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LEAST-SQUARES METHOD IN THE THEORY OF NONLINEAR BOUNDARY-VALUE PROBLEMS UNSOLVED WITH RESPECT TO THE DERIVATIVE

МЕТОД НАЙМЕНШИХ КВАДРАТІВ У ТЕОРІЇ НЕЛІНІЙНИХ КРАЙОВИХ ЗАДАЧ, НЕ РОЗВ'ЯЗАНИХ ЩОДО ПОХІДНОЇ

We establish constructive necessary and sufficient conditions of solvability and a scheme for the construction of solutions for a nonlinear boundary-value problem unsolved with respect to the derivative. We also suggest convergent iterative schemes for finding approximate solutions of this problem. As an example of application of the proposed iterative scheme, we find approximations to the solutions of periodic boundary-value problems for a Rayleigh-type equation unsolved with respect to the derivative, in particular, in the case of a periodic problem for the equation used to describe the motion of satellites on elliptic orbits.

Встановлено конструктивні необхідні й достатні умови розв'язності та схему побудови розв'язків для нелінійної крайової задачі, не розв'язаної щодо похідної. Запропоновано збіжні ітераційні схеми для знаходження наближених розв'язків цієї задачі. Як приклад застосування запропонованої ітераційної схеми знайдено наближення до розв'язків періодичних крайових задач для рівняння типу Релея, не розв'язаного щодо похідної, зокрема, у випадку періодичної задачі для рівняння, що описує рух супутників на еліптичних орбітах.

1. Statement of the problem. We investigate the problem of finding the solutions

$$z(t,\varepsilon)\colon z(\cdot,\varepsilon)\in C^1[a,b],\quad z(t,\cdot)\in C[0,\varepsilon_0]$$

of a boundary-value problem [1 – 3]

$$dz/dt = A(t)z + f(t) + \varepsilon Z(z,z',t,\varepsilon), \tag{1}$$

$$\ell z(\cdot,\varepsilon) = \alpha + \varepsilon J\big(z(\cdot,\varepsilon), z'(\cdot,\varepsilon),\varepsilon\big) \tag{2}$$

in a small neighborhood of the solution of the generating Noetherian $(m \neq n)$ boundary-value problem

$$dz_0/dt = A(t)z_0 + f(t), \qquad \ell z_0(\cdot) = \alpha, \quad \alpha \in \mathbb{R}^m. \tag{3}$$

Here, $A(t)$ is a real $(n\times n)$-matrix, $f(t)$ is an n-dimensional column vector whose elements are real functions continuous on the segment $[a,b]$ and $\ell z(\cdot)$ is a linear bounded vector functional $\ell z(\cdot)\colon \mathbb{C}[a,b]\to\mathbb{R}^m$. The nonlinearities $Z(z,z',t,\varepsilon)$ and $J\big(z(\cdot,\varepsilon),z'(\cdot,\varepsilon),\varepsilon\big)$ of problem (1), (2) are continuously differentiable with respect to the unknown z and its derivative z' in a small neighborhood of the generating solution and its derivative and, in addition, with respect to a small parameter ε

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in a small positive neighborhood of zero. Moreover, we assume that the nonlinear vector function $Z(z, z', t, \varepsilon)$ is continuous with respect to the independent variable t on the segment $[a, b]$.

In this article, we will consider boundary-value problem (1), (2) unsolved with respect to the derivative, of the form (1). In the general case, such boundary-value problem (1), (2) can be solved with respect to the derivative, but at the same time we get computational problems with finding their solutions in the case of obtaining nonlinearities that are not integrable in elementary functions. Example such a situation is in the articles [2–4]. Therefore the study of solvability of nonautonomous boundary-value problems (1), (2), unsolved with respect to the derivative is a sufficiently important problem.

In the article [3], constructive necessary and sufficient solvability conditions and a scheme for constructing solutions to a nonlinear boundary-value problem unsolved with respect to the derivative using the Newton–Kantorovich method are provided. Convergent iterative schemes for finding approximations to solutions of a nonlinear boundary-value problem unsolved with respect to the derivative were constructed but this required significant computing power and did not provide high accuracy in practice. Therefore, the aim of this article will be to find constructive solvability conditions and a scheme for constructing solutions to a nonautonomous nonlinear boundary-value problem unsolved with respect to the derivative using the least squares method.

Denote $X(t)$ is the normal $(X(a) = I_n)$ fundamental matrix of the homogeneous part of system (3), $Q = \ell X(\cdot)$ is an $(m \times n)$-matrix, $\operatorname{rank} Q = n_1,$ P_{Q^*} is the $(m \times m)$-orthoprojector, $P_{Q^*}: \mathbb{R}^m \to \mathbb{N}(Q^*)$,

$$K\big[f(s)\big](t) = X(t) \int\limits_a^t X^{-1}(s) f(s) ds$$

is the Green operator of the Cauchy problem for system (3).

We study the critical case $(P_{Q^*} \neq 0)$, under the condition

$$P_{Q_d^*}\big\{\alpha - \ell K\big[f(s)\big](\cdot)\big\} = 0. \tag{4}$$

In this case, the generating problem (3) possesses a family of solutions

$$z_0(t, c_r) = X_r(t) c_r + G\big[f(s); \alpha\big](t), \qquad r := n - n_1, \quad c_r \in \mathbb{R}^r.$$

Here, $X_r(t) = X(t) P_{Q_r}$, P_{Q_r} is the $(n \times r)$-matrix formed by r linearly independent columns of the $(n \times n)$-orthoprojector P_Q, $P_{Q_d^*}$ is the $(d \times m)$-matrix formed by $d := m - n_1$ linearly independent rows of the $(m \times m)$-orthoprojector $P_{Q^*}: \mathbb{R}^m \to \mathbb{N}(Q^*)$,

$$G[f(s); \alpha](t) = K\big[f(s)\big](t) + X(t) Q^+ \big\{\alpha - \ell K\big[f(s)\big](\cdot)\big\}$$

is the generalized Green operator of the boundary-value problem (3) and Q^+ is the Moore–Penrose pseudoinverse matrix [1].

2. Conditions of solvability. Necessary and sufficient conditions for the existence of a solution

$$z(t, \varepsilon) = z_0(t, c_r) + x(t, \varepsilon)$$

of the problem (1), (2) in the critical case defines the equality

$$P_{Q_d^*}\Big\{J\big(z(\cdot,\varepsilon), z'(\cdot,\varepsilon),\varepsilon\big) - \ell K\big[Z(z(s,\varepsilon), z'(s,\varepsilon), s,\varepsilon)\big](\cdot)\Big\} = 0.$$

The nonlinearities $Z(z, z', t, \varepsilon)$ and $J\big(z(\cdot,\varepsilon), z'(\cdot,\varepsilon),\varepsilon\big)$ of the problem (1), (2) are continuous with respect to the small parameter ε in a small positive neighborhood of zero. Thus if the problem (1), (2) has a development, which for $\varepsilon = 0$ transforms into a generator $z_0(t, c_r^*)$, then the vector $c_r^* \in \mathbb{R}^r$ satisfies the equation [1, 3, 5, 6]

$$F(c_r) := P_{Q_d^*}\Big\{J\big(z_0(\cdot,c_r), z_0'(\cdot,c_r),0\big) - \ell K\big[Z(z_0(s,c_r), z_0'(s,c_r), s,0)\big](\cdot)\Big\} = 0. \tag{5}$$

Necessary conditions for the existence of a solution to the problem (1), (2) in the critical case are determined by the next lemma.

