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ON THE SYMPLECTIC STRUCTURE DEFORMATIONS RELATED TO THE MONGE – AMPÈRE EQUATION ON THE KÄHLER MANIFOLD $P_2(\mathbb{C})$

ПРО СИМПЛЕКТИЧНУ СТРУКТУРУ ДЕФОРМАЦІЙ, ПОВ'ЯЗАНИХ З РІВНЯННЯМ МОНЖА – АМПЕРА НА КЬОЛЕРІВСЬКОМУ МНОГОВИДІ $P_2(\mathbb{C})$

We analyze the cohomology structure of the fundamental two-form deformation related to a modified Monge – Ampère type on the complex Kähler manifold $P_2(\mathbb{C})$. Based on the Levi-Civita connection and the related vector-field deformation of the fundamental two-form, we construct a hierarchy of bilinear symmetric forms on the tangent bundle of the Kähler manifold $P_2(\mathbb{C})$, that generate Hermitian metrics on it and corresponding solutions to the Monge – Ampère-type equation. The classical fundamental two-form construction on the complex Kähler manifold $P_2(\mathbb{C})$ is generalized and the related metric deformations are discussed.

Проаналізовано когомологічну структуру фундаментальної двоформної деформації, що пов'язана з модифікованим типом Монжа – Ампера на комплексному кьолерівському многовиді $P_2(\mathbb{C})$. На основі зв'язності Леві-Чивіта і пов'язаної з нею деформації векторного поля фундаментальної 2-форми побудовано ієрархію білінійних симетричних форм на дотичному розшаруванні кьолерівському многовиду $P_2(\mathbb{C})$, що породжує на ній ермітові метрики і відповідні розв'язки досліджуваного рівняння типу Монжа – Ампера. Узагальнено конструкцію класичної фундаментальної 2-форми на комплексному кьолерівському многовиді $P_2(\mathbb{C})$ та обговорено відповідні її метричні деформації.

1. Introduction. Let us consider a compact complex n-dimensional manifold $M^n_{\mathbb{C}}$, endowed with the Kähler [1, 3, 27] fundamental symplectic two-form $\omega \in \Lambda^2(M^n_{\mathbb{C}})$. The related Monge – Ampère equation, describes a deformation of this symplectic structure

$$(\omega + i\bar{\partial}\partial\varphi)^n = (\exp f)\omega^n \tag{1.1}$$

under the normalizing conditions

$$\int\limits_{M^n_{\mathbb{C}}} (\exp f)\omega^n = \int\limits_{M^n_{\mathbb{C}}} \omega^n, \qquad \int\limits_{M^n_{\mathbb{C}}} \varphi\omega^n = 0,$$

where $\varphi \in C^\infty(M^n_{\mathbb{C}}; \mathbb{R})$ is a real valued function on $M^n_{\mathbb{C}}$ and $\bar{\partial}$ is the complex ∂-bar differential, corresponding to the standard differential splitting $d = \partial \oplus \bar{\partial} \colon \Lambda(M^n_{\mathbb{C}}) \to \Lambda(M^n_{\mathbb{C}})$ on the complex manifold $M^n_{\mathbb{C}}$. In a general case it was established in [28] that if the two-form $\big(\omega + i\partial\bar{\partial}\varphi\big) \in \Lambda^2(M^n_{\mathbb{C}})$ is real valued and the first Chern class $c_1(M^n_{\mathbb{C}}) = 0$ of a Kähler manifold $M^n_{\mathbb{C}}$, then there exists a Riemannian metric $g \colon T(M^n_{\mathbb{C}}) \times T(M^n_{\mathbb{C}}) \to \mathbb{C}$ of the Calabi – Yau type, whose holonomy group

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[9, 10, 12, 13] coincides with a subgroup of the Lie group $\mathrm{SU}(2)$, generating, in particular, a so-called Einsteinian metric. The equation (1.1) is always [28] solvable, yet its holonomy groups, in general, not classified and its unitarity remains to be open.

One can also mention here that if a Kähler manifold $M^2_{\mathbb{C}}$ is compact with the Chern class $c_1(M^2_{\mathbb{C}}) = 0$, it is well-known [24] that it is then hyper-Kähler, possessing exactly three Kähler fundamental forms ω_I, ω_J and $\omega_K \in \Lambda^2(M^2_{\mathbb{C}})$, corresponding to three compatible complex structures I, J and $K : T(M^2_{\mathbb{C}}) \to T(M^2_{\mathbb{C}})$. As for the compact projective two-dimensional Kähler manifold $M^2_{\mathbb{C}} = P_2(\mathbb{C})$ the Chern class $c_1(M^2_{\mathbb{C}}) \neq 0$, it is not hyper-Kähler, and its holomorphic volume two-form $\Omega \in \Lambda^2_{hol}(M^2_{\mathbb{C}})$ is not composed of the symplectic forms ω_J and $\omega_K \in \Lambda^2(M^2_{\mathbb{C}})$.

