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ON PERTURBATION OF DRAZIN INVERTIBLE LINEAR RELATIONS

ПРО ЗБУРЕННЯ ОБЕРНЕНИХ ЛІНІЙНИХ СПІВВІДНОШЕНЬ ДРАЗІНА

We study the stability of regular, finite ascent, and finite descent linear relations defined in Banach spaces under commuting nilpotent operator perturbations. As an application, we give the invariance theorem of Drazin invertible spectrum under these perturbations. We also focus on the study of some properties of the left and right Drazin invertible linear relations. It is proved that, for bounded and closed left (resp., right) Drazin invertible linear relation with nonempty resolvent set, 0 is an isolated point of the associated approximate point spectrum (resp., surjective spectrum).

Досліджено стабільність регулярних лінійних співвідношень скінченного підйому та скінченного спуску, що визначені в банахових просторах для комутуючих збурень нільпотентного оператора. Як застосування наведено теорему про інваріантність оберненого спектра Дразіна при таких збуреннях. Також вивчаються деякі властивості лівих і правих обернених лінійних співвідношень Дразіна. Доведено, що для обмеженого та замкненого лівого (відповідно, правого) оберненого лінійного співвідношення Дразіна з непорожньою резольвентною множиною, 0 є ізольованою точкою відповідного наближеного точкового спектра (відповідно, сюр'єктивного спектра).

Introduction. Let A be a complex Banach algebra. An element a of A is called relatively regular if there exists $x \in A$ such that $axa = a$. If a is relatively regular, then it has a generalized inverse, which is an element $b \in A$ satisfying the equations $aba = a$ and $bab = b$ (see [27]). A relation between a relatively regular element and its generalized inverse is reflexive in the sense that if b is a generalized inverse of a, then a is a generalized inverse of b. M. P. Drazin introduced in [14], another form of a generalized inverse in associative rings and semigroups that does not have the reflexivity property but commutes with the element. In fact, for a and b two elements of a semigroup, b is said to be a Drazin inverse of a, if $ab = ba, b = ab^2$ and $a^k b = a^{k+1} b$ for some nonnegative integer k. This notion is investigated in the setting of bounded linear operators on complex Banach algebra by several authors [9, 15, 23, 27]. For X a Banach space, a bounded operator $T \in \mathcal{L}(X)$ is Drazin invertible if there exists an operator $T^D \in \mathcal{L}(X)$ called the Drazin inverse of T such that $TT^D = T^D T, T^D T D^D = T^D$ and $T^{k+1} T^D = T^k$ for some $k \in \mathbb{N}$. In [23], C. F. King gives another characterization of Drazin invertible operators by means of ascent and descent as follows: An operator T is Drazin invertible if and only if $asc(T) = des(T) < \infty$, where $asc(T)$ and $des(T)$ define the smallest nonnegative integer n such that $N(T^n) = N(T^{n+1})$ and $R(T^n) = R(T^{n+1})$, respectively. Later the notion of Drazin invertibility was extended in [2] as follows: $T \in \mathcal{L}(X)$ is said to be left Drazin invertible if $p = asc(T) < \infty$ and $R(T^{p+1})$ is closed while T is said to be right Drazin invertible if $q = des(T) < \infty$ and $R(T^q)$ is closed. The study of perturbation of these classes have been carried by many authors recently [16, 17, 22].

Linear relations made their appearance in functional analysis in von Neumann [1] motivated by the need to consider adjoints of nondensely defined operators used in applications to the theory of generalized equations [8] and also by the need to consider the inverses of certain operators, used, for example, in the study of some Cauchy problems associated with parabolic type equations in Banach

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spaces [19]. We can cite the following example: if $T \in \mathcal{L}(X)$ and $N(T) \neq \{0\}$, then T^{-1} is a linear relation in $\mathcal{LR}(X)$. In particular, an ordinary differential operator $T : C^{(n)}[a,b] \subset C[a,b] \longrightarrow C[a,b]$ of the kind

$$(Tx)(t) = x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \ldots + a_n(t)x(t),$$

where $a_k \in C[a,b],\ k = 1,\ldots,n,$ that acts in the Banach space $C[a,b]$ of bounded continuous complex functions on $[a,b] \subset \mathbb{R}$ has a finite-dimensional kernel $N(T)$ of dimension $n \geq 1$, and therefore $T^{-1} \in \mathcal{LR}(C[a,b])$ is a linear relation. If $Tx = x'$, then $T^{-1}x = \displaystyle\int x(t)dt,\ x \in C[a,b]$.