Lemma. *Suppose that the critical case* $(P_{Q^*} \neq 0)$ *is realized for the boundary-value problem* (1), (2) *and the condition of solvability* (4) *of the generating problem* (3) *is satisfied. Assume, in addition, that problem* (1), (2) *possesses a solution which turns into the generating solution* $z_0(t, c_r^*)$ *for* $\varepsilon = 0$. *Then the vector* $c_r^* \in \mathbb{R}^r$ *satisfies the equation* (5).

By analogy with weakly nonlinear boundary-value problems in the critical case [1], we say that equation (5) is an equation for generating constants of the boundary-value problem (1), (2), unsolved with respect to the derivative. Assume that equation (5) has real roots and does not turn into a trivial identity [7, 8]. We fix one of the solutions $c_r^* \in \mathbb{R}^r$ of equation (5) and arrive at the problem of finding the solutions

$$x(t,\varepsilon) : x(\cdot,\varepsilon) \in \mathbb{C}^1[a,b], \quad x(t,\cdot) \in \mathbb{C}[0,\varepsilon_0]$$

of the boundary-value problem

$$dx/dt = A(t)x + \varepsilon\, Z(z_0 + x, z_0' + x', t, \varepsilon), \tag{6}$$

$$\ell x(\cdot,\varepsilon) = \varepsilon\, J\big(z_0(\cdot,c_r^*) + x(\cdot,\varepsilon), z_0'(\cdot,c_r^*) + x'(\cdot,\varepsilon),\varepsilon\big). \tag{7}$$

In a small neighborhood of the generating solution $z_0(t, c_r^*)$, we get the following expansion:

$$Z\big(z_0(t,c_r^*) + x(t,\varepsilon), z_0'(t,c_r^*) + x'(t,\varepsilon), t,\varepsilon\big) = Z\big(z_0(t,c_r^*), z_0'(t,c_r^*), t, 0\big) +$$
$$+\, A_1(t)x(t,\varepsilon) + A_2(t)x'(t,\varepsilon) + \varepsilon\, A_3(t) + R_1(z_0(t,c_r^*) + x(t,\varepsilon), z_0'(t,c_r^*) + x'(t,\varepsilon), t,\varepsilon).$$

Here,

$$A_1(t) := \frac{\partial Z(z,z',t,\varepsilon)}{\partial z}\bigg|_{\substack{z=z_0(t,c_r^*)\\ z'=z_0'(t,c_r^*)\\ \varepsilon=0}}, \qquad A_2(t) := \frac{\partial Z(z,z',t,\varepsilon)}{\partial z'}\bigg|_{\substack{z=z_0(t,c_r^*)\\ z'=z_0'(t,c_r^*)\\ \varepsilon=0}},$$

$$A_3(t) := \frac{\partial Z(z,z',t,\varepsilon)}{\partial \varepsilon}\bigg|_{\substack{z=z_0(t,c_r^*)\\ z'=z_0'(t,c_r^*)\\ \varepsilon=0}}.$$

In view of the continuous Fréchet differentiability with respect to the three arguments of the vector functional $J\big(z_0(\cdot,c_r^*) + x(\cdot,\varepsilon), z_0'(\cdot,c_r^*) + x'(\cdot,\varepsilon),\varepsilon\big)$, we select the linear parts of this functional $\ell_1 x(\cdot,\varepsilon)$, $\ell_2 x'(\cdot,\varepsilon)$ and $\varepsilon\,\ell_3\big(z_0(\cdot,c_r^*)\big)$ and the term $J\big(z_0(\cdot,c_r^*), z_0'(\cdot,c_r^*),0\big) = J\big(z(\cdot,0), z'(\cdot,0),0\big)$ of order zero with respect to ε in a neighborhood of the points $x = 0$, $x' = 0$ and $\varepsilon = 0$:

$$J(z_0(\cdot, c_r^*) + x(\cdot, \varepsilon), z_0'(\cdot, c_r^*) + x'(\cdot, \varepsilon), \varepsilon) = J(z_0(\cdot, c_r^*), z_0'(\cdot, c_r^*), 0) + \ell_1 x(\cdot, \varepsilon) +$$
$$+ \ell_2 x'(\cdot, \varepsilon) + \varepsilon \ell_3(z_0(\cdot, c_r^*)) + J_1(z_0(\cdot, c_r^*) + x(\cdot, \varepsilon), z_0'(\cdot, c_r^*) + x'(\cdot, \varepsilon), \varepsilon).$$

Let
$$B_0 := P_{Q_d^*}\Big\{\ell_1 X_r(\cdot) + \ell_2 X_r'(\cdot) - \ell K\big[A_1(s)X_r(s) + A_2(s)X_r'(s)\big](\cdot)\Big\}$$
be a $(d \times r)$-matrix. By using the obtained expansions and the equations for generating constants (5), we arrive at the following operator system, which is equivalent to the problem of finding the solutions of system (6), that satisfy the boundary condition (7):

$$x(t,\varepsilon) = X_r(t)c_r + x^{(1)}(t,\varepsilon), \qquad x^{(2)}(t,\varepsilon) := \Big(x^{(1)}(t,\varepsilon)\Big)',$$
$$B_0\, c_r = -P_{Q_d^*}\Big\{\ell_1 x^{(1)}(\cdot,\varepsilon) + \ell_2 x^{(2)}(\cdot,\varepsilon) + \varepsilon \ell_3(z_0(\cdot, c_r^*)) +$$
$$+ J_1(z_0(\cdot, c_r^*) + x(\cdot,\varepsilon), z_0'(\cdot, c_r^*) + x'(\cdot,\varepsilon), \varepsilon) - \ell K\Big[A_1(s)x^{(1)}(s,\varepsilon) +$$
$$+ A_2(s)x^{(2)}(s,\varepsilon) + \varepsilon A_3(s) + R_1(z_0(s, c_r^*) + x(s,\varepsilon), z_0'(s, c_r^*) + x'(s,\varepsilon), s, \varepsilon)\Big](\cdot)\Big\}, \tag{8}$$
$$x^{(1)}(t,\varepsilon) = \varepsilon\, G\Big[Z(z_0(s, c_r^*) + x(s,\varepsilon), z_0'(s, c_r^*) + x'(s,\varepsilon), s, \varepsilon);$$
$$J\big(z_0(\cdot, c_r^*) + x(\cdot,\varepsilon), z_0'(\cdot, c_r^*) + x'(\cdot,\varepsilon), \varepsilon\big)\Big](t).$$

In article [3], under condition
$$P_{B_0^*} P_{Q_d^*} = 0, \tag{9}$$
we found constructive necessary and sufficient conditions of solvability and schemes of constructing solutions for a nonlinear boundary-value problem unsolved with respect to the derivative, with using the simple iteration method and the Newton – Kantorovich method. Under condition (9), this is said to be the critical case of the first order for the boundary-value problem (1), (2) unsolved with respect to the derivative. For finding the constructive conditions for the solution and the scheme for constructing solutions to a nonautonomous nonlinear boundary-value problem unsolved with respect to the derivative, under the condition (9), we use the least-squares method [9, 10].

3. Iteration scheme. Let
$$\varphi^{(1)}(t),\ \varphi^{(2)}(t),\ \ldots,\ \varphi^{(k)}(t),\ \ldots, \quad k \in \mathbb{N},$$
be a system of linearly independent continuously differentiable n dimensional vector functions. Let
$$\varphi_1(t) = \Big[\varphi^{(1)}(t) \quad \varphi^{(2)}(t) \quad \ldots \quad \varphi^{(p_1)}(t)\Big], \quad p_1 \in \mathbb{N},$$
denote an $(n \times p_1)$-matrix.