We should mention that there exists a slightly different modified Monge – Ampère type equation

$$(\omega + dJ^* d\varphi)^n = (\exp f)\omega^n, \tag{1.2}$$

on a real symplectic manifold $M^{2n} \simeq M^n_{\mathbb{C}}$, where $f \in C^\infty(M^{2n}; \mathbb{R})$ and $J : T(M^{2n}) \to T(M^{2n})$, $J^2 = -I$, is a suitably chosen nonintegrable quasicomplex structure on the manifold M^{2n} and $J^* : T^*(M^{2n}) \to T^*(M^{2n})$ denotes its conjugate. It was proved [5] that if the structure $J : T(M^{2n}) \to T(M^{2n})$ is integrable, then the equation (1.2) reduces to the Monge – Ampère equation (1.1) on the related complex manifold $M^n_{\mathbb{C}} \simeq M^{2n}$ owing to the classical Newlander – Nirenberg [16] criterion. Otherwise, if the equation (1.2) is solvable for an arbitrarily chosen right-hand side, then the quasicomplex structure $J : T(M^{2n}) \to T(M^{2n})$ has to be [5, 15, 17, 19] a complex one, once again reducing the equation (1.2) to the Monge – Ampère equation (1.1).

In current article we are interested in the following "symplectic" deformation

$$(\omega + dd^s\psi)^2 = (\exp f)\omega^2 \tag{1.3}$$

of the Monge – Ampère (1.1) on the complex two-dimensional Kähler manifold $M^2_{\mathbb{C}} = P_2(\mathbb{C})$, whose Chern class $c_1(M^2_{\mathbb{C}}) \neq 0$, by a symplectic deformation

$$\omega \to \omega + dd^s\psi, \tag{1.4}$$

where $\psi \in \Lambda^2(M^n_{\mathbb{C}})$ is a two-form and $d^s := (-1)^{k+1} \star_s d\star_s$ denotes the adjoint [6, 10, 12, 24] symplectic Hodge-type differentiation, satisfying the bilinear scalar relationship

$$\left(\alpha^{(k)} \mid d\beta^{(m)}\right)_s := \left(d^s\alpha^{(k)} \mid \beta^{(m)}\right)_s \tag{1.5}$$

for all differential k-forms $\alpha, \beta \in \Lambda^k(M^{2n})$, $k = \overline{1, 2n}$, as well as the identity $dd^s = -d^s d$. Here

$$\left(\alpha^{(k)} \mid \beta^{(m)}\right)_s := \delta_{km} \int\limits_{M^n} \left\langle \bar{\alpha}^{(k)} \mid \beta^{(m)} \right\rangle_s d\mu := \delta_{km} \int\limits_{M^n} \bar{\alpha}^{(k)} \wedge \star_s \beta^{(m)} \tag{1.6}$$

for any differential k-form $\alpha \in \Lambda^k(M^{2n})$ and differential m-form $\beta \in \Lambda^m(M^{2n})$, where $d\mu :=$ $:= \omega^n/n!$ is the volume measure on M^{2n} and the symplectic Hodge-star mapping $\star_s : \Lambda(M^{2n}) \to$ $\to \Lambda(M^{2n})$ acts on differential k-forms via the identity

$$\alpha \wedge \star_s\beta = \langle \alpha \mid \beta \rangle_s \omega^n/n! \tag{1.7}$$

with the “symplectic” bilinear form $\langle\cdot\mid\cdot\rangle_s:\Lambda^k(M^{2n})\times\Lambda^k(M^{2n})\to\mathbb{R}$,

$$\langle\alpha|\beta\rangle_s:=\frac{1}{k!}\sum_{i_l,j_m,l,m=\overline{1,k}}^{2n}\omega^{i_1j_1}\omega^{i_2j_2}\ldots\omega^{i_kj_k}\alpha_{i_1i_2\ldots i_k}\beta_{j_1j_2\ldots j_k},$$

for

$$\alpha:=\frac{1}{k!}\sum_{i_m,m=\overline{1,k}}^{2n}\alpha_{i_1i_2\ldots i_k}dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_k},$$

$$\beta:=\frac{1}{k!}\sum_{i_m,m=\overline{1,k}}^{2n}\beta_{i_1i_2\ldots i_k}dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_k}\in\Lambda^k(M^{2n}),$$

$$\omega:=\frac{1}{2}\sum_{ij=1}^{2n}\omega_{ij}dx^i\wedge dx^j\in\Lambda^2(M^{2n})$$