This concept has been developed intensively in the last years since it has applications in many problems in physics and other areas of applied mathematical: game theory and mathematical economics, discontinuous differential equations which occur in the biological sciences (for example, population in dynamics and epidemiology), optimal control and digital imaging. A systematic bibliography on these applications including references to other and more recent contributions can be found in [18].

As the interest of studying linear relations is relevant, the above concepts have been naturally extended in the setting of linear relations in [21] as follows. A linear relation T defined on X is said to be Drazin invertible of degree $k \in \mathbb{N}$ if T is everywhere defined and there exists a bounded operator $T^D \in \mathcal{L}(X)$ such that

$$TT^D = T^DT + T(0), \qquad T^DTT^D = T^D \quad \text{and} \quad T^{k+1}T^D = T^k + T^{k+1}(0).$$

If $\rho(T) \neq \varnothing$, the above definition is equivalent to the finiteness of the ascent and descent of T (see [21, Theorem 3.3]). The relation T is said to be left Drazin invertible if $asc(T) = p < \infty$ and $R(T^{p+1})$ is closed and is right Drazin invertible if $q = des(T) < \infty$ and $R(T^q)$ is closed. Furthermore, if $\rho(T) \neq \varnothing$, then Drazin invertible linear relations are exactly those that are both left and right Drazin invertible. The purpose of the present paper is to extend some perturbation results given by B. P. Duggal in the context of linear relations. More precisely, in [17], sufficient conditions for invariance of Drazin invertible operators under perturbation by commuting nilpotent operator are given. We prove that the mentioned results remain valid in the general context of linear relations.

This paper is organized as follows. In Section 1, some notations and auxiliary results which are needed in the sequel, are presented. In Section 2, some results involving ascent and descent of linear relations are established. Precisely, if T is a closed everywhere defined linear relation and N a nilpotent operator which commutes with T and such that $asc(T)$ (resp., $des(T)$) is finite, then $asc(T+N)$ (resp., $des(T+N)$) remains finite too. As a consequence, perturbation theorems for the corresponding spectra are deduced. Section 3, concerns regular linear relations. We set up a perturbation theorem for this class, as well for bounded below linear relations, by commuting nilpotent operators. In Section 4, the sets of core and quasinilpotent part of linear relations are mentioned. We gather some properties of these notions for left and right Drazin invertible linear relations, in relation with their kernels and ranges. Then we prove that if T is a bounded and closed left (resp., right) Drazin invertible linear relation with nonempty resolvent set, then 0 is an isolated point of the associated approximate point (resp., surjective) spectrum of T. The last section is devoted to the study of perturbation of Drazin invertible linear relations by commuting nilpotent operators. For left and right Drazin invertible operators, analogous perturbations results have been proved by Duggal in [17], under supplementary conditions, that is, T and $T+N$ satisfy a specified property noted (**P**). Same invariance results are covered for left and right Drazin invertible linear relations, without the conditions below. In that way, results of [19] about operators in Banach spaces are improved. The stated results are applied to give the invariance theorem of Drazin invertible spectrum under commuting nilpotent operator perturbations.

1. Preliminaries. We adhered to the notations and terminology of the monographs [10, 30]. Let X be an infinite-dimensional Banach space over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. A multivalued linear operator $T\colon X\longrightarrow X$ or simply a **linear relation** is a mapping from a subspace $D(T)\subseteq X$, called the domain of T, into the collection of nonempty subsets of X such that $T(\alpha x+\beta y)=\alpha T(x)+\beta T(y)$ for all nonzero scalars α, β and $x,y\in D(T)$. We denote by $\mathcal{LR}(X)$ the class of all linear relations on X. If T maps the points of its domain to singletons, then T is said to be a single valued linear operator or simply an operator. A linear relation T in X is uniquely determined by its graph, $G(T)$, which is defined by

$$G(T):=\big\{(x,y)\in X\times X : x\in D(T), y\in Tx\big\}.$$

The inverse T^{-1} of T is given by $G(T^{-1}):=\big\{(y,x)\colon (x,y)\in G(T)\big\}$. For $T,S\in\mathcal{LR}(X)$, the linear relations $T+S$ and $T\hat{+}S$ are defined by

$$G(T+S):=\big\{(x,y+z)\in X\times X : (x,y)\in G(T),\ (x,z)\in G(S)\big\}$$

and

$$G(T\hat{+}S):=\big\{(x+u,y+v)\colon (x,y)\in G(T),(u,v)\in G(S)\big\},$$

the last sum is direct when $G(T)\cap G(S)=\{(0,0)\}$. In this case we write $T\oplus S$. The product TS is given by

$$G(TS):=\big\{(x,z)\in X\times X : (x,y)\in G(S) \text{ and } (y,z)\in G(T) \text{ for some } y\in X\big\}.$$