The first approximation to the solution of the boundary-value problem (6), (7)
$$x_1(t,\varepsilon) := \xi_1(t,\varepsilon) := \varphi_1(t)\, \gamma_1(\varepsilon)$$
will be sought as the best (in the sense of least squares) approximation solution of the boundary-value problem of the first approximation

$$dx_1(t,\varepsilon)/dt = A(t)x_1(t,\varepsilon) + \varepsilon\Big[Z(z_0(t,c_r^*), z_0'(t,c_r^*), t, 0) +$$
$$+ A_1(t)x_1(t,\varepsilon) + A_2(t)x_1'(t,\varepsilon) + \varepsilon\, A_3(t)\Big], \tag{10}$$

$$\ell x_1(\cdot,\varepsilon) = \varepsilon\Big[J(z_0(\cdot,c_r^*), z_0'(\cdot,c_r^*), 0) + \ell_1 x_1(\cdot,\varepsilon) + \ell_2 x_1'(\cdot,\varepsilon) + \varepsilon\,\ell_3\big(z_0(\cdot,c_r^*)\big)\Big]. \tag{11}$$

Generally speaking, the first approximation

$$\xi_1(t,\varepsilon) = \varphi_1(t)\gamma_1(\varepsilon), \qquad \gamma_1(\varepsilon) = \Big[\gamma_1^{(1)}(\varepsilon) \quad \gamma_1^{(2)}(\varepsilon) \quad \ldots \quad \gamma_1^{(p_1)}(\varepsilon)\Big]^*$$

is not a solution of the boundary-value problem (10), (11), so we will require that

$$F(\gamma_1(\varepsilon)) := \Big\|\big[I_n - \varepsilon A_2(t)\big]\xi_1'(t,\varepsilon) - \big[A(t) + \varepsilon A_1(t)\big]\xi_1(t,\varepsilon) -$$
$$- \varepsilon Z\big(z_0(t,c_r^*), z_0'(t,c_r^*), t, 0\big) - \varepsilon^2 A_3(t)\Big\|^2_{\mathbb{L}^2[a,b]} +$$
$$+ \Big\|\big[\ell - \varepsilon\ell_1\big]\xi_1(\cdot,\ \varepsilon) - \varepsilon\ell_2\,\xi_1'(\cdot,\varepsilon) - \varepsilon^2\,\ell_3\big(z_0(\cdot,c_r^*)\big) - \varepsilon J(z_0(\cdot,c_r^*), z_0'(\cdot,c_r^*), 0)\Big\|^2_{\mathbb{R}^m} \to \min$$

for a fixed matrix $\varphi_1(t)$. The necessary condition of minimization of the function $F(\gamma_1(\varepsilon))$ leads to the equation

$$\Big[\Gamma\big(\varphi_1(\cdot)\big) + \Gamma\big(\ell\varphi_1(\cdot)\big)\Big]\gamma_1(\varepsilon) = \varepsilon\int_a^b \Phi_1^*(t,\varepsilon)\Big(Z(z_0(t,c_r^*), z_0'(t,c_r^*), t, 0) + \varepsilon\, A_3(t)\Big)\,dt +$$
$$+ \varepsilon\,\Psi_1^*(\varepsilon)\,\big(J(z_0(\cdot,c_r*), z_0'(\cdot,c_r^*), 0) + \varepsilon\,\ell_3(z_0(\cdot,c_r^*))\big),$$

which is uniquely solvable with respect to the vector $\gamma_1(\varepsilon) \in \mathbb{R}^{p_1}$ under the condition of the nondegeneracy of the sum of the $(p_1 \times p_1)$ Gram matrices [9, 10]

$$\Gamma\big(\varphi_1(\cdot)\big) = \int_a^b \Phi_1^*(t,\varepsilon)\cdot\Phi_1(t,\varepsilon)\,dt, \qquad \Gamma\big(\ell\varphi_1(\cdot)\big) = \Psi_1(\varepsilon)^*\cdot\Psi_1(\varepsilon).$$

Here,

$$\Phi_1(t,\varepsilon) := \big[I_n - \varepsilon A_2(t)\big]\varphi_1'(t) - \big[A(t) + \varepsilon A_1(t)\big]\varphi_1(t),$$
$$\Psi_1(\varepsilon) := \big[\ell - \varepsilon\ell_1\big]\varphi_1(\cdot) - \varepsilon\ell_2\,\varphi_1'(\cdot).$$

Thus, under the condition

$$\det\Big[\Gamma\big(\varphi_1(\cdot)\big) + \Gamma\big(\ell\varphi_1(\cdot)\big)\Big] \neq 0, \quad 0 \le \varepsilon \le \varepsilon_* \le \varepsilon_0, \tag{12}$$

we find the vector

$$\gamma_1(\varepsilon) = \varepsilon\Big[\Gamma\big(\varphi_1(\cdot)\big) + \Gamma\big(\ell\varphi_1(\cdot)\big)\Big]^{-1}\int_a^b \Phi_1^*(t,\varepsilon)\big(Z(z_0(t,c_r^*), z_0'(t,c_r^*), t, 0) + \varepsilon\, A_3(t)\big)\,dt +$$
$$+ \varepsilon\,\Psi_1^*(\varepsilon)\,\big(J(z_0(\cdot,c_r*), z_0'(\cdot,c_r^*), 0) + \varepsilon\,\ell_3(z_0(\cdot,c_r^*))\big),$$

which defines the first approximation

$$x_1(t,\varepsilon) = \xi_1(t,\varepsilon) \approx \varphi_1(t)\gamma_1(\varepsilon)$$

to the solution of the boundary-value problem (10), (11). Here,

$$z_1(t,\varepsilon) = z_0(t,c_r^*) + x_1(t,\varepsilon)$$

is the best (in the sense of least squares) first approximation to the solution of the boundary-value problem (1), (2). Condition (12) is a necessary condition for minimizing the residual $F(\gamma_1(\varepsilon))$. A sufficient condition for minimizing the residual value $F(\gamma_1(\varepsilon))$ is ensured by the positive definiteness of the sum of Gram matrices $\Gamma(\varphi_1(\cdot))$ and $\Gamma(\ell\varphi_1(\cdot))$. The positive definiteness of the sum of Gram matrices $\Gamma(\varphi_1(\cdot))$ and $\Gamma(\ell\varphi_1(\cdot))$ is ensured by the fulfillment of the Sylvester criterion, namely, the positivity of the determinants of all square diagonal minors of the last sum [11].