and

$$\sum_{k=1}^{2n}\omega^{ik}\omega_{kj}:=\delta^i_j,\quad i,j=i,j=\overline{1,2n},$$

being naturally extended on the complex-valued differential forms. It is worth to mention here that in the case of Kähler manifolds the important equalities

$$\operatorname{Im}d\cap\ker d^s=\ker d\cap\operatorname{Im}d^s=\operatorname{Im}dd^s \tag{1.8}$$

hold, as consequences of the expression (1.5) and (1.7). The scalar product (1.6), can be extended [3, 27] to the complex valued differential forms on the complex manifold $M^n\simeq\bar{M}^{2n}$ and gives rise to the following symplectic scalar product on $\Lambda(M^n_{\mathbb{C}})$:

$$(\alpha^{(k)}|\beta^{(m)})_s:=\delta_{km}\int\limits_{M^n_{\mathbb{C}}}\langle\alpha^{(k)}|\bar{\beta}^{(m)}\rangle_s d\mu:=\delta_{km}\int\limits_{M^n_{\mathbb{C}}}\alpha^{(k)}\wedge\star_s\bar{\beta}^{(m)},$$

where $\alpha^{(k)}\in\Lambda^k(M^n_{\mathbb{C}}),\ \beta^{(m)}\in\Lambda^m(M^n_{\mathbb{C}}),\ k,m=\overline{1,2n}$, are complex-valued differential forms on $M^n_{\mathbb{C}}$ and the bar “$-$” denotes the usual complex conjugation. In addition, the identities

$$c(\omega\mid\alpha\wedge\beta)_s=(\alpha\mid\beta)_s=-\bar{\beta}(X_\alpha)=\bar{\alpha}(X_\beta),$$

$$(\omega+dd^s\psi\mid\alpha\wedge\beta)_s=-\bar{\beta}(X_\alpha)-\big(i_{\bar{X}_\alpha}d\psi\mid d\bar{\beta}\big)_s+\big(i_{\bar{X}_\beta}d\psi\mid d\bar{\alpha}\big)_s$$

hold for any one-forms $\alpha,\beta\in\Lambda^1(M^n_{\mathbb{C}})$, where vector fields $X_\alpha,X_\beta\in\Gamma(T(M^n_{\mathbb{C}}))$ are defined via the relationships $i_{X_\alpha}\omega:=\alpha,\ i_{X_\beta}\omega:=\beta$, respectively.

It is worth to mention also that in general case, when the Chern class $c_1(M^2_{\mathbb{C}})\neq 0$, notwithstanding this fact, based on the equalities (1.8) and the well-known [3, 4, 26, 27] relationship $\star_s\eta=-\eta$ for an arbitrary "*primitive*" holomorphic volume two-form $\eta \in \Lambda^2_{hol}(M^2_{\mathbb{C}})$, satisfying the additional condition $\eta \wedge \omega = 0$, one easily derives the existence of two cohomological "*primitive*" holomorphic volume two-forms Ω_1 and $\Omega_2 \in \Lambda^2_{hol}(M^2)$, for which $\Omega_1 \wedge \bar{\Omega}_1 = \omega = \Omega_2 \wedge \bar{\Omega}_2$. Moreover, the interesting relationship

$$\Omega_1 - \Omega_2 = dd^s \chi$$

holds for some smooth two-from $\chi \in \Lambda^2(M^2_{\mathbb{C}})$, solving the problem (1.3) for the case, when the fundamental symplectic structure $\omega \in \Lambda^2(M^2_{\mathbb{C}})$ is replaced by a holomorphic volume two-form $\Omega \in \Lambda^2_{hol}(M^2_{\mathbb{C}})$.

In this article by analyzing the cohomology structure of the deformed two-form expression $(\omega + dd^s\psi) \in \Lambda^2(M^2_{\mathbb{C}})$ and applied some generalized transformations that were suggested in the classical works by Enneper [7] and Weierstrass [25] about one and half century ago and recently developed in [11], we rewrite the "symplectic" modification of the Monge – Ampère (1.3) in specially constructed coordinates, that allow us to construct its special solutions. It is important to mention that in general the considered deformed structures are stable and preserving [2, 15, 20 – 23] their properties only locally.

