established and already known inequalities, which is very interesting for new readers. Some applications to special means of real numbers are given to make the results more valuable.

Обгрунтовано деякі нові нерівності q-Островського типу для диференційовних та обмежених функцій. Крім того, встановлено зв'язок між новими нерівностями та нерівностями, отриманими раніше, що дуже цікаво для нових читачів. Наведено деякі застосування до спеціальних середніх дійсних чисел, щоб зробити наші результати більш цінними.

1. Introduction. A. Ostrowski showed an inequality involving a function with bounded derivative in 1938, which became known as the Ostrowski inequality [26].

Theorem 1.1. *For a differentiable functions* $\mathfrak{F}\colon [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ *over* $(\mathfrak{y}_1, \mathfrak{y}_2)$ *with* $|\mathfrak{F}(\mathfrak{r})| \leq M$, *the following inequality holds*:

$$\left| \mathfrak{F}(\mathfrak{r}) - \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1} \int\limits_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{F}(t) dt \right| \leq M(\mathfrak{y}_2 - \mathfrak{y}_1) \left[\frac{(\mathfrak{r} - \mathfrak{y}_1)^2 + (\mathfrak{y}_2 - \mathfrak{r})^2}{2} \right], \tag{1.1}$$

where $\mathfrak{r} \in [\mathfrak{y}_1, \mathfrak{y}_2]$.

The following are two possible interpretations of the Ostrowski inequality:

(i) estimation of the functional value's deviation from its average value;

(ii) a rectangle is used to approximate the area under the curve.

On the other hand, Budak et al. proved quantum version of the inequality (1.1) as follows:

Theorem 1.2 [15]. *Let* $\mathfrak{F}\colon [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ *be a q-differentiable function. If* ${}_{\mathfrak{y}_1}D_q\mathfrak{F}$ *and* ${}^{\mathfrak{y}_2}D_q\mathfrak{F}$ *are continuous and integrable on* $[\mathfrak{y}_1, \mathfrak{y}_2]$ *with* $|{}_{\mathfrak{y}_1}D_q\mathfrak{F}|$, $|{}^{\mathfrak{y}_2}D_q\mathfrak{F}| \leq M$, *then the following inequality holds for* $\mathfrak{r} \in [\mathfrak{y}_1, \mathfrak{y}_2]$:

$$\left| \mathfrak{F}(\mathfrak{r}) - \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1} \left[\int\limits_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)_{\mathfrak{y}_1} d_q t + \int\limits_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)^{\mathfrak{y}_2} d_q t \right] \right| \leq \frac{qM}{\mathfrak{y}_2 - \mathfrak{y}_1} \left[\frac{(\mathfrak{r} - \mathfrak{y}_1)^2 + (\mathfrak{y}_2 - \mathfrak{r})^2}{[2]_q} \right].$$

[1] Corresponding author, e-mail: michal.feckan@fmph.uniba.sk.

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It is also well-known that $\mathfrak{F}$ is convex if and only if it satisfies the Hermite – Hadamard inequality, stated below:

$$\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right) \leq \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\int\limits_{\mathfrak{y}_1}^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})d\mathfrak{r} \leq \frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2}, \tag{1.2}$$

where $\mathfrak{F}\colon \mathfrak{I}\to\mathbb{R}$ is a convex function and $\mathfrak{y}_1,\mathfrak{y}_2\in\mathfrak{I}$ with $\mathfrak{y}_1<\mathfrak{y}_2$.

In [8], Alp et al. proved the following version of quantum Hermite – Hadamard type for convex functions using the left quantum integrals:

Theorem 1.3. *For any convex function* $\mathfrak{F}\colon[\mathfrak{y}_1,\mathfrak{y}_2]\to\mathbb{R}$, *the following inequality holds*:

$$\mathfrak{F}\left(\frac{q\mathfrak{y}_1+\mathfrak{y}_2}{[2]_q}\right) \leq \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\int\limits_{\mathfrak{y}_1}^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\ {}_{\mathfrak{y}_1}d_q\mathfrak{r} \leq \frac{q\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{[2]_q}. \tag{1.3}$$

Recently, Bermudo et al. [10] used the right quantum integrals and proved the following variant of Hermite – Hadamard type inequalities for convex functions:

Theorem 1.4. *For any convex function* $\mathfrak{F}\colon[\mathfrak{y}_1,\mathfrak{y}_2]\to\mathbb{R}$, *the following inequalities holds*:

$$\mathfrak{F}\left(\frac{\mathfrak{y}_1+q\mathfrak{y}_2}{[2]_q}\right) \leq \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\int\limits_{\mathfrak{y}_1}^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\ {}^{\mathfrak{y}_2}d_q\mathfrak{r} \leq \frac{\mathfrak{F}(\mathfrak{y}_1)+q\mathfrak{F}(\mathfrak{y}_2)}{[2]_q} \tag{1.4}$$

and

$$\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right) \leq \frac{1}{2(\mathfrak{y}_2-\mathfrak{y}_1)}\left[\int\limits_{\mathfrak{y}_1}^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\ {}_{\mathfrak{y}_1}d_q\mathfrak{r}+\int\limits_{\mathfrak{y}_1}^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\ {}^{\mathfrak{y}_2}d_q\mathfrak{r}\right] \leq \frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2}. \tag{1.5}$$

***Remark* 1.1.** It is obvious that if we take the limit as $q\to 1^-$ in (1.3) – (1.5), then we obtain the inequality (1.2).

For the left and right estimates of inequalities (1.3) and (1.4), one can consult [1, 5, 9, 11 – 13, 19, 23, 24, 31]. In [25], Noor et al. established a generalized version of (1.3). In [2, 3, 14, 21, 29], the authors used convexity and coordinated convexity to prove Simpson's- and Newton's-type inequalities via q-calculus. For the study of Ostrowski's inequalities, one can consult [4, 6, 30].

Inspired by the ongoing studies, we prove a new parameterized quantum integral identity involving left and right quantum derivatives to prove different variants of quantum integral inequalities for quantum differentiable convex functions. The main advantage of the newly established inequalities is that these can be turned into quantum Ostrowski's-type inequalities for convex functions [15], classical Ostrowski's-type inequalities for convex functions [16], several classical integral inequalities for convex functions [22] and several new quantum integral inequalities like midpoint type, trapezoidal type, Ostrowski's-type and Simpson's-type without having to prove each one separately.

This paper is organized as follows. Section 2 provides a brief overview of the fundamentals of q-calculus as well as other related studies in this field. In Section 3, we establish an identity that plays an essential role in developing the main results of this paper. The different variants of quantum integral inequalities for quantum differentiable convex functions are described in Section 4.

The relationship between the findings reported here and similar findings in the literature are also considered. In Section 5, we give some applications to special means of real numbers by using the newly established results. Section 6 concludes with some recommendations for future research.

2. Preliminaries of q-calculus and some inequalities. This section first presents the definitions and some properties of quantum derivatives and quantum integrals. We also mention some well-known inequalities for quantum integrals. Throughout this paper, let $0 < q < 1$ be a constant.

The q-number or q-analogue of $n \in \mathbb{N}$ is given by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}.$$

Definition 2.1 [28]. *Let $\mathfrak{F}\colon [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ be a continuous function. Then the left q-derivative of function $\mathfrak{F}$ at $\mathfrak{r} \in [\mathfrak{y}_1, \mathfrak{y}_2]$ is defined by*

$$ {}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}) = \begin{cases} \dfrac{\mathfrak{F}(\mathfrak{r}) - \mathfrak{F}(q\mathfrak{r} + (1-q)\mathfrak{y}_1)}{(1-q)(\mathfrak{r} - \mathfrak{y}_1)} & \text{for} \quad \mathfrak{r} \neq \mathfrak{y}_1, \\ \lim\limits_{\mathfrak{r} \to \mathfrak{y}_1} {}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}) & \text{for} \quad \mathfrak{r} = \mathfrak{y}_1. \end{cases} \tag{2.1}$$

The function $\mathfrak{F}$ is said to be q-differentiable function on $[\mathfrak{y}_1, \mathfrak{y}_2]$ if ${}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})$ exists for all $\mathfrak{r} \in [\mathfrak{y}_1, \mathfrak{y}_2]$.

Note that if $\mathfrak{y}_1 = 0$ and ${}_0D_q\mathfrak{F}(\mathfrak{r}) = D_q\mathfrak{F}(\mathfrak{r})$, then (2.1) reduces to

$$D_q\mathfrak{F}(\mathfrak{r}) = \begin{cases} \dfrac{\mathfrak{F}(\mathfrak{r}) - \mathfrak{F}(q\mathfrak{r})}{(1-q)\mathfrak{r}} & \text{for} \quad \mathfrak{r} \neq 0, \\ \lim\limits_{\mathfrak{r} \to 0} D_q\mathfrak{F}(\mathfrak{r}) & \text{for} \quad \mathfrak{r} = 0, \end{cases}$$

which is the q-Jackson derivative (see [18, 20, 28] for more details).