Denote $(n \times p_2)$-matrix

$$\varphi_2(t) = \Big[\varphi^{(1)}(t) \quad \varphi^{(2)}(t) \quad \ldots \quad \varphi^{(p_2)}(t)\Big], \quad p_2 \in \mathbb{N}.$$

The second approximation to the solution of the problem (6), (7) is sought as the best (in the sense of least squares) approximation to the solution in the form

$$x_2(t,\varepsilon) := \xi_1(t,\varepsilon) + \xi_2(t,\varepsilon), \qquad \xi_2(t,\varepsilon) := \varphi_2(t)\,\gamma_2(\varepsilon).$$

The expansion

$$Z\big(z_1(t,\varepsilon) + \xi_2(t,\varepsilon), z_1'(t,\varepsilon) + \xi_2'(t,\varepsilon), t, \varepsilon\big) = Z\big(z_1(t,\varepsilon), z_1'(t,\varepsilon), t, 0\big) +$$
$$+ \mathcal{A}_1\big(z_1(t,\varepsilon)\big)\xi_2(t,\varepsilon) + \mathcal{A}_2\big(z_1(t,\varepsilon)\big)\xi_2'(t,\varepsilon) + \varepsilon\,\mathcal{A}_3\big(z_1(t,\varepsilon)\big) +$$
$$+ \mathcal{R}_1\big(z_1(t,\varepsilon) + \xi_2(t,\varepsilon), z_1'(t,\varepsilon) + \xi_2'(t,\varepsilon), t, \varepsilon\big)$$

takes place in a small neighborhood of the first approximation $z_1(t,\varepsilon)$ to the solution of the boundary-value problem (1), (2). Here,

$$\mathcal{A}_1\big(z_1(t,\varepsilon)\big) := \frac{\partial Z(z,z',t,\varepsilon)}{\partial z}\Bigg|_{\substack{z=z_1(t,\varepsilon)\\ z'=z_1'(t,\varepsilon)\\ \varepsilon=0}}, \qquad \mathcal{A}_2\big(z_1(t,\varepsilon)\big) := \frac{\partial Z(z,z',t,\varepsilon)}{\partial z'}\Bigg|_{\substack{z=z_1(t,\varepsilon)\\ z'=z_1'(t,\varepsilon)\\ \varepsilon=0}},$$

$$\mathcal{A}_3\big(z_1(t,\varepsilon)\big) := \frac{\partial Z(z,z',t,\varepsilon)}{\partial \varepsilon}\Bigg|_{\substack{z=z_1(t,\varepsilon)\\ z'=z_1'(t,\varepsilon)\\ \varepsilon=0}}.$$

Then, using the continuous differentiability (in the Fréchet sense) of the vector functional

$$J\big(z_1(\cdot,\varepsilon) + \xi_2(\cdot,\varepsilon), z_1'(\cdot,\varepsilon) + \xi_2'(\cdot,\varepsilon), \varepsilon\big),$$

with respect to three arguments, we separate linear parts $\big(\ell_1(z_1(\cdot,\varepsilon))\big)\xi_2(\cdot,\varepsilon)$, $\big(\ell_2(z_1(\cdot,\varepsilon))\big)\xi_2'(\cdot,\varepsilon)$ and $\varepsilon\big(\ell_3(z_1(\cdot,\varepsilon))\big)$ of this functional and the term $J\big(z_1(\cdot,\varepsilon), z_1'(\cdot,\varepsilon), 0\big)$ of zero order with respect to ε in the neighborhoods of the points $\xi_2(t,\varepsilon) = 0$, $\xi_2'(t,\varepsilon) = 0$ and $\varepsilon = 0$:

$$J\big(z_1(\cdot,\varepsilon) + \xi_2(\cdot,\varepsilon), z_1'(\cdot,\varepsilon) + \xi_2'(\cdot,\varepsilon), \varepsilon\big) = J(z_1(\cdot,\varepsilon), z_1'(\cdot,\varepsilon), 0) + \big(\ell_1(z_1(\cdot,\varepsilon))\big)\xi_2(\cdot,\varepsilon) +$$
$$+ \big(\ell_2(z_1(\cdot,\varepsilon))\big)\xi_2'(\cdot,\varepsilon) + \varepsilon\big(\ell_3(z_1(\cdot,\varepsilon))\big) + \mathcal{J}_1\big(z_1(\cdot,\varepsilon) + \xi_2(\cdot,\varepsilon), z_1'(\cdot,\varepsilon) + \xi_2'(\cdot,\varepsilon), \varepsilon\big).$$

The second approximation $x_2(t,\varepsilon)$ to the solution of the boundary-value problem (6), (7) will be sought as a solution to the boundary-value problem of the second approximation

$$dx_2(t,\varepsilon)/dt = A(t)x_2(t,\varepsilon) + \varepsilon\Big[Z(z_1(t,\varepsilon), z_1'(t,\varepsilon), t, 0) +$$

$$+\mathcal{A}_1\big(z_1(t,\varepsilon)\big)\xi_2(t,\varepsilon) + \mathcal{A}_2\big(z_1(t,\varepsilon)\big)\xi_2'(t,\varepsilon) + \varepsilon\,\mathcal{A}_3\big(z_1(t,\varepsilon)\big)\Big],$$

$$\ell x_2(\cdot,\varepsilon) = \varepsilon\Big[J(z_1(\cdot,\varepsilon), z_1'(\cdot,\varepsilon), 0) + \big(\ell_1(z_1(\cdot,\varepsilon))\big)\xi_2(\cdot,\varepsilon) +$$

$$+\big(\ell_2(z_1(\cdot,\varepsilon))\big)\xi_2'(\cdot,\varepsilon) + \varepsilon\,(\ell_3(z_1(\cdot,\varepsilon)))\Big].$$

We require that

$$F(\gamma_2(\varepsilon)) := \Big\|\big[I_n - \varepsilon\,\mathcal{A}_2\big(z_1(t,\varepsilon)\big)\big]\xi_2'(t,\varepsilon) - \big[A(t) + \varepsilon\,\mathcal{A}_1\big(z_1(t,\varepsilon)\big)\big]\xi_2(t,\varepsilon) -$$

$$-\varepsilon Z(z_1(t,\varepsilon), z_1'(t,\varepsilon), t, 0) - \varepsilon^2\mathcal{A}_3\big(z_1(t,\varepsilon)\big)\Big\|^2_{\mathbb{L}^2[a,b]} +$$

$$+\Big\|\big[\ell - \varepsilon\,(\ell_1(z_1(\cdot,\varepsilon)))\big]\xi_2(\cdot,\ \varepsilon) - \varepsilon\big(\ell_1(z_1(\cdot,\varepsilon))\big)\xi_2'(\cdot,\varepsilon) - \varepsilon^2\big(\ell_3(z_1(\cdot,\varepsilon))\big) -$$

$$-\varepsilon\,J\big(z_1(\cdot,\varepsilon), z_1'(\cdot,\varepsilon), 0\big)\Big\|^2_{\mathbb{R}^m} \to \min$$

for the fixed matrix $\varphi_2(t)$. The necessary condition of minimization of the function $F(\gamma_2(\varepsilon))$ leads to the equation

$$\Big[\Gamma\big(\varphi_2(\cdot)\big) + \Gamma\big(\ell\varphi_2(\cdot)\big)\Big]\gamma_2(\varepsilon) = \varepsilon\int_a^b \Phi_2^*(t,\varepsilon)\Big\{Z(z_1(t,\varepsilon), z_1'(t,\varepsilon), t, 0) + \varepsilon\mathcal{A}_3\big(z_1(t,\varepsilon)\big) -$$

$$-\Big[Z(z_0(t,c_r^*), z_0'(t,c_r^*), t, 0) + A_1(t)x_1(t,\varepsilon) + A_2(t)x_1'(t,\varepsilon) + \varepsilon\,A_3(t)\Big]\Big\}\,dt +$$

$$+\varepsilon\,\Psi_2^*(\varepsilon)\Big\{J(z_1(\cdot,\varepsilon), z_1'(\cdot,\varepsilon), 0) + \varepsilon\,(\ell_3(z_1(\cdot,\varepsilon))) - \Big[J(z_0(\cdot,c_r^*), z_0'(\cdot,c_r^*), 0) +$$

$$+\ell_1 x_1(\cdot,\varepsilon) + \ell_2 x_1'(\cdot,\varepsilon) + \varepsilon\,\ell_3(z_0(\cdot,c_r^*))\Big]\Big\},$$

which is uniquely solvable with respect to the vector $\gamma_2(\varepsilon) \in \mathbb{R}^{p_2}$ under the condition of the nondegeneracy of the sum of the $(p_2 \times p_2)$ Gram matrices [9, 10]

$$\Gamma\big(\varphi_2(\cdot)\big) = \int_a^b \Phi_2^*(t,\varepsilon)\cdot\Phi_2(t,\varepsilon)\,dt, \qquad \Gamma\big(\ell\varphi_2(\cdot)\big) = \Psi_2(\varepsilon)^*\cdot\Psi_2(\varepsilon).$$