2. Canonical metric on $P_2(\mathbb{C})$ and the related fundamental symplectic form. Let $z := (z^0, z^1, z^2)^\intercal \in \mathbb{C}^3$ be a uniform coordinate frame on the Kähler complex manifold $M^2_{\mathbb{C}} := P_2(\mathbb{C})$ and define a related linear connection mapping

$$E^3 \ni u \ \to d_f(u) := du + \vartheta_f u \in E^3 \otimes \Lambda^1(M^2_{\mathbb{C}}), \tag{2.1}$$

where $E^3 := \big(\mathbb{C}^3, \pi, M^2_{\mathbb{C}}; \mathrm{SO}(3) \times \mathbb{S}^1\big)$ is a one-dimensional complex vector bundle over $M^2_{\mathbb{C}} \simeq \mathbb{E}^3/\big(\mathrm{SO}(3) \times \mathbb{S}^1\big)$ with the structure group $\mathrm{SO}(3) \times \mathbb{S}^1$, completely specified by means of the local holomorphic basis frame vector $f(z) := z \in E^3$, and $\vartheta : E^3 \to E^3 \otimes \Lambda^1(M^2_{\mathbb{C}})$ denotes the corresponding connection form. As the basis frame vector $f(z) \in E^3$ makes it possible to define on the Kähler manifold $M^2_{\mathbb{C}}$ the canonical Hermitian metric

$$g_f(A(z), B(z)) := \bar{f}(z)^\intercal f(z) a(z) \bar{b}(z) \tag{2.2}$$

for any vectors $A(z) = a(z)f(z) \in E^3$ and $B(z) = a(z)f(z) \in E^3$ at a point $p(z) \in M^2_{\mathbb{C}}$, one can construct easily the holomorphic connection form $\vartheta_f := \vartheta : E^3 \to E^3 \otimes \Lambda^{1,0}(M^2_{\mathbb{C}})$, compatible with the metric (2.2) by means of the well-known [27] relationship

$$\vartheta^{1,0}(z) := \left[\bar{f}(z)^\intercal f(z)\right]^{-1} \bar{f}(z)^\intercal \partial f(z),$$

where, by definition, $\vartheta(f(z)) := \vartheta^{1,0}(z) f(z) \in E^3 \otimes \Lambda^1(M^2_{\mathbb{C}})$. As the iterated mapping $d^2_f = d_f \circ d_f : E^3 \otimes \Lambda(M^2_{\mathbb{C}}) \to E^3 \otimes \Lambda(M^2_{\mathbb{C}})$, being a linear homomorphism on the module $\Lambda(M^2_{\mathbb{C}})$, determines [3, 27] the closed global curvature two-form

$$\Omega(z) = \frac{i}{2} d^2_f = \frac{i}{2}\big(d\vartheta^{1,0}(z) + \vartheta^{1,0}(z) \wedge \vartheta^{1,0}(z)\big) =$$

$$= \frac{i}{2}\bar{\partial}\Big(\left[\bar{f}(z)^\intercal f(z)\right]^{-1} \partial \bar{f}(z)^\intercal f(z)\Big) =$$

$$= \frac{i}{2}\big(|z|^{-2}\langle dz| \wedge dz\rangle - |z|^{-4}\langle dz|z\rangle \wedge \langle z|dz\rangle\big) \tag{2.3}$$

at arbitrary point $p(z) \in M^2_{\mathbb{C}}$, generating the first nontrivial Chern class $c_1(M^2_{\mathbb{C}}) := [\Omega] \subset H^2(M^2_{\mathbb{C}};\mathbb{Z})$ of the Kähler complex manifold $M^2_{\mathbb{C}}$. The obtained curvature two-form $\Omega \in \Lambda^2(M^2_{\mathbb{C}})$, being nondegenerate, can be identified with the fundamental symplectic two-form $\omega \in \Lambda^2(M^2_{\mathbb{C}})$, that is,

$$\omega = \frac{i}{2}\big(|z|^{-2}\langle dz| \wedge dz\rangle - |z|^{-4}\langle dz|z\rangle \wedge \langle z|dz\rangle\big), \tag{2.4}$$

naturally determining on the Kähler complex manifold $M^2_{\mathbb{C}}$ the compatible positive definite Fubini–Study [3, 27] metric

$$g(dz, dz) := |z|^{-2}\langle dz|dz\rangle - |z|^{-4}\langle dz|z\rangle\langle z|dz\rangle \tag{2.5}$$

at $p(z) \in M^2_{\mathbb{C}}$.