Theorem 2.1 [28]. *If $\mathfrak{F}, g\colon J \to \mathbb{R}$ are q-differentiable functions, then the following identities hold*:

(i) *the product $\mathfrak{F}g\colon [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ is q-differentiable on $[\mathfrak{y}_1, \mathfrak{y}_2]$ with*

$${}_{\mathfrak{y}_1}D_q(\mathfrak{F}g)(\mathfrak{r}) = \mathfrak{F}(\mathfrak{r})_{\mathfrak{y}_1}D_q g(\mathfrak{r}) + g(q\mathfrak{r} + (1-q)\mathfrak{r})_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}) =$$
$$= g(\mathfrak{r})_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}) + \mathfrak{F}(q\mathfrak{r} + (1-q)\mathfrak{r})_{\mathfrak{y}_1}D_q g(\mathfrak{r});$$

(ii) *if $g(\mathfrak{r})g(q\mathfrak{r} + (1-q)\mathfrak{r}) \neq 0$, then $\mathfrak{F}/g$ is q-differentiable on $[\mathfrak{y}_1, \mathfrak{y}_2]$ with*

$${}_{\mathfrak{y}_1}D_q\left(\frac{\mathfrak{F}}{g}\right)(\mathfrak{r}) = \frac{g(\mathfrak{r})_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r}) - \mathfrak{F}(\mathfrak{r})_{\mathfrak{y}_1}D_q g(\mathfrak{r})}{g(\mathfrak{r})g(q\mathfrak{r} + (1-q)\mathfrak{r})}.$$

Definition 2.2 [28]. *Let $\mathfrak{F}\colon [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ be a continuous function. Then the left q-integral of function $\mathfrak{F}$ at $z \in [\mathfrak{y}_1, \mathfrak{y}_2]$ is defined by*

$$\int\limits_{\mathfrak{y}_1}^{z} \mathfrak{F}(\mathfrak{r})_{\mathfrak{y}_1}d_q\mathfrak{r} = (1-q)(z - \mathfrak{y}_1)\sum_{n=0}^{\infty} q^n \mathfrak{F}(q^n z + (1-q^n)\mathfrak{y}_1). \tag{2.2}$$

The function $\mathfrak{F}$ is said to be q-integrable function on $[\mathfrak{y}_1, \mathfrak{y}_2]$ if $\int_{\mathfrak{y}_1}^{z} \mathfrak{F}(\mathfrak{r})_{\mathfrak{y}_1}d_q\mathfrak{r}$ exists for all $z \in [\mathfrak{y}_1, \mathfrak{y}_2]$.

Note that if $\mathfrak{y}_1 = 0$, then (2.2) reduces to

$$\int_0^z \mathfrak{F}(\mathfrak{r})_0 d_q\mathfrak{r} = \int_0^z \mathfrak{F}(\mathfrak{r}) d_q\mathfrak{r} = (1-q)z\sum_{n=0}^{\infty} q^n \mathfrak{F}(q^n z),$$

which is the q-Jackson integral (see [18, 20, 28] for more details).

Theorem 2.2 [28]. *If* $\mathfrak{F}\colon [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ *is a continuous function and* $z \in [\mathfrak{y}_1, \mathfrak{y}_2]$, *then the following identities hold*:

(i) ${}_{\mathfrak{y}_1}D_q \int_{\mathfrak{y}_1}^{z} \mathfrak{F}(\mathfrak{r})_{\mathfrak{y}_1} d_q\mathfrak{r} = \mathfrak{F}(z)$;

(ii) $\int_c^z {}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})_{\mathfrak{y}_1} d_q\mathfrak{r} = \mathfrak{F}(z) - \mathfrak{F}(c)$ *for* $c \in (\mathfrak{y}_1, z)$.

On the other hand, Bermudo et al. [10] defined new quantum derivative and quantum integral which are called right q-derivative and right q-integral as follows:

Definition 2.3 [10]. *The right q-derivative of function* $\mathfrak{F}\colon [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ *is defined as*

$$^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r}) = \frac{\mathfrak{F}(q\mathfrak{r} + (1-q)\mathfrak{y}_2) - \mathfrak{F}(\mathfrak{r})}{(1-q)(\mathfrak{y}_2 - \mathfrak{r})}, \quad \mathfrak{r} \neq \mathfrak{y}_2.$$

If $\mathfrak{r} = \mathfrak{y}_2$, *we define* $^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2) = \lim_{\mathfrak{r}\to\mathfrak{y}_2} {}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})$ *if it exists and it is finite.*

Definition 2.4 [10]. *The right q-definite integral of function* $\mathfrak{F}\colon [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ *on* $[\mathfrak{y}_1, \mathfrak{y}_2]$ *is defined as*

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})^{\mathfrak{y}_2} d_q\mathfrak{r} = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{k=0}^{\infty} q^k \mathfrak{F}\big(q^k\mathfrak{y}_1 + (1-q^k)\mathfrak{y}_2\big).$$

Lemma 2.1 [7]. *For continuous functions* $\mathfrak{F}, g\colon [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$, *the following equality true*:

$$\int_0^c g(t)_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{y}_2 + (1-t)\mathfrak{y}_1) d_qt =$$

$$= \left.\frac{g(t)\mathfrak{F}(t\mathfrak{y}_2 + (1-t)\mathfrak{y}_1)}{\mathfrak{y}_2 - \mathfrak{y}_1}\right|_0^c - \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\int_0^c D_qg(t)\mathfrak{F}(qt\mathfrak{y}_2 + (1-qt)\mathfrak{y}_1) d_qt.$$

Lemma 2.2 [27]. *For continuous functions* $\mathfrak{F}, g\colon [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$, *the following equality true*:

$$\int_0^c g(t)^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{y}_1 + (1-t)\mathfrak{y}_2) d_qt =$$

$$= \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\int_0^c D_qg(t)\mathfrak{F}(qt\mathfrak{y}_1 + (1-qt)\mathfrak{y}_2) d_qt - \left.\frac{g(t)\mathfrak{F}(t\mathfrak{y}_1 + (1-t)\mathfrak{y}_2)}{\mathfrak{y}_2 - \mathfrak{y}_1}\right|_0^c.$$

3. Identities. We deal with identities necessary to attain our main estimations in this section. We first establish an identity based on two steps kernel in the following lemma.

Lemma 3.1. *Let* $\mathfrak{F}\colon [\mathfrak{y}_1, \mathfrak{y}_2] \to \mathbb{R}$ *be a q-differentiable function. If* ${}_{\mathfrak{y}_1}D_q\mathfrak{F}$ *and* ${}^{\mathfrak{y}_2}D_q\mathfrak{F}$ *are continuous and integrable on* $[\mathfrak{y}_1, \mathfrak{y}_2]$, *then, for* $\mathfrak{r} \in [\mathfrak{y}_1, \mathfrak{y}_2]$ *and* $\lambda \geq 0$, *one has the identity*

$$(1-\lambda)\mathfrak{F}(\mathfrak{r}) + \lambda\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2} - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt + \int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right] =$$

$$= \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int_0^1 \left(qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right) {}_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_1)d_qt -$$

$$- \frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int_0^1 \left(qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\right) {}^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_2)d_qt. \tag{3.1}$$

Proof. We have

$$I_1 = \int_0^1 \left(qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right) {}_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_1)d_qt.$$

By using Lemma 2.1, we have

$$I_1 = \left(qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right)\frac{\mathfrak{F}(t\mathfrak{r}+(1-t)\mathfrak{y}_1)}{\mathfrak{r}-\mathfrak{y}_1}\bigg|_0^1 - \frac{q}{\mathfrak{r}-\mathfrak{y}_1}\int_0^1 \mathfrak{F}(qt\mathfrak{r}+(1-qt)\mathfrak{y}_1)d_qt =$$

$$= \left(q - \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right)\frac{\mathfrak{F}(\mathfrak{r})}{\mathfrak{r}-\mathfrak{y}_1} + \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{(\mathfrak{r}-\mathfrak{y}_1)^2}\mathfrak{F}(\mathfrak{y}_1) - \frac{q}{\mathfrak{r}-\mathfrak{y}_1}\int_0^1 \mathfrak{F}(qt\mathfrak{r}+(1-qt)\mathfrak{y}_1)d_qt =$$

$$= \left(q - \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right)\frac{\mathfrak{F}(\mathfrak{r})}{\mathfrak{r}-\mathfrak{y}_1} + \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{(\mathfrak{r}-\mathfrak{y}_1)^2}\mathfrak{F}(\mathfrak{y}_1) -$$

$$- \frac{q}{\mathfrak{r}-\mathfrak{y}_1}\left\{\frac{1-q}{q}\sum_{n=0}^{\infty} q^n\mathfrak{F}(q^n\mathfrak{r}+(1-q^n)\mathfrak{y}_1) - \frac{1-q}{q}\mathfrak{F}(\mathfrak{r})\right\} =$$

$$= \left(q - \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right)\frac{\mathfrak{F}(\mathfrak{r})}{\mathfrak{r}-\mathfrak{y}_1} + \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{(\mathfrak{r}-\mathfrak{y}_1)^2}\mathfrak{F}(\mathfrak{y}_1) -$$

$$- \frac{q}{\mathfrak{r}-\mathfrak{y}_1}\left\{\frac{1}{q(\mathfrak{r}-\mathfrak{y}_1)}\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt - \frac{1-q}{q}\mathfrak{F}(\mathfrak{r})\right\} =$$

$$= \frac{2(\mathfrak{r}-\mathfrak{y}_1)-\lambda(\mathfrak{y}_2-\mathfrak{y}_1)}{2(\mathfrak{r}-\mathfrak{y}_1)^2}\mathfrak{F}(\mathfrak{r}) + \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{(\mathfrak{r}-\mathfrak{y}_1)^2}\mathfrak{F}(\mathfrak{y}_1) - \frac{1}{(\mathfrak{r}-\mathfrak{y}_1)^2}\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt. \tag{3.2}$$