Here,

$$\Phi_2(t,\varepsilon) := \big[I_n - \varepsilon\mathcal{A}_2\big(z_1(t,\varepsilon)\big)\big]\varphi_2'(t) - \Big[A(t) + \varepsilon\mathcal{A}_1\big(z_1(t,\varepsilon)\big)\Big]\varphi_2(t),$$

$$\Psi_2(\varepsilon) := \big[\ell - \varepsilon\big(\ell_1(z_1(\cdot,\varepsilon))\big)\big]\varphi_2(\cdot) - \big(\ell_2(z_1(\cdot,\varepsilon))\big)\varphi_2'(\cdot).$$

Thus, under the condition

$$\det\Big[\Gamma\big(\varphi_2(\cdot)\big) + \Gamma\big(\ell\varphi_2(\cdot)\big)\Big] \neq 0, \quad 0 \le \varepsilon \le \varepsilon_* \le \varepsilon_0,$$

we find the vector

$$\gamma_2(\varepsilon) = \varepsilon\Big[\Gamma\big(\varphi_2(\cdot)\big) + \Gamma\big(\ell\varphi_2(\cdot)\big)\Big]^{-1} \int\limits_a^b \Phi_2^*(t,\varepsilon)\Big\{Z(z_1(t,\varepsilon), z_1'(t,\varepsilon), t, 0) +$$

$$+ \varepsilon\,\mathcal{A}_3\big(z_1(t,\varepsilon)\big) - \Big[Z(z_0(t,c_r^*), z_0'(t,c_r^*), t, 0) + A_1(t)x_1(t,\varepsilon) + A_2(t)x_1'(t,\varepsilon) + \varepsilon\,A_3(t)\Big]\Big\}\,dt +$$

$$+ \varepsilon\,\Psi_2^*(\varepsilon)\Big\{J(z_1(\cdot,\varepsilon), z_1'(\cdot,\varepsilon), 0) + \varepsilon\big(\ell_3(z_1(\cdot,\varepsilon))\big) - \Big[J(z_0(\cdot,c_r^*), z_0'(\cdot,c_r^*), 0) +$$

$$+ \ell_1 x_1(\cdot,\varepsilon) + \ell_2 x_1'(\cdot,\varepsilon) + \varepsilon\,\ell_3(z_0(\cdot,c_r^*))\Big]\Big\},$$

which defines the second approximation

$$x_2(t,\varepsilon) = \xi_1(t,\varepsilon) + \xi_2(t,\varepsilon), \qquad \xi_2(t,\varepsilon) \approx \varphi_2(t)\gamma_2(\varepsilon)$$

to the solution of the boundary-value problem (6), (7) and the second approximation

$$z_2(t,\varepsilon) = z_0(t,c_r^*) + x_2(t,\varepsilon)$$

to the solution of the boundary-value problem (1), (2) under the condition of the positive definiteness of the sum of Gram matrices $\Gamma\big(\varphi_2(\cdot)\big)$ and $\Gamma\big(\ell\varphi_2(\cdot)\big)$. The best approximation is understood in the sense of least squares.

Denote the $(n \times p_{k+1})$-matrix

$$\varphi_{k+1}(t) = \big[\varphi^{(1)}(t) \quad \varphi^{(2)}(t) \quad \ldots \quad \varphi^{(p_{k+1})}(t)\big], \quad p_{k+1} \in \mathbb{N}.$$

Suppose further that a $(k+1)$-term approximation

$$x_{k+1}(t,\varepsilon) = \xi_1(t,\varepsilon) + \ldots + \xi_{k+1}(t,\varepsilon), \qquad \xi_{k+1}(t,\varepsilon) \approx \varphi_{k+1}(t)\gamma_{k+1}(\varepsilon), \quad k = 1, 2, \ldots,$$

of the boundary-value problem (6), (7) is found. And, accordingly, a $k+1$ approximation

$$z_{k+1}(t,\varepsilon) = z_0(t,c_r^*) + x_{k+1}(t,\varepsilon)$$

to the solution of the boundary-value problem (1), (2), which is the best in the least squares sense, is found. The next $(k+2)$-term approximation to the solution of the boundary-value problem (6), (7) is sought in the form

$$x_{k+2}(t,\varepsilon) = \xi_1(t,\varepsilon) + \ldots + \xi_{k+2}(t,\varepsilon), \qquad \xi_{k+2}(t,\varepsilon) \approx \varphi_{k+2}(t)\gamma_{k+2}(\varepsilon), \quad k = 1, 2, \ldots.$$

In a small neighborhood of the approximation $z_{k+1}(t,\varepsilon)$ to the solution of the boundary-value problem (1), (2), consider the following expansion:

$$Z(z_{k+1}(t,\varepsilon) + \xi_{k+2}(t,\varepsilon), z_{k+1}'(t,\varepsilon) + \xi_{k+2}'(t,\varepsilon), t, \varepsilon) = Z(z_{k+1}(t,\varepsilon), z_{k+1}'(t,\varepsilon), t, 0) +$$

$$+ \mathcal{A}_1\big(z_{k+1}(t,\varepsilon)\big)\xi_{k+2}(t,\varepsilon) + \mathcal{A}_2\big(z_{k+1}(t,\varepsilon)\big)\xi_{k+2}'(t,\varepsilon) + \varepsilon\,\mathcal{A}_3\big(z_{k+1}(t,\varepsilon)\big) +$$

$$+ \mathcal{R}_{k+1}(z_{k+1}(t,\varepsilon) + \xi_{k+2}(t,\varepsilon), z_{k+1}'(t,\varepsilon) + \xi_{k+2}'(t,\varepsilon), t, \varepsilon), \quad k = 1, 2, \ldots.$$

Here,

$$\mathcal{A}_1\big(z_{k+1}(t,\varepsilon)\big) := \frac{\partial Z(z,z',t,\varepsilon)}{\partial z}\bigg|_{\substack{z=z_{k+1}(t,\varepsilon)\\ z'=z'_{k+1}(t,\varepsilon)\\ \varepsilon=0}}, \qquad \mathcal{A}_2\big(z_{k+1}(t,\varepsilon)\big) := \frac{\partial Z(z,z',t,\varepsilon)}{\partial z'}\bigg|_{\substack{z=z_{k+1}(t,\varepsilon)\\ z'=z'_{k+1}(t,\varepsilon)\\ \varepsilon=0}},$$

$$\mathcal{A}_3\big(z_{k+1}(t,\varepsilon)\big) := \frac{\partial Z(z,z',t,\varepsilon)}{\partial \varepsilon}\bigg|_{\substack{z=z_{k+1}(t,\varepsilon)\\ z'=z'_{k+1}(t,\varepsilon)\\ \varepsilon=0}}.$$