The Fubini–Study metric (2.5) is compatible with the canonical symplectic structure (2.4), is generated by the curvature two-form (2.3), corresponding to the canonical connection (2.1) on the one-dimensional vector bundle $E^3 := (\mathbb{C}^3, \pi, M^2_{\mathbb{C}}; \mathrm{SO}(3) \times \mathbb{S}^1)$. The latter depends on a coordinate frame $f(z) \in E^3$ at point $p(z) \in M^2_{\mathbb{C}}$, which was chosen to be trivial as $f(z) := z \in E^3$. It is evident that this choice is not unique and any other coordinate frame $f(z) \in E^3$ will provide a suitable connection $d_f : E^3 \to E^3 \otimes \Lambda^1(M^2_{\mathbb{C}})$ on the Kähler complex manifold $M^2_{\mathbb{C}}$, whose a priori closed curvature two-form $\Omega_f \in \Lambda^2(M^2_{\mathbb{C}})$, if none-degenerate, can be interpreted as a symplectic two-form $\omega_f \in \Lambda^2(M^2_{\mathbb{C}})$, based on which one can present the related bilinear and symmetric form $g_f : T(M^2_{\mathbb{C}}) \times T(M^2_{\mathbb{C}}) \to \mathbb{C}$. If, moreover, a suitably chosen coordinate frame $f(z) \in E^3$ makes this bilinear form positive definite, we will arrive at some symplectic deformation $\omega_f \in \Lambda^2(M^2_{\mathbb{C}})$ of the canonical symplectic two-form $\omega \in \Lambda^2(M^2_{\mathbb{C}})$, constructed before.

To realize this scheme analytically, we will make use below of the classical constructions by Enneper [7] and Weierstrass [25] about one and half century ago and recently developed in the general case [4, 8, 11, 29] of the Kähler manifold $P_n(\mathbb{C})$, $n \in \mathbb{N}$.

Let us consider the linear projective type mapping

$$Q_V : E^3 \ni f \to \langle df \mid V\rangle - f|f|^{-2}\langle df \mid V\rangle \in E^3$$

for a fixed nontrivial vector field $V \in \Gamma(T(E^3))$ and iterate it, starting from some holomorphic coordinate frame function $f = f_0 \in E^3$. A sequence $f_j \in E$, $j = \overline{0,2}$, obtained in this way, is parameterized by a fixed nontrivial vector $V \in \Gamma(T(E^3))$ and characterized by the following lemma.

Lemma 2.1. *Let vectors $f_j := f_j(z) \in E$, $j = \overline{0,2}$, be defined for any $n \in \mathbb{Z}_+$ as*

$$f_0 := f, \quad f_1 := Q_V f_0, \quad f_2 := Q_V f_1 - f_0|f_0|^{-2}\langle df_1(V)|f_0\rangle,$$

$$f_3 := Q_V f_2 - f_1|f_1|^{-2}\langle df_2(V)|f_1\rangle - f_0|f_0|^{-2}\langle df_2(V)|f_0\rangle, \tag{2.6}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$f_{n+1} := Q_V f_n - \sum_{j=\overline{0,n}-1} f_j|f_j|^{-2}\langle df_n(V)|f_j\rangle$$

at each point $p(z) \in M^2_{\mathbb{C}}$. Then $f_j = 0$ for all $j \geq 3$ and three vectors $f_j \in E^3$, $j \in \overline{0,2}$, are biorthogonal: $\langle f_s \mid f_k\rangle = 0$ for $k \neq s = \overline{0,2}$ at all $p(z) \in M^2_{\mathbb{C}}$.

Proof. It is easy to check that $\langle f_0|f_1\rangle = 0$. Now, by induction, we assume that all vectors $f_j \in E^3$, $j = \overline{0,n}$, defined by (2.6), are biorthogonal to each other, that is, $\langle f_j|f_k\rangle = 0$ for all

$j \neq k = \overline{0, n-1}$. Concerning the vector $f_{n+1} \in \mathbb{E}^3$ we can calculate that

$$\langle f_{n+1}|f_k\rangle := \langle Q_V f_n|f_k\rangle - \sum_{j=\overline{0,n-1}} \langle f_j|f_k\rangle \langle df_n(V)|f_j\rangle |f_j|^{-2} =$$

$$= \langle Q_V f_n|f_k\rangle - \langle df_n(V)|f_k\rangle = \langle |df_n(V)|f_k\rangle - \langle df_n(V)|f_k\rangle = 0$$

for all $k = \overline{0, n-1}$. In the last case $k = n$, we have

$$\langle f_{n+1}|f_n\rangle = \langle Q_V f_n|f_k\rangle = \langle df_n(V) - f_n|f_n|^{-2}\langle df_n(V)|f_n\rangle |f_n\rangle =$$

$$= \langle df_n(V)|f_n\rangle - \langle df_n(V)|f_n\rangle = 0.$$

Taking now into account that $\dim\ \mathbb{C}^3 = 3$, we derive that all $f_j = 0$ for $j = \overline{3, n}$, proving the lemma.

The lemma above makes it possible to describe effectively possible deformations of the fundamental symplectic structure $\omega \in \Lambda^2(M^2_{\mathbb{C}})$, parameterized by two parameters: tangent vector $V \in T(M)$ and a suitably defined element $f \in \mathbb{E}^3$.