Since the product of linear relations is clearly associative, if $n\in\mathbb{Z}$, T^n is defined as usual with $T^0=I$ and $T^1=T$.

For a given closed subspace M of X, we denote by T_M the linear relation given by $G(T_M)= =G(T)\bigcap(M\times X)$ and Q_T stands for the quotient map from X onto $X/\overline{T(0)}$. Clearly Q_TT is a linear operator and hence we can define the quantity $\|T\|=\|Q_TT\|$.

The kernel of a linear relation T is the subspace $N(T):=T^{-1}(0)$. The subspace $R(T):= :=T(D(T))$ is called the range of T. T is called **injective** if $N(T)=\{0\}$ and **surjective** if $R(T)=X$. When T is both injective and surjective, we say that T is **bijective**. We define the generalized kernel and the generalized range of T respectively by

$$N^\infty(T)=\bigcup_{n\geq 1}N(T^n) \quad\text{and}\quad R^\infty(T)=\bigcap_{n\geq 1}R(T^n).$$

We say that $T\in\mathcal{LR}(X)$ has a trivial singular chain manifold if $R_c(T)=\{0\}$, where $R_c(T)= =N^\infty(T)\cap R^\infty(T)$. A linear relation $T\in\mathcal{LR}(X)$ is said to be **closed** if its graph is a closed subspace of $X\times X$. The class of such linear relations will be denoted by $\mathcal{CR}(X)$. T is **continuous** if for each open set $U\in R(T)$, $T^{-1}(U)$ is an open set in $D(T)$. We say that T is **open** if T^{-1} is continuous, equivalently $\gamma(T)>0$, where $\gamma(T)$ is the minimum modulus of T defined by

$$\gamma(T) = \sup\Big\{\lambda \geq 0 \colon \lambda d\big(x, N(T)\big) \leq \|Tx\|, x \in D(T)\Big\}.$$

Continuous everywhere defined linear relations are referred as **bounded** relations. The class of all bounded and closed linear relations on X is denoted by $\mathcal{BCR}(X)$. The **resolvent set** of $T \in \mathcal{CR}(X)$ is defined by

$$\rho(T) = \{\lambda \in \mathbb{C} \colon \lambda I - T \text{ is injective and surjective}\}$$

and $\sigma(T) = \mathbb{C}\backslash\rho(T)$ is called the spectrum of T.

For two bounded linear relations T and S, we say that S commutes with T, if $ST \subset TS$ and T and S commute mutually if $TS = ST$.

We begin by recalling some basics results from the theory of linear relations in normed linear spaces.

Lemma 1.1 [10, Proposition I.3.1]. *Let X be a normed space, $T \in \mathcal{LR}(X)$ and M be a subset in X. Then*:

1) $TT^{-1}(M) = M \cap R(T) + T(0)$,

2) $T^{-1}T(M) = M \cap D(T) + T^{-1}(0)$.

Lemma 1.2 [10, Definition II.5.1 and Theorem III.4.2], [20, Lemma 3.1]. *Let X be a normed and $T \in \mathcal{LR}(X)$. Then*

1. *If T is closed, then $T(0)$ and $N(T)$ are closed subspaces.*
2. *If T is continuous and $D(T)$ and $T(0)$ are closed, then T is closed.*
3. *If T is closed, then T is continuous if and only if $D(T)$ is closed.*
4. *If T is closed with $\rho(T) \neq \varnothing$, then T^n is closed for all $n \in \mathbb{N}$.*

Lemma 1.3 [10, Corollary II.3.13], [3, Lemma 14]. *Let X be a Banach space and $T, S \in \in \mathcal{LR}(X)$. Then*

1. *If T and S are continuous with $T(0) \subset D(S)$, then ST is continuous.*
2. *If T and S are closed and continuous with $S(0) \subset T(0)$ and $D(T) \subset D(S)$, then $T + S$ is closed.*