By using Lemma 2.2, we obtain

$$I_2 = \int\limits_0^1 \left(qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_2 - \mathfrak{r}}\right) {}^{\mathfrak{y}_2}D_q \mathfrak{F}(t\mathfrak{r} + (1-t)\mathfrak{y}_2) d_q t =$$

$$= \frac{q}{\mathfrak{y}_2 - \mathfrak{r}} \int\limits_0^1 \mathfrak{F}(qt\mathfrak{r} + (1-qt)\mathfrak{y}_2) d_q t - \left(qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_2 - \mathfrak{r}}\right) \frac{\mathfrak{F}(t\mathfrak{r} + (1-t)\mathfrak{y}_2)}{\mathfrak{y}_2 - \mathfrak{r}}\Bigg|_0^1 =$$

$$= \frac{q}{\mathfrak{y}_2 - \mathfrak{r}} \left\{ \frac{1-q}{q} \sum_{n=0}^{\infty} q^n \mathfrak{F}(q^n \mathfrak{r} + (1-q^n)\mathfrak{y}_2) - \frac{1-q}{q}\mathfrak{F}(\mathfrak{r}) \right\} -$$

$$- q - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_2 - \mathfrak{r}} \frac{\mathfrak{F}(\mathfrak{r})}{\mathfrak{y}_2 - \mathfrak{r}} - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{(\mathfrak{y}_2 - \mathfrak{r})^2}\mathfrak{F}(\mathfrak{y}_2) =$$

$$= \frac{1}{(\mathfrak{y}_2 - \mathfrak{r})^2} \int\limits_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\, {}^{\mathfrak{y}_2}d_q t - \frac{1-q}{\mathfrak{y}_2 - \mathfrak{y}_1}\mathfrak{F}(\mathfrak{r}) - \left(q - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_2 - \mathfrak{r}}\right)\frac{\mathfrak{F}(\mathfrak{r})}{\mathfrak{y}_2 - \mathfrak{r}} - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{(\mathfrak{y}_2 - \mathfrak{r})^2}\mathfrak{F}(\mathfrak{y}_2) =$$

$$= \frac{1}{(\mathfrak{y}_2 - \mathfrak{r})^2} \int\limits_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\, {}^{\mathfrak{y}_2}d_q t - \left(1 - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_2 - \mathfrak{r}}\right)\frac{\mathfrak{F}(\mathfrak{r})}{\mathfrak{y}_2 - \mathfrak{r}} - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{(\mathfrak{y}_2 - \mathfrak{r})^2}\mathfrak{F}(\mathfrak{y}_2). \tag{3.3}$$

Thus, we obtain the required equality (3.1) by subtracting (3.3) from (3.2) after multiplying $\frac{(\mathfrak{r} - \mathfrak{y}_1)^2}{\mathfrak{y}_2 - \mathfrak{y}_1}$ and $\frac{(\mathfrak{y}_2 - \mathfrak{r})^2}{\mathfrak{y}_2\mathfrak{y}_1}$ with (3.2) and (3.3), respectively.

Lemma 3.1 is proved.

***Remark* 3.1.** In Lemma 3.1, we have:

(i) if we take the limit as $q \to 1^-$, then we obtain Lemma 1 of [22];

(ii) if we set $\lambda = 0$, then we obtain the equality

$$\mathfrak{F}(\mathfrak{r}) - \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\left[\int\limits_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)_{\mathfrak{y}_1} d_q t + \int\limits_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)^{\mathfrak{y}_2} d_q t\right] =$$

$$= \frac{q(\mathfrak{r} - \mathfrak{y}_1)^2}{\mathfrak{y}_2 - \mathfrak{y}_1} \int\limits_0^1 t_{\mathfrak{y}_1} D_q \mathfrak{F}(t\mathfrak{r} + (1-t)\mathfrak{y}_1) d_q t -$$

$$- \frac{q(\mathfrak{y}_2 - \mathfrak{r})^2}{\mathfrak{y}_2 - \mathfrak{y}_1} \int\limits_0^1 t^{\mathfrak{y}_2} D_q \mathfrak{F}(t\mathfrak{r} + (1-t)\mathfrak{y}_2) d_q t,$$

which is given by Budak et al. [15].

4. Ostrowski's inequalities. In this section, we prove some new generalizations of Ostrowski's, midpoint and trapezoidal type inequalities for differentiable convex functions and bounded differentiable bounded functions. For brevity, we start this section with some following notations which will be used in new results:

$$A_1(q;\mathfrak{r};\lambda)=\int\limits_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right|td_qt=$$

$$=\begin{cases}\dfrac{\lambda(\mathfrak{y}_2-\mathfrak{y}_1)}{2(\mathfrak{r}-\mathfrak{y}_1)[2]_q}-\dfrac{q}{[3]_q} & \text{for}\quad 0<q<\dfrac{\lambda}{2}\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1},\\ \dfrac{1}{[3]_q}-\dfrac{1}{2}\lambda\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{(\mathfrak{r}-\mathfrak{y}_1)[2]_q}+\dfrac{1}{4}\lambda^3\dfrac{(\mathfrak{y}_2-\mathfrak{y}_1)^3}{(\mathfrak{r}-\mathfrak{y}_1)^3([4]_q+q^2+q)} & \text{for}\quad \dfrac{\lambda}{2}\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\le q<1,\end{cases} \tag{4.1}$$

$$A_2(q;\mathfrak{r};\lambda)=\int\limits_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right|(1-t)d_qt=$$

$$=\begin{cases}\dfrac{1}{2}q\lambda\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{(\mathfrak{r}-\mathfrak{y}_1)[2]_q}-\dfrac{q^3}{[4]_q+q+q^2} & \text{for}\quad 0<q<\dfrac{\lambda}{2}\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1},\\ \dfrac{q^3+q^2-1}{[4]_q+q+q^2}-\dfrac{1}{4}\lambda^3\dfrac{(\mathfrak{y}_2-\mathfrak{y}_1)^3}{(\mathfrak{r}-\mathfrak{y}_1)^3([4]_q+q+q^2)}+ & \\ \qquad+\dfrac{1}{2}\lambda^2\dfrac{(\mathfrak{y}_2-\mathfrak{y}_1)^2}{(\mathfrak{r}-\mathfrak{y}_1)^2[2]_q}-\dfrac{1}{2}q\lambda\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{(\mathfrak{r}-\mathfrak{y}_1)[2]_q} & \text{for}\quad \dfrac{\lambda}{2}\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\le q<1,\end{cases} \tag{4.2}$$

$$A_3(q;\mathfrak{r};\lambda)=\int\limits_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\right|td_qt=$$

$$=\begin{cases}\dfrac{\lambda(\mathfrak{y}_2-\mathfrak{y}_1)}{2(\mathfrak{y}_2-\mathfrak{r})[2]_q}-\dfrac{q}{[3]_q} & \text{for}\quad 0<q<\dfrac{\lambda}{2}\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}},\\ \dfrac{1}{[3]_q}-\dfrac{1}{2}\lambda\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{(\mathfrak{y}_2-\mathfrak{r})[2]_q}+\dfrac{1}{4}\lambda^3\dfrac{(\mathfrak{y}_2-\mathfrak{y}_1)^3}{(\mathfrak{y}_2-\mathfrak{r})^3([4]_q+q^2+q)} & \text{for}\quad \dfrac{\lambda}{2}\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\le q<1,\end{cases} \tag{4.3}$$

$$A_4(q;\mathfrak{r};\lambda)=\int\limits_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\right|(1-t)d_qt=$$

$$=\begin{cases}\dfrac{1}{2}q\lambda\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{(\mathfrak{y}_2-\mathfrak{r})[2]_q}-\dfrac{q^3}{[4]_q+q+q^2} & \text{for}\quad 0<q<\dfrac{\lambda}{2}\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}},\\ \dfrac{q^3+q^2-1}{[4]_q+q+q^2}-\dfrac{1}{4}\lambda^3\dfrac{(\mathfrak{y}_2-\mathfrak{y}_1)^3}{(\mathfrak{y}_2-\mathfrak{r})^3([4]_q+q+q^2)}+ & \\ \qquad+\dfrac{1}{2}\lambda^2\dfrac{(\mathfrak{y}_2-\mathfrak{y}_1)^2}{(\mathfrak{y}_2-\mathfrak{r})^2[2]_q}-\dfrac{1}{2}q\lambda\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{(\mathfrak{y}_2-\mathfrak{r})[2]_q} & \text{for}\quad \dfrac{\lambda}{2}\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\le q<1,\end{cases} \tag{4.4}$$

$$A_5(q;\mathfrak{r};\lambda)=\int\limits_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right|d_qt=$$

$$= \begin{cases} \dfrac{\lambda(\mathfrak{y}_2-\mathfrak{y}_1)}{2(\mathfrak{r}-\mathfrak{y}_1)} - \dfrac{1}{[2]_q} & \text{for} \quad 0 < q < \dfrac{\lambda}{2}\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}, \\ \dfrac{q}{[2]_q} - \lambda\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{2(\mathfrak{r}-\mathfrak{y}_1)} + \dfrac{1}{2}\lambda^2\dfrac{(\mathfrak{y}_2-\mathfrak{y}_1)^2}{(\mathfrak{r}-\mathfrak{y}_1)^2[2]_q} & \text{for} \quad \dfrac{\lambda}{2}\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1} \leq q < 1, \end{cases} \tag{4.5}$$

$$A_6(q;\mathfrak{r};\lambda) = \int_0^1 \left| qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}} \right| d_qt =$$

$$= \begin{cases} \dfrac{\lambda(\mathfrak{y}_2-\mathfrak{y}_1)}{2(\mathfrak{y}_2-\mathfrak{r})} - \dfrac{1}{[2]_q} & \text{for} \quad 0 < q < \dfrac{\lambda}{2}\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}, \\ \dfrac{q}{[2]_q} - \lambda\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{2(\mathfrak{y}_2-\mathfrak{r})} + \dfrac{1}{2}\lambda^2\dfrac{(\mathfrak{y}_2-\mathfrak{y}_1)^2}{(\mathfrak{y}_2-\mathfrak{r})^2[2]_q} & \text{for} \quad \dfrac{\lambda}{2}\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}} \leq q < 1. \end{cases} \tag{4.6}$$