3. The deformed Kähler fundamental form and induced Monge – Ampère equation. Consider now a t-parametric deformation of the symplectic form $\omega \in \Lambda^2(M^2_{\mathbb{C}})\colon \omega \to \omega_t := td\alpha + \omega$, where $t \in [0, 1]$, $\alpha \in \Lambda^1(M^2_{\mathbb{C}})$ is some one-form and $\varphi_t \colon M^2_{\mathbb{C}} \to M^2_{\mathbb{C}}$ a one-parametric group of diffeomorphisms of the Kähler manifold $M^2_{\mathbb{C}}$. Now we need the following Moser theorem [15].

Theorem 3.1. *Let* $(M^{2n}; \omega)$ *be oriented symplectic manifold and some symplectic deformation* $\omega \to \omega_t := \omega - \omega t \in \Lambda^2(M^{2n})$, $t \in [0, 1]$, *with fixed two-dimensional periods*, *that is*,

$$\int_\sigma \omega_t = \int_\sigma \omega \tag{3.1}$$

for every two-cycle $\sigma \in H_2(M^{2n}; \mathbb{R})$. *Then there exists a diffeomorphism* $\varphi_t \colon M^{2n} \to M^{2n}$ *such that*

$$\varphi_t^* \omega = \omega_t$$

for all $t \in [0, 1]$.

As the symplectic deformation $\omega \to \omega_t := td\alpha + \omega \in \omega \in \Lambda^2(M^2_{\mathbb{C}})$ a priori satisfies for all $t \in [0, 1]$ the condition (3.1), as a consequence of Theorem 3.1 one easily derives by differentiation with respect to the parameter $t \in [0, 1]$ that there exists a vector field $K \in \Gamma(T(M^2_{\mathbb{C}}))$ such that the Lie derivative $L_K\omega = d\alpha$, where the one-form $\alpha \in \Lambda^1(M^2_{\mathbb{C}})$ can be $\alpha = d^s\psi$ on the Kähler manifold $M^2_{\mathbb{C}}$. Taking into account the symplectic deformation (1.4), the latter means that

$$L_K\omega = di_K\omega = dd^s\psi,$$

from which one ensues the equivalence

$$i_K\omega = d^s\psi \bmod d\Lambda^0(M^2_{\mathbb{C}})$$

on the Kähler manifold $M^2_{\mathbb{C}}$. The written above mod-equivalence can be easily omitted, if to take into account that the vector field $K \in \Gamma(T(M^2_{\mathbb{C}}))$ is taken to be equivalent to the naturally related set $\mathcal{H} = \{H \in \Gamma(T(M^2_{\mathbb{C}}))\colon L_H\omega = 0\}$ of the Hamiltonian vector fields on the manifold $M^2_{\mathbb{C}}$, reducing our problem (1.3) to the following slightly simpler form:

$$i_K\omega = d^s\psi. \tag{3.2}$$

Now the initial problem reduces to the next two tasks: the first one assumes solving the equation (3.2) with respect to the corresponding two-form $\psi \in \Lambda^2(M^2_{\mathbb{C}})$, and the second one is a description of vector fields $K \in \Gamma(T(M^2_{\mathbb{C}}))$ on $M^2_{\mathbb{C}}$, for which the two-form $\omega + di_K\omega \in \Lambda^2(M^2_{\mathbb{C}})$ generates a Hermitian structure $h\colon T(M^2_{\mathbb{C}}) \times T(M^2_{\mathbb{C}}) \to \mathbb{C}$ on the Kähler manifold $M^2_{\mathbb{C}}$.

First of all, take now into account the Kodaira theorem [3, 27] that any two-form $\psi \in \Lambda^2(M^2_{\mathbb{C}})$ on the complex manifold $M^2_{\mathbb{C}}$ satisfies the conditions $\psi = *\psi \ (\mathrm{mod}\, d\Lambda^1(M^2_{\mathbb{C}})$ and $\psi \wedge \omega = 0$. The latter makes it possible upon applying to this condition the operation $\alpha \otimes i_K$ to obtain the identity

$$\alpha(K) = -(\alpha \wedge \psi | d\psi)_s \tag{3.3}$$

for any real one-form $\alpha = \bar{\alpha} \in \Lambda^1(M^2_{\mathbb{C}})$. The identity (3.3) is equivalent to a representation of the searched for vector field $K \in \Gamma(T(M^2_{\mathbb{C}}))$ as some solvable quadratic differential expression on the two-form $\psi \in \Lambda^2(M^2_{\mathbb{C}})$. Thereby, we can formulate the obtained above result as the following proposition.