The **adjoint** of a linear relation T is defined by $G(T^*) := G(-T^{-1})^{\perp}$, that is,

$$(y', x') \in G(T^*) \quad \text{if and only if, for all} \quad (x, y) \in G(T), \quad y'y = x'x.$$

Lemma 1.4 [10, Chapter III], [13, Lemma 9]. *Let X be a normed space and $T \in \mathcal{LR}(X)$. Then*

1. *$N(T^*) = R(T)^{\perp}$ and $N(\overline{T}) = R(T^*)^{\top}$.*
2. *If T is closed, then T is open if and only if $R(T)$ is closed if and only if $R(T^*)$ is closed.*
3. *If T is closed and bounded relation, then $T^{*n} = T^{n*}$ for all $n \in \mathbb{N}$.*

Lemma 1.5 [12, Lemma 2.3]. *Let T be an everywhere defined linear relation in a Banach space X and let M be a closed subspace of X such that $T(0) \subset M$. Then $T^{-1}(M)$ is closed.*

***Remark* 1.1.** If T has a nonempty resolvent set, then $N(T^n) \cap T^m(0) = \{0\}$ for all $m, n \in \mathbb{N}$, which implies that $R_c(T) = \{0\}$. The proof of such implication can be found in [29, Lemma 6.1].

Lemma 1.6 [5, Lemma 2.4]. *Let X be a Banach space and A, $B \in \mathcal{LR}(X)$. Let A self-commutes, i.e., $A(D(A)) \subset D(A)$ and B is an everywhere defined operator which commutes mutually with A. Then $R_c(A) = \{0\}$ if and only if $R_c(A + B) = \{0\}$.*

***Remark* 1.2.** The equivalence given in Lemma 1.6, holds if the relation A and the operator B verify just the inclusion $BA \subset AB$ instead of A and B commute mutually.

Lemma 1.7 [4, Lemma 20]. *Let X be a Banach space and $T \in \mathcal{LR}(X)$. Then*

1. *If $\lambda \in \mathbb{K}^*$, $N(T - \lambda) \subset R^{\infty}(T)$.*

2. *If* $\lambda, \mu \in \mathbb{K}$ *are distinct, then* $N\big((T-\lambda)^n\big) \subset R^\infty(T-\mu)$ *for all* $n \in \mathbb{N}$.

3. *If there exists* $d \in \mathbb{N}$ *such that* $N(T) \cap R(T^d) = N(T) \cap R(T^{n+d})$ *for all* $n \in \mathbb{N}$, *then* $T(D(T) \cap R^\infty(T)) = R^\infty(T)$.

4. *If* $N(T) \subset R^\infty(T)$ *or* $\dim N(T) < \infty$, *then* $T(D(T) \cap R^\infty(T)) = R^\infty(T)$.

Recall that $T \in \mathcal{CR}(X)$ is said to be **upper semi-Fredholm** if its range is closed and has finite dimensional null space, **lower semi-Fredholm** if its range is finite codimensional. T is said **Fredholm** if it is both upper and lower semi-Fredholm. The set of upper semi-Fredholm and lower semi-Fredholm are denoted respectively by $R\phi_+(X)$ and $R\phi_-(X)$.

Lemma 1.8 [6, Lemma 3.5]**.** *Let* X *be a Banach space,* $T \in \mathcal{LR}(X)$ *and* $n \in \mathbb{N}$. *Then*

1. *If* $T \in R\phi_+(X)$, *then* $T^n \in R\phi_+(X)$.

2. *If* $T \in R\phi_-(X)$ *such that* $\overline{D(T)} = X$ *and* $\rho(T) \neq \varnothing$, *then* $T^n \in R\phi_-(X)$.

In the following, we gather some properties concerning the sum and product of two commuting linear relations T and N where T is everywhere defined and N is a bounded operator which verifies $TN = NT + T(0)$. We begin by recalling this useful lemma.

Proposition 1.1 [10, Proposition I.4.2]**.** *Let* X *be a normed spaces,* T, S *and* $R \in \mathcal{LR}(X)$. *Then*

1) $(R+S)Tx \subset (RT+ST)x$ *for all* $x \in X$, *with equality if* T *is single valued,*

2) $TR + TS \subset T(R+S)$, *with equality if* $D(T)$ *contains the ranges of both* R *and* S.