Then, using the continuous differentiability (in the Fréchet sense) of the vector functional

$$J\big(z_{k+1}(\cdot,\varepsilon)+\xi_{k+2}(\cdot,\varepsilon), z'_{k+1}(\cdot,\varepsilon)+\xi'_{k+2}(\cdot,\varepsilon),\varepsilon\big),$$

with respect to three arguments, we separate the linear parts $\big(\ell_1(z_{k+1}(\cdot,\varepsilon))\big)\xi_{k+2}(\cdot,\varepsilon)$, $\big(\ell_2(z_{k+1}(\cdot,\varepsilon))\big)\xi'_{k+2}(\cdot,\varepsilon)$ and $\varepsilon\big(\ell_3(z_1(\cdot,\varepsilon))\big)$ of this functional and the term $J\big(z_{k+1}(\cdot,\varepsilon), z'_{k+1}(\cdot,\varepsilon),0\big)$ of zeroth order with respect to ε in the neighborhoods of the points $\xi_{k+2}(t,\varepsilon)=0$, $\xi'_{k+2}(t,\varepsilon)=0$ and $\varepsilon=0$:

$$J\big(z_{k+1}(\cdot,\varepsilon)+\xi_{k+2}(\cdot,\varepsilon), z'_{k+1}(\cdot,\varepsilon)+\xi'_{k+2}(\cdot,\varepsilon),\varepsilon\big) =$$
$$= J(z_{k+1}(\cdot,\varepsilon), z'_{k+1}(\cdot,\varepsilon),0) + (\ell_1(z_{k+1}(\cdot,\varepsilon)))\,\xi_{k+2}(\cdot,\varepsilon) +$$
$$+ (\ell_2(z_{k+1}(\cdot,\varepsilon)))\,\xi'_{k+2}(\cdot,\varepsilon) + \varepsilon\big(\ell_3(z_{k+1}(\cdot,\varepsilon))\big) +$$
$$+ \mathcal{J}_1\big(z_{k+1}(\cdot,\varepsilon)+\xi_{k+2}(\cdot,\varepsilon), z'_{k+1}(\cdot,\varepsilon)+\xi'_{k+2}(\cdot,\varepsilon),\varepsilon\big).$$

The approximation $x_{k+2}(t,\varepsilon)$ to the solution of the boundary-value problem (6), (7) will be sought as a solution to the boundary-value problem

$$dx_{k+2}(t,\varepsilon)/dt = A(t)x_{k+2}(t,\varepsilon) + \varepsilon\Big[Z(z_{k+1}(t,\varepsilon), z'_{k+1}(t,\varepsilon),t,0) +$$
$$+ \mathcal{A}_1\big(z_{k+1}(t,\varepsilon)\big)\,\xi_{k+2}(t,\varepsilon) + \mathcal{A}_2\big(z_{k+1}(t,\varepsilon)\big)\,\xi'_{k+2}(t,\varepsilon) + \varepsilon\,\mathcal{A}_3\big(z_{k+1}(t,\varepsilon)\big)\Big],$$
$$\ell x_{k+2}(\cdot,\ \varepsilon) = \varepsilon\Big[J(z_{k+1}(\cdot,\varepsilon), z'_{k+1}(\cdot,\varepsilon),0) + (\ell_1(z_{k+1}(\cdot,\varepsilon)))\,\xi_{k+2}(\cdot,\varepsilon) +$$
$$+ (\ell_2(z_{k+1}(\cdot,\varepsilon)))\,\xi'_{k+2}(\cdot,\varepsilon)\ + \varepsilon\,\big(\ell_3(z_{k+1}(\cdot,\varepsilon))\big)\Big].$$

Denote the $(n\times p_{k+2})$-matrix

$$\varphi_{k+2}(t) = \Big[\varphi^{(1)}(t)\ \ \varphi^{(2)}(t)\ \ \ldots\ \ \varphi^{(p_{k+2})}(t)\Big], \quad p_{k+2}\in\mathbb{N}.$$

We require that

$$F(\gamma_{k+2}(\varepsilon)) := \Big\|\Big[I_n - \varepsilon\,\mathcal{A}_2\big(z_{k+1}(t,\varepsilon)\big)\Big]\xi'_{k+2}(t,\varepsilon) - \big[A(t)+\varepsilon\,\mathcal{A}_1\big(z_{k+1}(t,\varepsilon)\big)\big]\xi_{k+2}(t,\varepsilon) -$$
$$- \varepsilon Z(z_{k+1}(t,\varepsilon), z'_{k+1}(t,\varepsilon),t,0) - \varepsilon^2\mathcal{A}_3\big(z_{k+1}(t,\varepsilon)\big)\Big\|^2_{\mathbb{L}^2[a,b]} +$$

$$+ \Big\| \big[\ell - \varepsilon \big(\ell_1(z_{k+1}(\cdot,\varepsilon))\big)\big]\xi_{k+2}(\cdot,\varepsilon) - \varepsilon\big(\ell_1(z_{k+1}(\cdot,\varepsilon))\big)\xi'_{k+2}(\cdot,\varepsilon) - \varepsilon^2\big(\ell_3(z_{k+1}(\cdot,\varepsilon))\big) -$$

$$- \varepsilon\, J\big(z_{k+1}(\cdot,\varepsilon), z'_{k+1}(\cdot,\varepsilon), 0\big)\Big\|^2_{\mathbb{R}^m} \to \min$$

for the fixed matrix $\varphi_{k+2}(t)$. The necessary condition of minimization of the function $F(\gamma_{k+2}(\varepsilon))$ leads to the equation

$$\Big[\Gamma\big(\varphi_{k+2}(\cdot)\big) + \Gamma\big(\ell\varphi_{k+2}(\cdot)\big)\Big]\,\gamma_{k+2}(\varepsilon) =$$

$$= \varepsilon \int\limits_a^b \Phi^*_{k+2}(t,\varepsilon)\Big\{Z(z_{k+1}(t,\varepsilon), z'_{k+1}(t,\varepsilon), t, 0) + \varepsilon\mathcal{A}_3\big(z_{k+1}(t,\varepsilon)\big) -$$

$$- \Big[Z(z_k(t,\varepsilon), z'_k(t,\varepsilon), t, 0) + \mathcal{A}_1(z_k(t,\varepsilon))\xi_{k+1}(t,\varepsilon) + \mathcal{A}_2(z_k(t,\varepsilon))\xi'_{k+1}(t,\varepsilon) +$$

$$+ \varepsilon\,\mathcal{A}_3(z_k(t,\varepsilon))\Big]\Big\}\,dt + \varepsilon\,\Psi^*_{k+2}(\varepsilon)\,\Big\{J(z_{k+1}(\cdot,\varepsilon), z'_{k+1}(\cdot,\varepsilon), 0) + \varepsilon\,\big(\ell_3(z_{k+1}(\cdot,\varepsilon))\big) -$$

$$- \Big[J(z_k(\cdot,\varepsilon), z'_k(\cdot,\varepsilon), 0) + \ell_1\,\xi_{k+1}(\cdot,\varepsilon) + \ell_2\,\xi'_{k+1}(\cdot,\varepsilon) + \varepsilon\,\ell_3(z_k(\cdot,\varepsilon))\Big]\Big\},$$

which is uniquely solvable with respect to the vector $\gamma_{k+2}(\varepsilon) \in \mathbb{R}^{p_{k+2}}$ under the condition of the nondegeneracy of the sum of $(p_{k+2} \times p_{k+2})$ measurable Gram matrices [9, 10]

$$\Gamma\big(\varphi_{k+2}(\cdot)\big) = \int\limits_a^b \Phi^*_{k+2}(t,\varepsilon)\cdot\Phi_{k+2}(t,\varepsilon)\,dt, \qquad \Gamma\big(\ell\varphi_{k+2}(\cdot)\big) = \Psi_{k+2}(\varepsilon)^*\cdot\Psi_{k+2}(\varepsilon).$$