Theorem 4.1. *Under the assumption of Lemma* 3.1, *if* $|{}_{\mathfrak{y}_1}D_q\mathfrak{F}|$ *and* $|{}^{\mathfrak{y}_2}D_q\mathfrak{F}|$ *are convex mappings over* $[\mathfrak{y}_1,\mathfrak{y}_2]$. *Then, for* $\mathfrak{r} \in \left[\mathfrak{y}_1 + \lambda\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{2}, \mathfrak{y}_2 - \lambda\frac{\mathfrak{y}_2-\mathfrak{y}_1}{2}\right]$, *we have the following inequality*:

$$\left| (1-\lambda)\mathfrak{F}(\mathfrak{r}) + \lambda\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2} - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt + \int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right] \right| \leq$$

$$\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\big(A_1(q;\mathfrak{r};\lambda)|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})| + A_2(q;\mathfrak{r};\lambda)|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|\big) +$$

$$+ \frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\big(A_3(q;\mathfrak{r};\lambda)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})| + A_4(q;\mathfrak{r};\lambda)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|\big),$$

where $A_1(q;\mathfrak{r};\lambda) - A_4(q;\mathfrak{r};\lambda)$ *defined in* (4.1) – (4.4).

Proof. On taking modulus in Lemma 3.1, because of the properties of modulus, we find that

$$\left| (1-\lambda)\mathfrak{F}(\mathfrak{r}) + \lambda\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2} - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt + \int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right] \right| \leq$$

$$\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int_0^1 \left| qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right| |{}_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r}-(1-t)\mathfrak{y}_1)|d_qt +$$

$$+ \frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int_0^1 \left| qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\right| |{}^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{r}-(1-t)\mathfrak{y}_2)|d_qt \leq$$

$$\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int_0^1 \left| qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right| [t|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})| + (1-t)|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|]d_qt +$$

$$+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int\limits_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\right|[t|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|+(1-t)|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|]d_qt=$$

$$=\frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}(A_1(q;\mathfrak{r};\lambda)|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})|+A_2(q;\mathfrak{r};\lambda)|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|)+$$

$$+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}(A_3(q;\mathfrak{r};\lambda)|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|+A_4(q;\mathfrak{r};\lambda)|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|).$$

Theorem 4.1 is proved.

***Corollary* 4.1.** *In Theorem* 4.1, *we have*:

(i) *if we set* $|_{\mathfrak{y}_1}D_q\mathfrak{F}|,\ |^{\mathfrak{y}_2}D_q\mathfrak{F}|\leq M$, *then we obtain the following inequality*:

$$\left|(1-\lambda)\mathfrak{F}(\mathfrak{r})+\lambda\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2}-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int\limits_{\mathfrak{y}_1}^{\mathfrak{r}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int\limits_{\mathfrak{r}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\leq$$

$$\leq\frac{M}{\mathfrak{y}_2-\mathfrak{y}_1}\Big[(\mathfrak{r}-\mathfrak{y}_1)^2(A_1(q;\mathfrak{r};\lambda)+A_2(q;\mathfrak{r};\lambda))+(\mathfrak{y}_2-\mathfrak{r})^2(A_3(q;\mathfrak{r};\lambda)+A_4(q;\mathfrak{r};\lambda))\Big]; \qquad (4.7)$$

(ii) *if we set* $\mathfrak{r}=\dfrac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, *then we obtain the following new inequality*:

$$\left|(1-\lambda)\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)+\lambda\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2}-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int\limits_{\mathfrak{y}_1}^{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int\limits_{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\leq$$

$$\leq\frac{\mathfrak{y}_2-\mathfrak{y}_1}{4}\left[\left(A_1\left(q;\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2};\lambda\right)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|+A_2\left(q;\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2};\lambda\right)|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|\right)+\right.$$

$$\left.+\left(A_3\left(q;\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2};\lambda\right)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|+A_4\left(q;\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2};\lambda\right)|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|\right)\right].$$

***Remark* 4.1.** In Theorem 4.1, we have:

(i) if we set $\lambda=0$, then we obtain the following Ostrowski inequality:

$$\left|\mathfrak{F}(\mathfrak{r})-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int\limits_{\mathfrak{y}_1}^{\mathfrak{r}}\mathfrak{F}(t)_{\mathfrak{y}_1}d_qt+\int\limits_{\mathfrak{r}}^{\mathfrak{y}_2}\mathfrak{F}(t)^{\mathfrak{y}_2}d_qt\right]\right|\leq$$

$$\leq\frac{q}{(\mathfrak{y}_2-\mathfrak{y}_1)[2]_q[3]_q}\Big[(\mathfrak{r}-\mathfrak{y}_1)^2\big([2]_q|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})|+q^2|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|\big)+$$

$$+(\mathfrak{y}_2-\mathfrak{r})^2\big([2]_q|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|+q^2|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|\big)\Big],$$

which is proved by Budak et al. in [15];

(ii) if we set $\lambda = 1$, then we obtain the following new version of quantum trapezoidal type inequality for differentiable convex functions:

$$\left|\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2} - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int\limits_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt + \int\limits_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right| \leq$$

$$\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(A_1(q;\mathfrak{r};1)|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})| + A_2(q;\mathfrak{r};1)|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|\right) +$$

$$+ \frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(A_3(q;\mathfrak{r};1)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})| + A_4(q;\mathfrak{r};1)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|\right).$$

***Remark* 4.2.** In Corollary 4.1(i), we have:

(i) if we take the limit as $q \to 1^-$, then we obtain Theorem 6 for $s = q = 1$ of [22];

(ii) if we set $\mathfrak{r} = \dfrac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$ and take the limit as $q \to 1^-$, then we obtain Corollary 2 for $s = q = 1$ of [22];

(iii) if we set $\lambda = 1$ and $q \to 1^-$, then we obtain inequality (5) in [22] (Remark 2 for $q = 1$);

(iv) if we set $\lambda = 0$, $\mathfrak{r} = \dfrac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$ and $q \to 1^-$, then we obtain inequality (6) in [22] (Remark 2 for $q = 1$).

Theorem 4.2. *Under the assumption of Lemma* 3.1, *if* $|{}_{\mathfrak{y}_1}D_q\mathfrak{F}|^s$ *and* $|{}^{\mathfrak{y}_2}D_q\mathfrak{F}|^s$, *where* $s \geq 1$ *are convex mappings over* $[\mathfrak{y}_1, \mathfrak{y}_2]$. *Then, for* $\mathfrak{r} \in \left[\mathfrak{y}_1 + \lambda\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{2}, \mathfrak{y}_2 - \lambda\dfrac{\mathfrak{y}_2-\mathfrak{y}_1}{2}\right]$, *we have the following inequality*:

$$\left|(1-\lambda)\mathfrak{F}(\mathfrak{r}) + \lambda\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2} - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int\limits_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(\mathfrak{r})\ {}_{\mathfrak{y}_1}d_q\mathfrak{r} + \int\limits_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})\ {}^{\mathfrak{y}_2}d_q\mathfrak{r}\right]\right| \leq$$

$$\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}A_5^{1-\frac{1}{s}}(q;\mathfrak{r};\lambda)\left[\left(A_1(q;\mathfrak{r};\lambda)|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})|^s + A_2(q;\mathfrak{r};\lambda)|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|^s\right)^{\frac{1}{s}}\right] +$$

$$+ \frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}A_6^{1-\frac{1}{s}}(q;\mathfrak{r};\lambda)\left[\left(A_3(q;\mathfrak{r};\lambda)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|^s + A_4(q;\mathfrak{r};\lambda)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^s\right)^{\frac{1}{s}}\right],$$

where $A_i(q;\mathfrak{r};\lambda)$, $i = 1,2,3,4,5,6$, *are defined in* (4.1) – (4.6).

Proof. By taking modulus in (3.1) and using power mean inequality, we have

$$\left|(1-\lambda)\mathfrak{F}(\mathfrak{r}) + \lambda\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2} - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int\limits_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(\mathfrak{r})\ {}_{\mathfrak{y}_1}d_q\mathfrak{r} + \int\limits_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})\ {}^{\mathfrak{y}_2}d_q\mathfrak{r}\right]\right| \leq$$

$$\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int\limits_0^1\left|qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right||{}_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r}-(1-t)\mathfrak{y}_1)|d_qt +$$

$$+ \frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\int\limits_0^1\left|qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\right||{}^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{r}-(1-t)\mathfrak{y}_2)|d_qt \leq$$

$$\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\int_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right|d_qt\right)^{1-\frac{1}{s}}\times$$

$$\times\left(\int_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right||_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r}-(1-t)\mathfrak{y}_1)|^s d_qt\right)^{\frac{1}{s}}+$$

$$+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\int_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\right|d_qt\right)^{1-\frac{1}{s}}\left(\int_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\right||^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{r}-(1-t)\mathfrak{y}_2)|^s d_qt\right)^{\frac{1}{s}}.$$

By using the convexity, we obtain

$$\left|(1-\lambda)\mathfrak{F}(\mathfrak{r})+\lambda\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2}-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}}\mathfrak{F}(\mathfrak{r})\ {}_{\mathfrak{y}_1}d_q\mathfrak{r}+\int_{\mathfrak{r}}^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\ {}^{\mathfrak{y}_2}d_q\mathfrak{r}\right]\right|\leq$$

$$\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\int_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right|d_qt\right)^{1-\frac{1}{s}}\times$$

$$\times\left(\int_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right|[t|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})|^s+(1-t)|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|^s]d_qt\right)^{\frac{1}{s}}+$$

$$+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\int_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\right|d_qt\right)^{1-\frac{1}{s}}\times$$

$$\times\left(\int_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\right|[t|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|^s+(1-t)|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^s]d_qt\right)^{\frac{1}{s}}=$$

$$=\frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}A_5^{1-\frac{1}{s}}(q;\mathfrak{r};\lambda)\Big[\big(A_1(q;\mathfrak{r};\lambda)|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})|^s+A_2(q;\mathfrak{r};\lambda)|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|^s\big)^{\frac{1}{s}}\Big]+$$

$$+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}A_6^{1-\frac{1}{s}}(q;\mathfrak{r};\lambda)\Big[\big(A_3(q;\mathfrak{r};\lambda)|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|^s+A_4(q;\mathfrak{r};\lambda)|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^s\big)^{\frac{1}{s}}\Big].$$

Theorem 4.2 is proved.