Lemma 1.9. *Let* X *be a Banach space,* $A, B \in \mathcal{LR}(X)$ *be everywhere defined and* C *be a bounded operator in* X. *Suppose that* A *and* B *commute mutually,* $A(0) \subset B(0)$ *and* $AC = CA + A(0)$. *Then*

$$A^n(B+C) = (B+C)A^n = A^nB + A^nC \quad \textit{for all} \quad n \in \mathbb{N}.$$

Proof. For $n = 1$, by applying Proposition 1.1, we get $A(B+C) = AB + AC$ and $(B+C)A \subset BA + CA \subset AB + AC = A(B+C)$. Now, let $(x, y) \in A(B+C)$, then $y \in ABx + CAx + A(0) \subset BAx + CAx + BA(0) = BAx + CAx$. This implies that $y \in (B+C)y_1 + C(y_2 - y_1)$ for some y_1, $y_2 \in Ax$. So, $y \in (B+C)Ax + CA(0)$ and as $CA(0) \subset A(0) \subset (B+C)(0)$, then $(x, y) \in (B+C)A$. Thus, $A(B+C) = (B+C)A$. Assume now that the desired equality holds for some integer n. Then $A^{n+1}(B+C) = AA^n(B+C) = A(B+C)A^n = (B+C)AA^n = (B+C)A^{n+1}$. Therefore, for all $n \in \mathbb{N}$, A^n and $B+C$ commute mutually.

Lemma 1.10. *Let* X *be a Banach space,* $T \in \mathcal{LR}(X)$ *be everywhere defined and* N *be a bounded operator in* X *such that* $TN = NT + T(0)$. *Then*:

1) *for all* $k, n \in \mathbb{N}$, $N^nT^k \subset T^kN^n$,

2) *for all* $k, n \in \mathbb{N}$, $T^kN^n = N^nT^k + T^k(0)$,

3) *for all* $n \in \mathbb{N}$, $(T+N)N^n = N^n(T+N) + T(0)$,

4) *for all* $k, n \in \mathbb{N}$, $(T+N)^nT^k = T^k(T+N)^n$,

5) *for all* $n \in \mathbb{N}$, $(T+N)^n(0) = T^n(0)$.

Proof. 1. The first assertion is evident since N commutes with T.

2. Let $k = 1$. For $n = 0, 1$, the equality is given. Let $n \geq 1$ and suppose that $TN^n = N^nT + T(0)$. Then $TN^{n+1} = [N^nT + T(0)]N = N^nTN + T(0) = N^n[NT + T(0)] + T(0) = N^{n+1}T + T(0)$, since $NT(0) \subset T(0)$. Hence, for all $n \in \mathbb{N}$, $TN^n = N^nT + T(0)$. Now, for $k \geq 1$ and $n \in \mathbb{N}$, suppose that $T^kN^n = N^nT^k + T^k(0)$. Then $T^{k+1}N = T[N^nT^k + T^k(0)] = TN^nT^k + T^{k+1}(0) = [N^nT + T(0)]T^k + T^{k+1}(0) \subset N^nT^{k+1} + T^{k+1}(0) \subset T^{k+1}N^n + T^{k+1}(0) = T^{k+1}N^n$. Thus, $T^{k+1}N^n = N^nT^{k+1} + T^{k+1}(0)$.

3. Let $n \in \mathbb{N}$. Then $(T+N)N^n = TN^n + N^{n+1} = N^nT + T(0) + N^{n+1} = N^n(T+N) + T(0)$.

4. The result follows immediately from Lemma 1.9.

5. The equality is obtained by induction, since T and $T+N$ commute mutually.

Proposition 1.2. *Let X be a Banach space, $T \in \mathcal{LR}(X)$ be everywhere defined and N be a bounded operator in X such that $TN = NT + T(0)$. Then*:

1) $(T+N)^n \subset \sum_{t=0}^{n} C_n^t N^t T^{n-t}$ *for all* $n \in \mathbb{N}$,

2) $T^n - N^n = \left(\sum_{t=0}^{n-1} T^{n-1-t} N^t\right)(T-N) = (T-N)\left(\sum_{t=0}^{n-1} T^{n-1-t} N^t\right)$ *for all* $n \geq 1$.