Proposition 3.1. *Any symplectic deformation* (1.4) *of the symplectic structure $\omega \in \Lambda^2(M^2_{\mathbb{C}})$ on the complex manifold $M^2_{\mathbb{C}}$ is generated by the real vector fields $K \in \Gamma(T(M^2_{\mathbb{C}}))$ on $M^2_{\mathbb{C}}$, satisfying the scalar quadratic functional identity* (3.3).

Recall now that the constructed above symplectic deformation (1.4) of the symplectic structure $\omega \in \Lambda^2(M^2_{\mathbb{C}})$ should generate an Hermitian metric on our Kähler manifold $M^2_{\mathbb{C}} = P_2(\mathbb{C})$, that imposes natural constraints on the generating vector field $K \in \Gamma(T(M^2_{\mathbb{C}}))$ on $M^2_{\mathbb{C}}$, some of which were described in Proposition 3.1. The Levi-Civita connection $\nabla\colon T(M^2_{\mathbb{C}}) \to T(M^2_{\mathbb{C}})$, corresponding to the fundamental symplectic structure $\omega \in \Lambda^2(M^2_{\mathbb{C}})$, leaves invariant the related complex-structure $J\colon T(M^2_{\mathbb{C}}) \to T(M^2_{\mathbb{C}})$, naturally extended from the complexified tangent space $T(M^4) \otimes_{\mathbb{R}} \mathbb{C}$ on the whole $T(M^2_{\mathbb{C}})$, that is, $\nabla J = 0$ on the complex Kähler manifold $M^2_{\mathbb{C}}$. Taking now into account that the related compatible Hermitian metric $g\colon T(M^2_{\mathbb{C}}) \times T(M^2_{\mathbb{C}}) \to \mathbb{C}$ on the Kähler manifold $M^2_{\mathbb{C}}$ satisfies the relation

$$g(X,Y) := \omega(X, JY), \tag{3.4}$$

as well as the Levi-Civita connection ∇-invariance

$$Kg(X,Y) = g(\nabla_K X, Y) + g(X, \nabla_K Y)$$

for any vector fields $X, Y \in \Gamma(T(M^2_{\mathbb{C}}))$ along the vector field $K \in \Gamma(T(M^2_{\mathbb{C}}))$ on $M^2_{\mathbb{C}}$ one can derive by means of easy, yet slightly cumbersome calculations, the following identity:

$$(L_K\omega)(X, JY) = g\big((\nabla_K - L_K)X, Y\big) + g\big(X, (\nabla_K + JL_KJ)Y\big). \tag{3.5}$$

The expression (3.5), together with (3.4), gives rise to the deformed metric $g_K\colon T(M^2_{\mathbb{C}}) \times T(M^2_{\mathbb{C}}) \to \mathbb{C}$ on the Kähler manifold $M^2_{\mathbb{C}}$, defined by means of the expression

$$g_K(X,Y) := g(X,Y) + (L_K\omega)(X,Y), \tag{3.6}$$

which should be for all $Y = X \in \Gamma(T(M^2_{\mathbb{C}}))$ positive definite, imposing suitable constraints on the vector field $K \in \Gamma(T(M^2_{\mathbb{C}}))$ on $M^2_{\mathbb{C}}$.

To specify this metric positivity constraint on the vector field $K \in \Gamma(T(M^2_{\mathbb{C}}))$, satisfying the additional quadratic relationship (3.3), we need, preliminarily, to construct the related Levi-Civita connection $d_{\mathcal{A}} : \Gamma(T(M^2_{\mathbb{C}})) \to \Gamma(T^*(M^2_{\mathbb{C}}) \otimes T(M^2_{\mathbb{C}})) \simeq \Gamma(\mathrm{End}(T(M^2_{\mathbb{C}})))$ on sections of $\Gamma(T(M^2_{\mathbb{C}}))$:

$$d_{\mathcal{A}} X := dX + \mathcal{A}^{(1)} X, \tag{3.7}$$

where $\mathcal{A}^{(1)} \in \Gamma(\mathrm{End}(T(M^2_{\mathbb{C}}))) \otimes \Lambda^1(M^2_{\mathbb{C}})$ is the corresponding connection matrix, which is compatible with the constructed above Fubini – Study metric (2.5). Since the latter is representable on sections $X, Y \in \Gamma(T(M^2_{\mathbb{C}}))$ as

$$g(X, Y) = \langle hX|Y \rangle,$$

where the Hermitian matrix $h \in \mathrm{End}(T(M^2_{\mathbb{C}}))$, satisfies the following differential relationship:

$$dh = h\mathcal{A}^{(1)} + \bar{\mathcal{A}}^{(1)\intercal} h,$$

its solution provides the related curvature matrix two-form

$$\Omega_{\mathcal{A}} = d\mathcal{A}^{(1)} + \mathcal{A}^{(1)} \wedge \mathcal{A}^{(1)},$$

whose matrix trace

$$\omega_{\mathcal{A}} := \frac{i}{2} \operatorname{tr} \Omega_{\mathcal{A}}$$

is a priori closed and belongs to the Chern class, that is $\omega_{\mathcal{A}} \in c_1(M^2_{\mathbb{C}})$. Moreover, the following proposition holds.