***Corollary* 4.2.** *In Theorem* 4.2, *we have*:

(i) *if we set* $|_{\mathfrak{y}_1}D_q\mathfrak{F}|, |^{\mathfrak{y}_2}D_q\mathfrak{F}|\leq M$, *then we obtain the following inequality*:

$$\left|(1-\lambda)\mathfrak{F}(\mathfrak{r})+\lambda\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2}-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\mathfrak{r}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\leq$$

$$\leq \frac{M(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1} A_5^{1-\frac{1}{s}}(q;\mathfrak{r};\lambda)\Big[(A_1(q;\mathfrak{r};\lambda)+A_2(q;\mathfrak{r};\lambda))^{\frac{1}{s}}\Big] +$$

$$+ \frac{M(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1} A_6^{1-\frac{1}{s}}(q;\mathfrak{r};\lambda)\Big[(A_3(q;\mathfrak{r};\lambda)+A_4(q;\mathfrak{r};\lambda))^{\frac{1}{s}}\Big]; \tag{4.8}$$

(ii) *if we set* $\mathfrak{r} = \dfrac{\mathfrak{y}_1+\mathfrak{y}_2}{2}$, *then we obtain the following new inequality*:

$$\left|(1-\lambda)\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)+\lambda\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2} - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt + \int_{\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right| \leq$$

$$\leq \frac{\mathfrak{y}_2-\mathfrak{y}_1}{4}\left[A_5^{1-\frac{1}{s}}\left(q;\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2};\lambda\right)\times\right.$$

$$\times\left\{\left(A_1\left(q;\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2};\lambda\right)\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|^s + A_2\left(q;\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2};\lambda\right)|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|^s\right)^{\frac{1}{s}}\right\} +$$

$$+ A_6^{1-\frac{1}{s}}\left(q;\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2};\lambda\right)\times$$

$$\left.\times\left\{\left(A_3\left(q;\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2};\lambda\right)\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}\left(\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2}\right)\right|^s + A_4\left(q;\frac{\mathfrak{y}_1+\mathfrak{y}_2}{2};\lambda\right)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^s\right)^{\frac{1}{s}}\right\}\right].$$

***Remark* 4.3.** In Theorem 4.2, we have:
(i) if we set $\lambda = 0$, then we obtain the following Ostrowski inequality:

$$\left|\mathfrak{F}(\mathfrak{r}) - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)_{\mathfrak{y}_1}d_qt + \int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)^{\mathfrak{y}_2}d_qt\right]\right| \leq$$

$$\leq \frac{q}{(\mathfrak{y}_2-\mathfrak{y}_1)[2]_q[3]_q}\Big[(\mathfrak{r}-\mathfrak{y}_1)^2\big([2]_q|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})| + q^2|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|\big) +$$

$$+ (\mathfrak{y}_2-\mathfrak{r})^2\big([2]_q|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})| + q^2|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|\big)\Big],$$

which is proved by Budak et al. in [15];

(ii) if we set $\lambda = 1$, then we obtain the following new version of quantum trapezoidal type inequality for differentiable convex functions:

$$\left|\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2} - \frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt + \int_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right| \leq$$

$$\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}(A_1(q;\mathfrak{r};1)|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})| + A_2(q;\mathfrak{r};1)|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|) +$$

$$+ \frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}(A_3(q;\mathfrak{r};1)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})| + A_4(q;\mathfrak{r};1)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|).$$

***Remark* 4.4.** In Corollary 4.2 (i), we have:

(i) if we take the limit as $q \to 1^-$, then we obtain Theorem 6 for $s = 1$ of [22];

(ii) if we set $\mathfrak{r} = \dfrac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$ and take the limit as $q \to 1^-$, then we obtain Corollary 2 for $s = 1$ of [22];

(iii) if we set $\lambda = 1$ and $q \to 1^-$, then we obtain inequality (5) in [22] (Remark 2);

(iv) if we set $\lambda = 0$, $\mathfrak{r} = \dfrac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$ and $q \to 1^-$, then we obtain inequality (6) in [22] (Remark 2).

Theorem 4.3. *Under the assumption of Lemma* 3.1, *if* $s > 1$ *is a real number and* $|_{\mathfrak{y}_1}D_q\mathfrak{F}|^s$ *and* $|^{\mathfrak{y}_2}D_q\mathfrak{F}|^s$ *are convex mappings over* $[\mathfrak{y}_1, \mathfrak{y}_2]$. *Then, for* $\mathfrak{r} \in \left[\mathfrak{y}_1 + \lambda\dfrac{\mathfrak{y}_2 - \mathfrak{y}_1}{2}, \mathfrak{y}_2 - \lambda\dfrac{\mathfrak{y}_2 - \mathfrak{y}_1}{2}\right]$, *we have following inequality*:

$$\left|(1-\lambda)\mathfrak{F}(\mathfrak{r}) + \lambda\frac{\mathfrak{F}(\mathfrak{y}_1) + \mathfrak{F}(\mathfrak{y}_2)}{2} - \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\left[\int\limits_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(\mathfrak{r})\ {}_{\mathfrak{y}_1}d_q\mathfrak{r} + \int\limits_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})\ {}^{\mathfrak{y}_2}d_q\mathfrak{r}\right]\right| \leq$$

$$\leq \frac{(\mathfrak{r} - \mathfrak{y}_1)^2}{\mathfrak{y}_2 - \mathfrak{y}_1} A_7^{\frac{1}{r}}(q;\mathfrak{r};\lambda;r)\left(\frac{|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})|^s + q|_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|^s}{[2]_q}\right)^{\frac{1}{s}} +$$

$$+ \frac{(\mathfrak{y}_2 - \mathfrak{r})^2}{\mathfrak{y}_2 - \mathfrak{y}_1} A_8^{\frac{1}{r}}(q;\mathfrak{r};\lambda;r)\left(\frac{|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|^s + q|^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^s}{[2]_q}\right)^{\frac{1}{s}},$$

where $\dfrac{1}{s} + \dfrac{1}{r} = 1$ *and*

$$A_7(q;\mathfrak{r};\lambda;r) = \int\limits_0^1 \left|qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{r} - \mathfrak{y}_1}\right|^r = (1-q)\sum_{n=0}^{\infty} q^n \left|q^{n+1} - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{r} - \mathfrak{y}_1}\right|^r$$

and

$$A_8(q;\mathfrak{r};\lambda;r) = \int\limits_0^1 \left|qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_2 - \mathfrak{r}}\right|^r = (1-q)\sum_{n=0}^{\infty} q^n \left|q^{n+1} - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_2 - \mathfrak{r}}\right|^r.$$

Proof. By taking modulus in (3.1) and applying Hölder inequality, we have

$$\left|(1-\lambda)\mathfrak{F}(\mathfrak{r}) + \lambda\frac{\mathfrak{F}(\mathfrak{y}_1) + \mathfrak{F}(\mathfrak{y}_2)}{2} - \frac{1}{\mathfrak{y}_2 - \mathfrak{y}_1}\left[\int\limits_{\mathfrak{y}_1}^{\mathfrak{r}} \mathfrak{F}(\mathfrak{r})\ {}_{\mathfrak{y}_1}d_q\mathfrak{r} + \int\limits_{\mathfrak{r}}^{\mathfrak{y}_2} \mathfrak{F}(\mathfrak{r})\ {}^{\mathfrak{y}_2}d_q\mathfrak{r}\right]\right| \leq$$

$$\leq \frac{(\mathfrak{r} - \mathfrak{y}_1)^2}{\mathfrak{y}_2 - \mathfrak{y}_1}\int\limits_0^1 \left|qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{r} - \mathfrak{y}_1}\right| |_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r} - (1-t)\mathfrak{y}_1)| d_qt +$$

$$+ \frac{(\mathfrak{y}_2 - \mathfrak{r})^2}{\mathfrak{y}_2 - \mathfrak{y}_1}\int\limits_0^1 \left|qt - \frac{\lambda}{2}\frac{\mathfrak{y}_2 - \mathfrak{y}_1}{\mathfrak{y}_2 - \mathfrak{r}}\right| |^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{r} - (1-t)\mathfrak{y}_2)| d_qt \leq$$

$$\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\int_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right|^r d_qt\right)^{\frac{1}{r}}\left(\int_0^1\left|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(t\mathfrak{r}-(1-t)\mathfrak{y}_1)\right|^s d_qt\right)^{\frac{1}{s}}+$$

$$+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\int_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\right|^r d_qt\right)^{\frac{1}{r}}\left(\int_0^1\left|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(t\mathfrak{r}-(1-t)\mathfrak{y}_2)\right|^s d_qt\right)^{\frac{1}{s}}.$$