Proof. 1. For $n = 0, 1$, the inclusion is obvious. Suppose that the result holds for some integer $n \geq 1$. Then

$$(T+N)^{n+1} \subset \sum_{t=0}^{n} C_n^t (T+N) N^t T^{n-t} \subset \sum_{t=0}^{n} C_n^t \big[N^t(T+N) + T(0)\big] T^{n-t} \subset$$

$$\subset \sum_{t=0}^{n} C_n^t N^t T^{n-t}(T+N) + T(0) \subset \sum_{t=0}^{n} C_n^t N^t T^{n-t} T + C_n^t N^t \big[N T^{n-t} + T^{n-t}(0)\big] + T(0) \subset$$

$$\subset T^{n+1} + \sum_{t=1}^{n} (C_n^t + C_n^{t-1}) N^t T^{n+1-t} + N^{n+1} + T^n(0) \subset \sum_{t=0}^{n+1} C_{n+1}^t N^t T^{n+1-t}.$$

Thus, the required equality is verified for all $n \in \mathbb{N}$.

2. The equality holds for $n = 1$. To get the desired result, we shall prove by induction, that, for all $n \geq 2$,

$$T(T^{n-1} + T^{n-2}N + \ldots + TN^{n-2}) = (T^{n-1} + T^{n-2}N + \ldots + TN^{n-2})T. \tag{1.1}$$

The case $n = 2$ is evident. Suppose that (1.1) holds for some $n \geq 2$. Then, since

$$T^n + T^{n-1}N + \ldots + TN^{n-1} = T(T^{n-1} + T^{n-2}N + \ldots + TN^{n-2} + N^{n-1}),$$

according to Lemma 1.9, with $A = T$, $B = T^{n-1} + T^{n-2}N + \ldots + TN^{n-2}$ and $C = N^{n-1}$, we get

$$T^n + T^{n-1}N + \ldots + TN^{n-1} = (T^{n-1} + \ldots + TN^{n-2} + N^{n-1})T.$$

It follows that

$$T(T^n + T^{n-1}N + \ldots + TN^{n-1}) = (T^n + T^{n-1}N + \ldots + TN^{n-1})T.$$

Therefore,

$$\left(\sum_{t=0}^{n-1} T^{n-1-t} N^t\right)(T-N) =$$

$$= (T^{n-1} + T^{n-2}N + \ldots + N^{n-1})T - (T^{n-1} + T^{n-2}N + \ldots + N^{n-1})N =$$

$$= T^n + T^{n-1}N + \ldots + TN^{n-1} - (T^{n-1}N + T^{n-2}N^2 + \ldots + N^n) =$$

$$= T^n - N^n + T^{n-1}(0) = T^n - N^n.$$

At present, proving the second inclusion. The result is obvious for $n = 1$. Let $n \geq 2$, we can see from (1.1) that

$$(T^{n-1} + T^{n-2}N + \ldots + TN^{n-2})T =$$
$$= T^n + T^{n-1}N + \ldots + T^2N^{n-2} =$$
$$= T^n + T^{n-2}[NT + T(0)] + \ldots + T[N^{n-2}T + T(0)] =$$
$$= T^n + T^{n-2}NT + \ldots + TN^{n-2}T + T^{n-1}(0) =$$
$$= T^n + T^{n-2}NT + \ldots + TN^{n-2}T.$$

Moreover, set $A = T - N$, $B = T^{n-1} + T^{n-2}N + \ldots + TN^{n-2}$ and $C = N^{n-1}$. Then $\left(\sum_{t=0}^{n-1} T^{n-1-t}N^t\right)(T - N) = (B + C)A$, where $A(0) \subset B(0)$, $AC = CA + A(0)$ and

$$AB = (T - N)(T^{n-1} + T^{n-2}N + \ldots + TN^{n-2}) =$$
$$= (T - N)T^{n-1} + (T - N)T^{n-2}N + \ldots + (T - N)TN^{n-2} =$$
$$= T^n - T^{n-1}N + T^{n-2}[N(T - N) + T(0)] + \ldots + T[N^{n-2}(T - N) + T(0)] =$$
$$= (T^{n-1} + T^{n-2}N + \ldots + TN^{n-2})T - (T^{n-1} + T^{n-2}N + \ldots + TN^{n-2})N =$$
$$= BT - BN = BA.$$

Therefore, Lemma 1.9 implies that, for all $n \geq 1$, $\sum_{t=0}^{n-1} T^{n-1-t}N^t$ and $T - N$ commute mutually.