Here,

$$\Phi_{k+2}(t,\varepsilon) := \Big[I_n - \varepsilon\,\mathcal{A}_2\big(z_{k+1}(t,\varepsilon)\big)\Big]\varphi'_{k+2}(t) - \Big[A(t) + \varepsilon\mathcal{A}_1\big(z_{k+1}(t,\varepsilon)\big)\Big]\varphi_{k+2}(t),$$

$$\Psi_{k+2}(\varepsilon) := \Big[\ell - \varepsilon\big(\ell_1(z_{k+1}(\cdot,\varepsilon))\big)\Big]\varphi_{k+2}(\cdot) - (\ell_2(z_{k+1}(\cdot,\varepsilon)))\,\varphi'_{k+2}(\cdot).$$

Thus, under the condition

$$\det\Big[\Gamma\big(\varphi_{k+2}(\cdot)\big) + \Gamma\big(\ell\varphi_{k+2}(\cdot)\big)\Big] \neq 0, \quad 0 \le \varepsilon \le \varepsilon_* \le \varepsilon_0, \tag{13}$$

we find the vector

$$\gamma_{k+2}(\varepsilon) = \varepsilon\,\Big[\Gamma\big(\varphi_{k+2}(\cdot)\big) + \Gamma\big(\ell\varphi_{k+2}(\cdot)\big)\Big]^{-1} \times$$

$$\times \int\limits_a^b \Phi^*_{k+2}(t,\varepsilon)\Big\{Z(z_{k+1}(t,\varepsilon), z'_{k+1}(t,\varepsilon), t, 0) + \varepsilon\mathcal{A}_3\big(z_{k+1}(t,\varepsilon)\big) -$$

$$- \Big[Z(z_k(t,\varepsilon), z'_k(t,\varepsilon), t, 0) + \mathcal{A}_1(z_k(t,\varepsilon))\xi_{k+1}(t,\varepsilon) + \mathcal{A}_2(z_k(t,\varepsilon))\xi'_{k+1}(t,\varepsilon) +$$

$$+ \varepsilon\,\mathcal{A}_3(z_k(t,\varepsilon))\Big]\Big\}\,dt + \varepsilon\,\Psi^*_{k+2}(\varepsilon)\,\Big\{J(z_{k+1}(\cdot,\varepsilon), z'_{k+1}(\cdot,\varepsilon), 0) + \varepsilon\,\big(\ell_3(z_{k+1}(\cdot,\varepsilon))\big) -$$

$$- \Big[J(z_k(\cdot,\varepsilon), z_k'(\cdot,\varepsilon), 0) + \ell_1\, \xi_{k+1}(\cdot,\varepsilon) + \ell_2\, \xi_{k+1}'(\cdot,\varepsilon) + \varepsilon\, \ell_3(z_k(\cdot,\varepsilon))\Big]\Big\}, \quad k = 1, 2, \ldots .$$

This vector defines the approximation

$$x_{k+2}(t,\varepsilon) = \xi_1(t,\varepsilon) + \ldots + \xi_{k+2}(t,\varepsilon), \quad \xi_{k+2}(t,\varepsilon) \approx \varphi_{k+2}(t)\gamma_{k+2}(\varepsilon)$$

to the solution of the boundary-value problem (6), (7) and the approximation

$$z_{k+2}(t,\varepsilon) = z_0(t,c_r^*) + x_{k+2}(t,\varepsilon), \qquad 0 \le \varepsilon \le \varepsilon_* \le \varepsilon_0, \quad k = 1, 2, \ldots,$$

to the solution of the boundary-value problem (1), (2) under the condition of the positive definiteness of the sum of Gram matrices $\Gamma(\varphi_{k+2}(\cdot))$ and $\Gamma(\ell\varphi_{k+2}(\cdot))$. The best approximation is understood in the of least squares sense.

To estimate the accuracy of iterations of the obtained approximations, we assume that the operator

$$\Theta\big(x_k(t,\varepsilon)\big) : x_k(t,\varepsilon) \to x_{k+1}(t,\varepsilon)$$

in some sufficiently small neighborhood Ω of the generating solution $z_0(t,c_r^*)$ is a contraction operator, and for arbitrary vector functions $\xi(t,\varepsilon),\ \zeta(t,\varepsilon) \in \Omega$ the inequality

$$\big\|\Theta\left(\xi(t,\varepsilon)\right) - \Theta\left(\zeta(t,\varepsilon)\right)\big\| \le \lambda\big\|\xi(t,\varepsilon) - \zeta(t,\varepsilon)\big\|, \quad 0 < \lambda < 1,$$

is fulfilled. According to the Caccioppoli – Banach principle [12, p. 605] in this case, in a sufficiently small neighborhood Ω of the generating solution $z_0(t,c_r^*)$. There exists a unique fixed point $x(t,\varepsilon)$ of the mapping $\Theta x_k(t,\varepsilon)$, which is the equilibrium position of the equation

$$x(t,\varepsilon) = \Theta\big(x(t,\varepsilon)\big).$$

To find a fixed point $x(t,\varepsilon)$ of the mapping $\Theta(x_k(t,\varepsilon))$, we use the iterative scheme

$$x_{k+1}(t,\varepsilon) = \Theta\big(z_0(t,c_r^*) + x_k(t,\varepsilon)\big), \qquad x_0(t,\varepsilon) \equiv 0, \quad k = 0, 1, \ldots . \tag{14}$$

It is natural to assume that the first approximation $x_1(t,\varepsilon)$ does not coincide with the fixed point $x(t,\varepsilon)$ of the mapping $\Theta(x_k(t,\varepsilon))$. In this case, the equality

$$\big\|x(t,\varepsilon) - x_1(t,\varepsilon)\big\| = \big\|\Theta(x(t,\varepsilon)) - \Theta(0)\big\| \le \lambda\big\|x(t,\varepsilon)\big\|$$

is satisfied. Denote

$$\delta_1(\varepsilon) = \frac{\big\|x_1(t,\varepsilon) - \xi_1(t,\varepsilon)\big\|}{\big\|x(t,\varepsilon)\big\|}.$$

The value of $\delta_1(\varepsilon)$ depends of the choice on the $(n \times p_1)$ dimensional matrix $\varphi_1(t)$. Using the triangle inequality, we get the estimate

$$\big\|x(t,\varepsilon) - \xi_1(t,\varepsilon)\big\| \le (\lambda + \delta_1(\varepsilon))\big\|x(t,\varepsilon)\big\|.$$

For the second approximation calculated according to the scheme (14), the following inequality holds:

$$\big\|x(t,\varepsilon) - x_2(t,\varepsilon)\big\| \le \lambda^2\big\|x(t,\varepsilon)\big\|.$$

Denote

$$\delta_2(\varepsilon) = \frac{\big\|x_2(t,\varepsilon) - \big(\xi_1(t,\varepsilon) + \xi_2(t,\varepsilon)\big)\big\|}{\big\|x^*(t,\varepsilon)\big\|}.$$

Using again the triangle inequality, we obtain the estimate

$$\Big\|x(t,\varepsilon) - \big(\xi_1(t,\varepsilon) + \xi_2(t,\varepsilon)\big)\Big\| \le (\lambda^2 + \delta_2(\varepsilon))\big\|x(t,\varepsilon)\big\|.$$

Similar, we get

$$\left\|x(t,\varepsilon) - \sum_{i=1}^{k} \xi_i(t,\varepsilon)\right\| \le (\lambda^k + \delta_k(\varepsilon))\big\|x(t,\varepsilon)\big\|.$$

Here,

$$\delta_k(\varepsilon) = \frac{\Big\|x_k(t,\varepsilon) - \big(\xi_1(t,\varepsilon) + \ldots + \xi_k(t,\varepsilon)\big)\Big\|}{\big\|x(t,\varepsilon)\big\|}.$$

For a sufficiently small value

$$\delta(\varepsilon) = \max_{0 \le i \le k} \delta_i(\varepsilon),$$

for $k \le \kappa$ the inequalities

$$\lambda^k + \delta(\varepsilon) < \lambda^{k-1} + \delta(\varepsilon) < \ldots < \lambda^2 + \delta(\varepsilon) < \lambda + \delta(\varepsilon) < 1$$

hold, which guarantee a sufficiently small value of the norm of the difference

$$\left\|x(t,\varepsilon) - \sum_{i=1}^{k} \xi_i(t,\varepsilon)\right\|.$$

In contrast to the method of simple iterations, the use of an iterative scheme constructed by the least-squares method makes it possible to find iterations of a sufficiently high order. However, the accuracy of the least-squares approximation is limited by a value of the order of $\delta(\varepsilon)$, which, in turn, depends on the choice of the $(n \times p_k)$ dimensional matrix $\varphi_k(t)$. In addition, the accuracy of the least-squares approximations is affected by errors in intermediate calculations. Thus, the following theorem is true.