By using the convexity, we obtain

$$\left|(1-\lambda)\mathfrak{F}(\mathfrak{r})+\lambda\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2}-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}}\mathfrak{F}(\mathfrak{r})\ {}_{\mathfrak{y}_1}d_q\mathfrak{r}+\int_{\mathfrak{r}}^{\mathfrak{y}_2}\mathfrak{F}(\mathfrak{r})\ {}^{\mathfrak{y}_2}d_q\mathfrak{r}\right]\right|\leq$$

$$\leq \frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\int_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{r}-\mathfrak{y}_1}\right|^r d_qt\right)^{\frac{1}{r}}\times$$

$$\times\left(\int_0^1[t|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})|^s+(1-t)|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|^s]d_qt\right)^{\frac{1}{s}}+$$

$$+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}\left(\int_0^1\left|qt-\frac{\lambda}{2}\frac{\mathfrak{y}_2-\mathfrak{y}_1}{\mathfrak{y}_2-\mathfrak{r}}\right|^r d_qt\right)^{\frac{1}{r}}\times$$

$$\times\left(\int_0^1[t|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|^s+(1-t)|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^s]d_qt\right)^{\frac{1}{s}}=$$

$$=\frac{(\mathfrak{r}-\mathfrak{y}_1)^2}{\mathfrak{y}_2-\mathfrak{y}_1}A_7^{\frac{1}{r}}(q;\mathfrak{r};\lambda;r)\left(\frac{|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{r})|^s+q|{}_{\mathfrak{y}_1}D_q\mathfrak{F}(\mathfrak{y}_1)|^s}{[2]_q}\right)^{\frac{1}{s}}+$$

$$+\frac{(\mathfrak{y}_2-\mathfrak{r})^2}{\mathfrak{y}_2-\mathfrak{y}_1}A_8^{\frac{1}{r}}(q;\mathfrak{r};\lambda;r)\left(\frac{|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{r})|^s+q|{}^{\mathfrak{y}_2}D_q\mathfrak{F}(\mathfrak{y}_2)|^s}{[2]_q}\right)^{\frac{1}{s}}.$$

Theorem 4.3 is proved.

***Corollary* 4.3.** *In Theorem* 4.3, *we have:*

(i) *if we set* $|{}_{\mathfrak{y}_1}D_q\mathfrak{F}|, |{}^{\mathfrak{y}_2}D_q\mathfrak{F}|\leq M$, *then we obtain the following inequality*:

$$\left|(1-\lambda)\mathfrak{F}(\mathfrak{r})+\lambda\frac{\mathfrak{F}(\mathfrak{y}_1)+\mathfrak{F}(\mathfrak{y}_2)}{2}-\frac{1}{\mathfrak{y}_2-\mathfrak{y}_1}\left[\int_{\mathfrak{y}_1}^{\mathfrak{r}}\mathfrak{F}(t)\ {}_{\mathfrak{y}_1}d_qt+\int_{\mathfrak{r}}^{\mathfrak{y}_2}\mathfrak{F}(t)\ {}^{\mathfrak{y}_2}d_qt\right]\right|\leq$$

$$\leq\frac{M}{\mathfrak{y}_2-\mathfrak{y}_1}\left[(\mathfrak{r}-\mathfrak{y}_1)^2A_7^{\frac{1}{r}}(q;\mathfrak{r};\lambda;r)+(\mathfrak{y}_2-\mathfrak{r})^2A_8^{\frac{1}{r}}(q;\mathfrak{r};\lambda;r)\right];\qquad(4.9)$$

(ii) *if we set* $\mathfrak{r} = \dfrac{\eta_1+\eta_2}{2}$, *then we obtain the following new inequality*:

$$\left|(1-\lambda)\mathfrak{F}\left(\frac{\eta_1+\eta_2}{2}\right)+\lambda\frac{\mathfrak{F}(\eta_1)+\mathfrak{F}(\eta_2)}{2}-\frac{1}{\eta_2-\eta_1}\left[\int\limits_{\eta_1}^{\frac{\eta_1+\eta_2}{2}}\mathfrak{F}(t)\,{}_{\eta_1}d_qt+\int\limits_{\frac{\eta_1+\eta_2}{2}}^{\eta_2}\mathfrak{F}(t)\,{}^{\eta_2}d_qt\right]\right|\leq$$

$$\leq\frac{\eta_2-\eta_1}{4}\left[A_7^{\frac{1}{r}}\left(q;\frac{\eta_1+\eta_2}{2};\lambda;r\right)\left(\frac{\left|{}_{\eta_1}D_q\mathfrak{F}\left(\frac{\eta_1+\eta_2}{2}\right)\right|^s+q|{}_{\eta_1}D_q\mathfrak{F}(\eta_1)|^s}{[2]_q}\right)^{\frac{1}{s}}+\right.$$

$$\left.+A_8^{\frac{1}{r}}\left(q;\frac{\eta_1+\eta_2}{2};\lambda;r\right)\left(\frac{\left|{}^{\eta_2}D_q\mathfrak{F}\left(\frac{\eta_1+\eta_2}{2}\right)\right|^s+q|{}^{\eta_2}D_q\mathfrak{F}(\eta_2)|^s}{[2]_q}\right)^{\frac{1}{s}}\right].$$

***Remark* 4.5.** In Theorem 4.3, we have:
(i) if we set $\lambda=0$, then we obtain the following Ostrowski inequality:

$$\left|\mathfrak{F}(\mathfrak{r})-\frac{1}{\eta_2-\eta_1}\left[\int\limits_{\eta_1}^{\mathfrak{r}}\mathfrak{F}(t)_{\eta_1}d_qt+\int\limits_{\mathfrak{r}}^{\eta_2}\mathfrak{F}(t)^{\eta_2}d_qt\right]\right|\leq$$

$$\leq\frac{(\mathfrak{r}-\eta_1)^2}{\eta_2-\eta_1}A_7^{\frac{1}{r}}(q;\mathfrak{r};0;r)\left(\frac{|{}_{\eta_1}D_q\mathfrak{F}(\mathfrak{r})|^s+q|{}_{\eta_1}D_q\mathfrak{F}(\eta_1)|^s}{[2]_q}\right)^{\frac{1}{s}}+$$

$$+\frac{(\eta_2-\mathfrak{r})^2}{\eta_2-\eta_1}A_8^{\frac{1}{r}}(q;\mathfrak{r};0;r)\left(\frac{|{}^{\eta_2}D_q\mathfrak{F}(\mathfrak{r})|^s+q|{}^{\eta_2}D_q\mathfrak{F}(\eta_2)|^s}{[2]_q}\right)^{\frac{1}{s}};$$

(ii) if we set $\lambda=1$, then we obtain the following new version of quantum trapezoidal type inequality for differentiable convex functions:

$$\left|\frac{\mathfrak{F}(\eta_1)+\mathfrak{F}(\eta_2)}{2}-\frac{1}{\eta_2-\eta_1}\left[\int\limits_{\eta_1}^{\mathfrak{r}}\mathfrak{F}(t)\,{}_{\eta_1}d_qt+\int\limits_{\mathfrak{r}}^{\eta_2}\mathfrak{F}(t)\,{}^{\eta_2}d_qt\right]\right|\leq$$

$$\leq\frac{(\mathfrak{r}-\eta_1)^2}{\eta_2-\eta_1}A_7^{\frac{1}{r}}(q;\mathfrak{r};1;r)\left(\frac{|{}_{\eta_1}D_q\mathfrak{F}(\mathfrak{r})|^s+q|{}_{\eta_1}D_q\mathfrak{F}(\eta_1)|^s}{[2]_q}\right)^{\frac{1}{s}}+$$

$$+\frac{(\eta_2-\mathfrak{r})^2}{\eta_2-\eta_1}A_8^{\frac{1}{r}}(q;\mathfrak{r};1;r)\left(\frac{|{}^{\eta_2}D_q\mathfrak{F}(\mathfrak{r})|^s+q|{}^{\eta_2}D_q\mathfrak{F}(\eta_2)|^s}{[2]_q}\right)^{\frac{1}{s}}.$$

***Remark* 4.6.** In Corollary 4.3 (i), we have:
(i) if we take the limit as $q\to1^-$, then we obtain Theorem 5 for $s=1$ of [22];
(ii) if we set $\mathfrak{r}=\dfrac{\eta_1+\eta_2}{2}$ and take the limit as $q\to1^-$, then we obtain Corollary 1 for $s=1$ of [22];

(iii) if we set $\lambda = 1$ and $q \to 1^-$, then we obtain inequality (2) in [22] (Remark 2);

(iv) if we set $\lambda = 0$, $\mathfrak{r} = \dfrac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$ and $q \to 1^-$, then we obtain inequality (3) in [22] (Remark 2).