***Remark* 1.3.** Let T is a bounded and closed linear relation with $\rho(T) \neq \varnothing$ and N a bounded operator such that $TN = NT + T(0)$. Then $(T + N)^n$ is closed for each $n \in \mathbb{N}$. Indeed, $\rho(T) \neq \varnothing$ ensures that T^n is closed for all $n \in \mathbb{N}$ and, hence, by Lemma 1.10, $(T + N)^n(0) = T^n(0)$ is closed. Moreover, according to Lemmas 1.2 and 1.3, $T + N$ is continuous and since $(T + N)^n(0) \subset$ $\subset D(T + N)$, then $(T + N)^n$ is also continuous with closed domain and such that $(T + N)^n(0)$ is closed. Thus, $(T + N)^n$ is a closed linear relation.

2. Ascent and descent perturbation. In [11, Corollary 4.1], it was shown that if F is a bounded operator, then $\sigma_{des}(T + F) = \sigma_{des}(T)$ for every $T \in K_F$ if and only if F^k is of finite rank for some integer k, where $K_F = \{T \in \mathcal{LR}(X) \colon D(T) = X, TF = FT \text{ and } T(0) \subset N(T)\}$. Later, in [12, Corollary 4.1], analogous perturbation result was proved for ascent spectrum, that is, $\sigma_{asc}(T + F) = \sigma_{asc}(T)$ for every $T \in K'_F$ if and only if F^k is of finite rank for some integer k, where $K'_F = \{T \in \mathcal{CR}(X) \colon D(T) = X, TF = FT, \rho(T) \neq \varnothing \text{ and } \rho(T + F) \neq \varnothing\}$. The above results includes strictly perturbations by nilpotent operators since every nilpotent operator is a power finite rank operator.

In this section, we give similar invariance results of ascent and descent spectrum under commuting nilpotent operators perturbations, without the assumptions $T(0) \subset N(T)$ for descent spectrum, and $\rho(T) \neq \varnothing$ and $\rho(T + F) \neq \varnothing$ for ascent spectrum.

Before this, we quote some properties of ascent and descent of linear relations needed in the sequel. Recall that, for $T \in \mathcal{LR}(X)$, we write $n(T) = \dim N(T)$ and $d(T) = \operatorname{codim} R(T)$ and the index is the quantity $i(T) = n(T) - d(T)$ provided that $n(T)$ and $d(T)$ are not both infinite. The kernels and the ranges of the iterates T^n, $n \in \mathbb{N}$, form two increasing and decreasing chains respectively, i.e., the chain of kernels $N(T^0) = 0 \subset N(T) \subset N(T^2) \subset \ldots$ and the chain of ranges $R(T^0) = X \supset R(T) \supset R(T^2) \supset \ldots$. The ascent and the descent of a linear relation T are respectively defined by

$$asc(T) := \inf\big\{n \in \mathbb{N} \colon N(T^n) = N(T^{n+1})\big\} \quad \text{and} \quad des(T) := \inf\big\{n \in \mathbb{N} \colon R(T^n) = R(T^{n+1})\big\}.$$

The corresponding spectra are defined by $\sigma_{asc}(T) := \big\{\lambda \in \mathbb{C} \colon asc(T-\lambda) = \infty\big\}$ and $\sigma_{des}(T) := := \big\{\lambda \in \mathbb{C} \colon des(T-\lambda) = \infty\big\}$.

In the following lemma, we recall some results relating nullity and defect to ascent and descent.

Lemma 2.1 [30]. *Let X be a Banach space and $T \in \mathcal{LR}(X)$. Then*

1. *If $R_c(T) = \{0\}$ and $asc(T) \le p$ for some $p \in \mathbb{N}$, then $N(T^k) \cap R(T^p) = \{0\}$ for all $k \in \mathbb{N}$.*
2. *If $R_c(T) = \{0\}$ and $asc(T) < \infty$, then $n(T) \le d(T)$.*
3. *If $D(T) = X$ and $des(T) < \infty$, then $d(T) \le n(T)$.*
4. *If $R_c(T) = \{0\}$, $asc(T) < \infty$ and $n(T) = d(T)$, then $asc(T) = des(T)$.*

Let T be a linear relation in a linear space X. For any $\lambda \in \mathbb{K}$ the notation $T - \lambda$ stands for $T - \lambda I$, i.e., $T - \lambda = \big\{(x, y - \lambda x) \colon x, y \in G(T)\big\}$.

The following proposition is important for future use.