Proposition 3.2. *The closed two-form $\omega_{\mathcal{A}} \in \Lambda^2(M^2_{\mathbb{C}})$ proves to be nondegenerate and defines an equivalent to $\omega \in c_1(M^2_{\mathbb{C}})$ fundamental two-form, generating a compatible Hermitian metric on the Kähler manifold $M^2_{\mathbb{C}}$.*

Return now to the deformed metric $g_K : T(M^2_{\mathbb{C}}) \times T(M^2_{\mathbb{C}}) \to \mathbb{C}$ on the Kähler manifold $M^2_{\mathbb{C}}$, defined by the expression (3.6) and depending on the covariant derivative $\nabla_K : T(M^2_{\mathbb{C}}) \to T(M^2_{\mathbb{C}})$, whose action on $X \in \Gamma(T(M^2_{\mathbb{C}}))$ can be now rewritten as

$$\nabla_K(X) = i_K dX + (i_K \mathcal{A}^{(1)})X. \tag{3.8}$$

Taking into account (3.8), we obtain the linear mappings

$$\nabla_K - L_K + I/2 = K_* + i_K \mathcal{A}^{(1)} + I/2,$$

$$\nabla_K + JL_K J + I/2 = -JK_* J + 2i_K \mathcal{A}^{(1)} + J(i_K \mathcal{A}^{(1)})J + I/2,$$

entering the deformed metric (3.6), where we made use of the covariant and Lie derivatives

$$\nabla_K J = J_* K + \left[J, i_K \mathcal{A}^{(1)}\right], \qquad L_K J = J_* K + [J, K_*],$$

respectively, and denoted by dash "$\prime$" the corresponding tangent mapping, making the tangent vector bundle diagrams

$$\begin{array}{ccc} T(M^2_{\mathbb{C}}) & \stackrel{K_*}{\to} & T(T(M^2_{\mathbb{C}})) \\ \downarrow & & \downarrow \\ M^2_{\mathbb{C}} & \stackrel{K}{\to} & T(M^2_{\mathbb{C}}) \end{array}, \qquad \begin{array}{ccc} T(M^2_{\mathbb{C}}) & \stackrel{J_*}{\to} & T(\mathrm{End}(T(M^2_{\mathbb{C}}))) \\ \downarrow & & \downarrow \\ M^2_{\mathbb{C}} & \stackrel{J}{\to} & \mathrm{End}(T(M^2_{\mathbb{C}})) \end{array}$$

commutative. Whence, the deformed metric (3.6) finally reduces to the bilinear symmetric expression

$$g_K(X,Y) := g((K_* + i_K\mathcal{A}^{(1)} + I/2)X, JY) +$$
$$+g\big(X, (-JK_*J + 2i_K\mathcal{A}^{(1)} + J(i_K\mathcal{A}^{(1)})J + I/2)Y\big) \tag{3.9}$$

on the product $T(M^2_{\mathbb{C}}) \times T(M^2_{\mathbb{C}})$. The obtained result we can formulate as the following preliminary theorem.

Theorem 3.2. *The deformed metric* (3.6) *is correctly defined on the complex Kähler manifold* $M^2_{\mathbb{C}}$ *as a bilinear symmetric form* (3.7) *on the product* $T(M^2_{\mathbb{C}}) \times T(M^2_{\mathbb{C}})$, *whose positive definiteness depends uniquely on a choice of the vector field* $K : M^2_{\mathbb{C}} \to T(M^2_{\mathbb{C}})$.

A detail Hermitian analysis and application of the deformed metric expression (3.9) to solving the Monge – Ampère type equation (1.3) will be presented in another article under preparation.

4. Conclusion. We analyzed the cohomology structure of the fundamental two-form deformation related to a modified Monge – Ampère type on the complex Kähler manifold $P_2(\mathbb{C})$. Based on the Levi-Civita connection together with the related vector field deformation of the fundamental two-form we construct a hierarchy of bilinear symmetric forms on the tangent bundle to the Kähler manifold $P_2(\mathbb{C})$, that generate Hermitian metric and solutions to the Monge – Ampère type equation. The classical fundamental two-form construction on the complex Kähler manifold $P_2(\mathbb{C})$, and its relation to the Hermitian metric deformations is discussed.

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