5. Applications to special means of real numbers. For any positive number $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$, we consider the following means:

(i) the arithmetic mean:

$$\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2) = \frac{\mathfrak{y}_1 + \mathfrak{y}_2}{2};$$

(ii) the harmonic mean

$$\mathcal{H}(\mathfrak{y}_1, \mathfrak{y}_2) = \frac{2\mathfrak{y}_1\mathfrak{y}_2}{\mathfrak{y}_1 + \mathfrak{y}_2};$$

(iii) the geometric mean

$$\mathcal{G}(\mathfrak{y}_1, \mathfrak{y}_2) = \sqrt{\mathfrak{y}_1\mathfrak{y}_2}.$$

Proposition 5.1. *For* $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ *with* $\mathfrak{y}_1 < \mathfrak{y}_2$, *the following inequality holds*:

$$\left| \mathcal{A}^k(\mathfrak{y}_1, \mathfrak{y}_2) - \frac{2}{\mathfrak{y}_2 - \mathfrak{y}_1} \mathcal{A}(\Theta_1, \Theta_2) \right| \leq$$

$$\leq \frac{M(\mathfrak{y}_2 - \mathfrak{y}_1)}{2} \left[\begin{array}{r} \mathcal{A}(A_1(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0), A_2(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0)) + \\ + \mathcal{A}(A_3(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0), A_4(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0)) \end{array} \right],$$

where

$$\Theta_1 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1) \sum_{n=0}^{\infty} q^n (q^n (\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_1)^k$$

and

$$\Theta_2 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1) \sum_{n=0}^{\infty} q^n (q^n (\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_2)^k.$$

Proof. The inequality (4.7) for function $\mathfrak{F}(t) = t^k$, $\lambda = 0$ and $\mathfrak{r} = \dfrac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$ leads to the required result.

Proposition 5.2. *For* $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ *with* $\mathfrak{y}_1 < \mathfrak{y}_2$, *the following inequality holds*:

$$\left| \mathcal{A}\left(\mathfrak{y}_1^k, \mathfrak{y}_2^k\right) - \frac{2}{\mathfrak{y}_2 - \mathfrak{y}_1} \mathcal{A}(\Theta_1, \Theta_2) \right| \leq$$

$$\leq \frac{M(\mathfrak{y}_2 - \mathfrak{y}_1)}{2} \left[\begin{array}{r} \mathcal{A}(A_1(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1), A_2(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1)) + \\ + \mathcal{A}(A_3(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1), A_4(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1)) \end{array} \right],$$

where

$$\Theta_1 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1) \sum_{n=0}^{\infty} q^n (q^n (\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_1)^k$$

and

$$\Theta_2 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1) \sum_{n=0}^{\infty} q^n (q^n (\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_2)^k.$$

Proof. The inequality (4.7) for function $\mathfrak{F}(t) = t^k$, $\lambda = 1$ and $\mathfrak{r} = \dfrac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$ leads to the required result.

Proposition 5.3. *For* $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ *with* $\mathfrak{y}_1 < \mathfrak{y}_2$, *the following inequality holds*:

$$\left| \frac{\mathcal{H}(\mathfrak{y}_1, \mathfrak{y}_2)}{\mathcal{G}(\mathfrak{y}_1, \mathfrak{y}_2)} - \frac{2}{\mathfrak{y}_2 - \mathfrak{y}_1} \mathcal{A}(\Theta_3, \Theta_4) \right| \leq$$

$$\leq \frac{M(\mathfrak{y}_2 - \mathfrak{y}_1)}{2} \left[\begin{array}{c} \mathcal{A}(A_1(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0), A_2(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0)) + \\ + \mathcal{A}(A_3(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0), A_4(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0)) \end{array} \right],$$

where

$$\Theta_3 = (1 - q)(\mathfrak{y}_2 - \mathfrak{y}_1) \sum_{n=0}^{\infty} \frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1 - q^n)\mathfrak{y}_1)}$$

and

$$\Theta_4 = (1 - q)(\mathfrak{y}_2 - \mathfrak{y}_1) \sum_{n=0}^{\infty} \frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1 - q^n)\mathfrak{y}_2)}.$$

Proof. The inequality (4.7) for function $\mathfrak{F}(t) = \dfrac{1}{t}$, $\lambda = 0$ and $\mathfrak{r} = \dfrac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$ leads to the required result.

Proposition 5.4. *For* $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ *with* $\mathfrak{y}_1 < \mathfrak{y}_2$, *the following inequality holds*:

$$\left| \mathcal{H}^{-1}(\mathfrak{y}_1, \mathfrak{y}_2) - \frac{2}{\mathfrak{y}_2 - \mathfrak{y}_1} \mathcal{A}(\Theta_3, \Theta_4) \right| \leq$$

$$\leq \frac{M(\mathfrak{y}_2 - \mathfrak{y}_1)}{2} \left[\begin{array}{c} \mathcal{A}(A_1(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1), A_2(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1)) + \\ + \mathcal{A}(A_3(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1), A_4(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1)) \end{array} \right],$$

where

$$\Theta_3 = (1 - q)(\mathfrak{y}_2 - \mathfrak{y}_1) \sum_{n=0}^{\infty} \frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1 - q^n)\mathfrak{y}_1)}$$

and

$$\Theta_4 = (1 - q)(\mathfrak{y}_2 - \mathfrak{y}_1) \sum_{n=0}^{\infty} \frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1 - q^n)\mathfrak{y}_2)}.$$

Proof. The inequality (4.7) for function $\mathfrak{F}(t) = \dfrac{1}{t}$, $\lambda = 1$ and $\mathfrak{r} = \dfrac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$ leads to the required result.

Proposition 5.5. *For* $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ *with* $\mathfrak{y}_1 < \mathfrak{y}_2$, *the following inequality holds*:

$$\left| \mathcal{A}^k(\mathfrak{y}_1, \mathfrak{y}_2) - \frac{2}{\mathfrak{y}_2 - \mathfrak{y}_1} \mathcal{A}(\Theta_1, \Theta_2) \right| \leq$$

$$\leq 2^{\frac{1-2s}{s}} M(\mathfrak{y}_2 - \mathfrak{y}_1) \left[A_5^{1-\frac{1}{s}}(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0) \left\{ (\mathcal{A}(A_1(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0), A_2(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0)))^{\frac{1}{s}} \right\} + \right.$$

$$\left. + A_6^{1-\frac{1}{s}}(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0) \left\{ (\mathcal{A}(A_3(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0), A_4(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0)))^{\frac{1}{s}} \right\} \right],$$

where

$$\Theta_1 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} q^n (q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_1)^k$$

and

$$\Theta_2 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} q^n (q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_2)^k.$$

Proof. The inequality (4.8) for function $\mathfrak{F}(t) = t^k,\ \lambda = 0$ and $\mathfrak{r} = \dfrac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$ leads to the required result.

Proposition 5.6. *For* $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ *with* $\mathfrak{y}_1 < \mathfrak{y}_2$, *the following inequality holds*:

$$\left|\mathcal{A}\Big(\mathfrak{y}_1^k, \mathfrak{y}_2^k\Big) - \frac{2}{\mathfrak{y}_2 - \mathfrak{y}_1}\mathcal{A}(\Theta_1, \Theta_2)\right| \le$$

$$\le 2^{\frac{1-2s}{s}} M(\mathfrak{y}_2 - \mathfrak{y}_1)\left[A_5^{1-\frac{1}{s}}(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1)\Big\{(\mathcal{A}(A_1(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1), A_2(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1)))^{\frac{1}{s}}\Big\} + \right.$$

$$\left. + A_6^{1-\frac{1}{s}}(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1)\Big\{(\mathcal{A}(A_3(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1), A_4(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1)))^{\frac{1}{s}}\Big\}\right],$$

where

$$\Theta_1 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} q^n (q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_1)^k$$

and

$$\Theta_2 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} q^n (q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_2)^k.$$

Proof. The inequality (4.8) for function $\mathfrak{F}(t) = t^k,\ \lambda = 1$ and $\mathfrak{r} = \dfrac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$ leads to the required result.

Proposition 5.7. *For* $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ *with* $\mathfrak{y}_1 < \mathfrak{y}_2$, *the following inequality holds*:

$$\left|\frac{\mathcal{H}(\mathfrak{y}_1, \mathfrak{y}_2)}{\mathcal{G}(\mathfrak{y}_1, \mathfrak{y}_2)} - \frac{2}{\mathfrak{y}_2 - \mathfrak{y}_1}\mathcal{A}(\Theta_3, \Theta_4)\right| \le$$

$$\le 2^{\frac{1-2s}{s}} M(\mathfrak{y}_2 - \mathfrak{y}_1)\left[A_5^{1-\frac{1}{s}}(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0)\Big\{(\mathcal{A}(A_1(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0), A_2(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0)))^{\frac{1}{s}}\Big\} + \right.$$

$$\left. + A_6^{1-\frac{1}{s}}(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0)\Big\{(\mathcal{A}(A_3(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0), A_4(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0)))^{\frac{1}{s}}\Big\}\right],$$

where

$$\Theta_3 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} \frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_1)}$$

and

$$\Theta_4 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} \frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_2)}.$$

Proof. The inequality (4.8) for function $\mathfrak{F}(t) = \dfrac{1}{t}$, $\lambda = 0$ and $\mathfrak{r} = \dfrac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$ leads to the required result.

Proposition 5.8. *For* $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ *with* $\mathfrak{y}_1 < \mathfrak{y}_2$, *the following inequality holds*:

$$\left|\mathcal{H}^{-1}(\mathfrak{y}_1, \mathfrak{y}_2) - \frac{2}{\mathfrak{y}_2 - \mathfrak{y}_1}\mathcal{A}(\Theta_3, \Theta_4)\right| \leq$$

$$\leq 2^{\frac{1-2s}{s}} M(\mathfrak{y}_2 - \mathfrak{y}_1)\left[A_5^{1-\frac{1}{s}}(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1)\left\{(\mathcal{A}(A_1(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1), A_2(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1)))^{\frac{1}{s}}\right\} +\right.$$

$$\left. + A_6^{1-\frac{1}{s}}(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1)\left\{(\mathcal{A}(A_3(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1), A_4(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1)))^{\frac{1}{s}}\right\}\right],$$

where

$$\Theta_3 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} \frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_1)}$$

and

$$\Theta_4 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} \frac{q^n}{(q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_2)}.$$

Proof. The inequality (4.8) for function $\mathfrak{F}(t) = \dfrac{1}{t}$, $\lambda = 1$ and $\mathfrak{r} = \dfrac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$ leads to the required result.