Theorem. *Suppose that the boundary-value problem* (1), (2) *unsolved with respect to the derivative corresponds to the critical case* $(P_{Q^*} \ne 0)$ *and the condition of solvability* (4) *of the generating problem* (3) *is satisfied. Assume that equation* (5) *does not turn into a trivial identity and has real solutions. Then, for each root* $c_r^* \in \mathbb{R}^r$ *of equation* (5), *under the condition* (9) *the boundary-value problem* (1), (2) *unsolved with respect to the derivative possesses at least one solution*

$$z(t,\varepsilon)\colon z(\cdot,\varepsilon) \in \mathbb{C}^1[a,b], \quad z(t,\cdot) \in \mathbb{C}[0,\varepsilon_0].$$

For $\varepsilon = 0$ *this solution turns into the generating solution*

$$z_0(t,c_r^*) = X_r(t)c_r^* + G[f(s);\alpha](t).$$

In the case (13), *when the sums of Gram matrices* $\Gamma(\varphi_{k+2}(\cdot))$ *and* $\Gamma(\ell\varphi_{k+2}(\cdot))$ *are positive definiteness, the best solution*, *by least-squares*, *of the boundary-value problem* (1), (2) *unsolved with respect to the derivative*, *can be determined by using the following iterative process*:

$$z_1(t,\varepsilon) = z_0(t,c_r^*) + x_1(t,\varepsilon), \qquad x_1(t,\varepsilon) = \xi_1(t,\varepsilon) \approx \varphi_1(t)\gamma_1(\varepsilon),$$

$$\gamma_1(\varepsilon) = \varepsilon\Big[\Gamma\big(\varphi_1(\cdot)\big) + \Gamma\big(\ell\varphi_1(\cdot)\big)\Big]^{-1}\int\limits_a^b \Phi_1^*(t,\varepsilon)\big(Z(z_0(t,c_r^*), z_0'(t,c_r^*), t, 0) + \varepsilon\, A_3(t)\big)\, dt \, +$$

$$+\,\varepsilon\,\Psi_1^*(\varepsilon)\,\big(J(z_0(\cdot,c_r*), z_0'(\cdot,c_r^*),0) + \varepsilon\,\ell_3(z_0(\cdot,c_r^*))\big), \ldots,$$

$$z_{k+2}(t,\varepsilon) = z_0(t,c_r^*) + x_{k+2}(t,\varepsilon), \qquad x_{k+2}(t,\varepsilon) = \xi_1(t,\varepsilon) + \ldots + \xi_{k+2}(t,\varepsilon),$$

$$\xi_{k+2}(t,\varepsilon) \approx \varphi_{k+2}(t)\gamma_{k+2}(\varepsilon), \qquad \gamma_{k+2}(\varepsilon) = \varepsilon\,\Big[\Gamma\big(\varphi_{k+2}(\cdot)\big) + \Gamma\big(\ell\varphi_{k+2}(\cdot)\big)\Big]^{-1} \times \tag{15}$$

$$\times \int\limits_a^b \Phi_{k+2}^*(t,\varepsilon)\Big\{Z(z_{k+1}(t,\varepsilon), z_{k+1}'(t,\varepsilon), t, 0) + \varepsilon\mathcal{A}_3\big(z_{k+1}(t,\varepsilon)\big) \,-$$

$$-\,\Big[Z(z_k(t,\varepsilon), z_k'(t,\varepsilon), t, 0) + \mathcal{A}_1(z_k(t,\varepsilon))\xi_{k+1}(t,\varepsilon) + \mathcal{A}_2(z_k(t,\varepsilon))\xi_{k+1}'(t,\varepsilon) \,+$$

$$+\,\varepsilon\,\mathcal{A}_3(z_k(t,\varepsilon))\Big]\Big\}\, dt + \varepsilon\,\Psi_{k+2}^*(\varepsilon)\,\Big\{J(z_{k+1}(\cdot,\varepsilon), z_{k+1}'(\cdot,\varepsilon), 0) + \varepsilon\,\big(\ell_3(z_{k+1}(\cdot,\varepsilon))\big) \,-$$

$$-\,\Big[J(z_k(\cdot,\varepsilon), z_k'(\cdot,\varepsilon), 0) + \ell_1\,\xi_{k+1}(\cdot,\varepsilon) + \ell_2\,\xi_{k+1}'(\cdot,\varepsilon) + \varepsilon\,\ell_3(z_k(\cdot,\varepsilon))\Big]\Big\},\ k = 1, 2, \ldots,$$

which converges for $0 \le \varepsilon \le \varepsilon_* \le \varepsilon_0$.

The proved theorem generalizes the results of [1, 2] to the case of a nonlinear boundary-value problem unsolved with respect to the derivative. Under the condition (13), in the case of positive definiteness of the sum of Gram matrices $\Gamma(\varphi_{k+2}(\cdot))$ and $\Gamma(\ell\varphi_{k+2}(\cdot))$, the length of the segment $[0,\varepsilon_*]$, on which the iterative scheme is applicable (15), can be estimated using the Lyapunov smearing equations [1, 13]. Also, similar to [14], the length of the segment $[0,\varepsilon_*]$ can be estimated using the compression condition of the operator $\Theta(x_k(t,\varepsilon))$.

Conclusions. The scheme proposed in the article for constructing solutions to a nonlinear boundary-value problem (1), (2), unsolved with respect to the derivative, can be used for a nonlinear periodic boundary-value problem for a Rayleigh-type equation, unsolved with respect to derivative [2, 3], in particular, in the case of a periodic problem for the equation that determines the motion of a satellite in an elliptical orbit. The proposed scheme can also be transferred to nonlinear matrix boundary-value problems [15], including, problems with delay [16], nonlinear differential-algebraic boundary-value problems [17], as well as nonlinear boundary-value problems with switching unsolved with respect to the derivative [21, 22].

Unlike numerous studies of various boundary-value problems unsolved with respect to the derivative [18 – 20], the article is devoted to constructing solutions to a nonlinear boundary-value problem for an equation unsolved with respect to the derivative, and not to obtaining conditions for solving boundary-value problems problems unsolved with respect to the derivative. Therefore, the scheme proposed in the article for constructing solutions of a nonlinear boundary-value problem for an equation unsolved with respect to the derivative continues our studies of a periodic boundary-value problem for a Duffing-type equation unsolved with respect to the derivative [5], as well as the study carried out under the guidance of Academician of the National Academy of Sciences of Ukraine A. M. Samoilenko [3].

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