Proposition 5.9. *For* $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ *with* $\mathfrak{y}_1 < \mathfrak{y}_2$, *the following inequality holds*:

$$\left|\mathcal{A}^k(\mathfrak{y}_1, \mathfrak{y}_2) - \frac{2}{\mathfrak{y}_2 - \mathfrak{y}_1}\mathcal{A}(\Theta_1, \Theta_2)\right| \leq$$

$$\leq \frac{M(\mathfrak{y}_2 - \mathfrak{y}_1)}{2}\left[\mathcal{A}\left(A_7^{\frac{1}{r}}(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0; r), A_8^{\frac{1}{r}}(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 0; r)\right)\right],$$

where

$$\Theta_1 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} q^n(q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_1)^k$$

and

$$\Theta_2 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1)\sum_{n=0}^{\infty} q^n(q^n(\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_2)^k.$$

Proof. The inequality (4.9) for function $\mathfrak{F}(t) = t^k$, $\lambda = 0$ and $\mathfrak{r} = \dfrac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$ leads to the required result.

Proposition 5.10. *For* $\mathfrak{y}_1, \mathfrak{y}_2 \in \mathbb{R}$ *with* $\mathfrak{y}_1 < \mathfrak{y}_2$, *the following inequality holds*:

$$\left|\mathcal{A}\left(\mathfrak{y}_1^k, \mathfrak{y}_2^k\right) - \frac{2}{\mathfrak{y}_2 - \mathfrak{y}_1}\mathcal{A}(\Theta_1, \Theta_2)\right| \leq$$

$$\leq \frac{M(\mathfrak{y}_2 - \mathfrak{y}_1)}{2}\left[\mathcal{A}\left(A_7^{\frac{1}{r}}(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1; r), A_8^{\frac{1}{r}}(q; \mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2); 1; r)\right)\right],$$

where

$$\Theta_1 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1) \sum_{n=0}^{\infty} q^n (q^n (\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_1)^k$$

and

$$\Theta_2 = (1-q)(\mathfrak{y}_2 - \mathfrak{y}_1) \sum_{n=0}^{\infty} q^n (q^n (\mathcal{A}(\mathfrak{y}_1, \mathfrak{y}_2)) + (1-q^n)\mathfrak{y}_2)^k.$$

Proof. The inequality (4.9) for function $\mathfrak{F}(t) = t^k$, $\lambda = 1$ and $\mathfrak{r} = \dfrac{\mathfrak{y}_1 + \mathfrak{y}_2}{2}$ leads to the required result.

6. Conclusions. In this paper, we demonstrate multiple forms of quantum integral inequality for quantum differentiable convex functions using a new parameterized quantum integral identity, including left and right quantum derivatives. We also showed that the newly established inequalities could be transformed into quantum Ostrowski's-type inequalities for convex functions [15], classical Ostrowski's-type inequalities for convex functions [16], several classical integral inequalities for convex functions [22], and several new quantum integral inequalities such as midpoint type, trapezoidal type, Ostrowski's-type, and Simpson's type without having to prove each one separately. To illustrate the findings, some applications to special means of real numbers were given. The researcher can find similar inequalities for various types of convexity and coordinated convexity in their future studies, which is a new and exciting idea.

References

1. M. A. Ali, H. Budak, M. Abbas, Y.-M. Chu, *Quantum Hermite – Hadamard-type inequalities for functions with convex absolute values of second $q^{\mathfrak{y}_2}$-derivatives*, Adv. Different. Equat., **2021**, 1 – 12 (2021).
2. M. A. Ali, H. Budak, Z. Zhang, H. Yildrim, *Some new Simpson's type inequalities for co-ordinated convex functions in quantum calculus*, Math. Methods Appl. Sci., **44**, 4515 – 4540 (2021).
3. M. A. Ali, M. Abbas, H. Budak, P. Agarwal, G. Murtaza, Y.-M. Chu, *New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions*, Adv. Different. Equat., **2021**, 1 – 21 (2021).
4. M. A. Ali, Y.-M. Chu, H. Budak, A. Akkurt, H. Yildrim, *Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables*, Adv. Different. Equat., **2021**, 1 – 26 (2021).
5. M. A. Ali, N. Alp, H. Budak, Y.-M. Chu, Z. Zhang, *On some new quantum midpoint type inequalities for twice quantum differentiable convex functions*, Open Math., **19**, 427 – 439 (2021).
6. M. A. Ali, H. Budak, A. Akkurt, Y.-M. Chu, *Quantum Ostrowski type inequalities for twice quantum differentiable functions in quantum calculus*, Open Math., **19**, 440 – 449 (2021).
7. M. A. Ali, H. Budak, K. Nanlaopon, Z. Abdullah, *Simpson's and Newton's inequalities for (α, m)-convex functions via quantum calculus* (2021).
8. N. Alp, M. Z. Sarikaya, M. Kunt, İ. İşcan, *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, J. King Saud Univ. Sci., **30**, 193 – 203 (2018).
9. N. Alp, M. Z. Sarikaya, *Hermite Hadamard's type inequalities for co-ordinated convex functions on quantum integral*, Appl. Math. E-Notes, **20**, 341 – 356 (2020).
10. S. Bermudo, P. Kórus, J. N. Valdés, *On q-Hermite – Hadamard inequalities for general convex functions*, Acta Math. Hungar., **162**, 364 – 374 (2020).
11. K.Brahim, S. Taf, L. Rihahi, *Some result for Hadamard-type inequalities in quantum calculus*, Matematiche, **69**, 243 – 258 (2014).
12. H. Budak, *Some trapezoid and midpoint type inequalities for newly defined quantum integrals*, Proyecciones, **40**, 199 – 215 (2021).
13. H. Budak, M. A. Ali, M. Tarhanaci, *Some new quantum Hermite – Hadamard-like inequalities for coordinated convex functions*, J. Optim. Theory and Appl., **186**, 899 – 910 (2020).

14. H. Budak, S. Erden, M. A. Ali, *Simpson and Newton type inequalities for convex functions via newly defined quantum integrals*, Math. Methods Appl. Sci., **44**, 378 – 390 (2020).
15. H. Budak, M. A. Ali, N. Alp, Y.-M. Chu, *Quantum Ostrowski type integral inequalities*, J. Math. Inequal. (2021) (to appear).
16. P. Cerone, S. S. Dragomir, *Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions*, Demonstr. Math., **37**, 299 – 308 (2004).
17. H. Kavurmaci, M. Avci, M. E. Özdemir, *New inequalities of Hermite – Hadamard type for convex functions with applications*, J. Inequal. and Appl., **2011**, Article 86 (2011).
18. F. H. Jackson, *On a q-definite integrals*, Quart. J. Pure Appl. Math., **41**, 193 – 203 (1910).
19. S. Jhanthanam, J. Tariboon, S. K. Ntouyas, K. Nonlaopon, *On q-Hermite – Hadamard inequalities for differentiable convex functions*, Mathematics, **7**, Article 632 (2019).
20. V. Kac, P. Cheung, *Quantum calculus*, Springer (2001).
21. H. Kalsoom, J.-D. Wu, S. Hussain, M. A. Latif, *Simpson's type inequalities for co-ordinated convex functions on quantum calculus*, Symmetry, **11**, Article 768 (2019).
22. Z. Liu, *A note on Ostrowski type inequalities related to some s-convex functions in the second sense*, Bull. Korean Math. Soc., **49**, 775 – 785 (2012).
23. W. Liu, Z. Hefeng, *Some quantum estimates of Hermite – Hadamard inequalities for convex functions*, J. Appl. Anal. and Comput., **7**, 501 – 522 (2016).
24. M. A. Noor, K. I. Noor, M. U. Awan, *Some quantum estimates for Hermite – Hadamard inequalities*, Appl. Math. and Comput., **251**, 675 – 679 (2015).
25. M. A. Noor, K. I. Noor, M. U. Awan, *Some quantum integral inequalities via preinvex functions*, Appl. Math. and Comput., **269**, 242 – 251 (2015).
26. A. Ostrowski, *Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert*, Comment. Math. Helv., **10**, 226 – 227 (1938).
27. I. B. Sial, S. Mei, M. A. Ali, K. Nanlaopon, *On some generalized Simpson's and Newton's inequalities for (α, m)-convex functions in q-calculus*, Mathematics, **2021**, Article 3266 (2021).
28. J. Tariboon, S. K. Ntouyas, *Quantum calculus on finite intervals and applications to impulsive difference equations*, Adv. Different. Equat., **2013**, 1 – 19 (2013).
29. M. Vivas-Cortez, M. A. Ali, A. Kashuri, I. B. Sial, Z. Zhang, *Some new Newton's type integral inequalities for co-ordinated convex functions in quantum calculus*, Symmetry, **12**, Article 1476 (2020).
30. M. Vivas-Cortez, A. Kashuri, R. Liko, J. E. Hernádez, *Quantum estimates of Ostrowski inequalities for generalized ϕ-convex functions*, Symmetry, **11**, Article 1513 (2019).
31. P. P. Wang, T. Zhu, T. S. Du, *Some inequalities using s-preinvexity via quantum calculus*, J. Interdisciplinary Math., **24**, 613 – 636 (2021).

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