Proposition 2.1. *Let X be a Banach space, $T \in \mathcal{CR}(X)$ be everywhere defined and N be a nilpotent operator of degree m such that $TN = NT + T(0)$. If $asc(T) = p < \infty$, then $asc(T+N) \le \le m + p - 1$.*

Proof. Let $x \in N\big((T+N)^{m+p}\big)$. Then $(T+N)^{m+p}(x) = (T+N)^{m+p}(0) = T^{m+p}(0)$. However, using Proposition 1.2, we obtain

$$(T+N)^{m+p}(x) \subset \sum_{t=0}^{m-1} C_{m+p}^t N^t T^{m+p-t}(x) +$$

$$+ \sum_{t=m}^{m+p} C_{m+p}^t N^t T^{m+p-t}(x) \subset \sum_{t=0}^{m-1} C_{m+p}^t N^t T^{m+p-t}(x).$$

Thus,

$$T^{m+p}(0) = (T+N)^{m+p}(x) \subset \sum_{t=0}^{m-1} C_{m+p}^t T^{m+p-t} N^t(x). \tag{2.1}$$

It follows that

$$0 \in N^{m-1}(T+N)^{m+p}(x) \subset \sum_{t=0}^{m-1} C_{m+p}^t T^{m+p-t} N^{m-1+t}(x) \subset$$

$$\subset T^{m+p} N^{m-1}(x) + T^{m+p-1}(0) \subset T^{m+p} N^{m-1}(x).$$

Hence, $N^{m-1}(x) \in N(T^{m+p}) = N(T^{m+p-1}) = N(T^p)$.

Similarly,

$$0 \in N^{m-2}(T+N)^{m+p}(x) \subset \sum_{t=0}^{m-1} C_{m+p}^t T^{m+p-t} N^{m-2+t}(x) \subset$$

$$\subset T^{m+p}N^{m-2}(x)+C^1_{m+p}T^{m+p-1}N^{m-1}(x)+T^{m+p-2}(0)\subset$$

$$\subset T^{m+p}N^{m-2}(x)+C^1_{m+p}T^{m+p-1}N^{m-1}(x).$$

Since $N^{m-1}(x)\in N(T^{m+p-1})$, then $N^{m-2}(x)\in N(T^{m+p})=N(T^p)$. Using the same technique, we can prove that, for all $1\le k\le m-1$,

$$N^{m-k}(x)\in N(T^p)=N(T^{p+k})=N(T^{p+k-1}),$$

and, as by (2.1),

$$0\in T^{m+p}(x)+\sum_{t=1}^{m-1}C^t_{m+p}T^{m+p-t}N^t(x)=T^{m+p}(x)+T^{m+p-1}(0)=T^{m+p}(x),$$

then $x\in N(T^{m+p})=N(T^{m+p-1})=N(T^p)$. Now, since

$$(T+N)^{m+p-1}(x)\subset T^{m+p-1}(x)+\sum_{t=1}^{m-1}C^t_{m+p-1}T^{m+p-1-t}N^t(x)\subset$$

$$\subset T^{m+p-1}(x)+T^{m+p-2}(0)\subset T^{m+p-1}(x)\subset(T+N)^{m+p-1}(0),$$

then $x\in N\big((T+N)^{m+p-1}\big)$ and hence $asc(T+N)\le m+p-1$.

In a similar way, we state the following proposition.

Proposition 2.2. *Let X be a Banach space, $T\in\mathcal{CR}(X)$ be everywhere defined and N be a nilpotent operator of degree m such that $TN=NT+T(0)$. If $des(T)=p<\infty$, then $des(T+N)\le$ $\le m+p-1$.*

Proof. Assume without loss of generality that $p\ge m$. We first claim that, for all $n\ge m+p-1$, $R\big((T+N)^n\big)\subset R(T^p)$. For this, let $n\ge m+p-1$ and $y\in R((T+N)^n)$. Then there exists $x\in X$ such that $y\in(T+N)^n(x)$. Furthermore, we have

$$(T+N)^n(x)\subset\sum_{t=0}^{m-1}C^t_nT^{n-t}N^t(x)\subset$$

$$\subset T^n(x)+C^1_nT^{n-1}N(x)+C^2_nT^{n-2}N^2(x)+\ldots+C^{m-1}_nT^{n-m+1}N^{m-1}(x)\subset$$

$$\subset R(T^n)+R(T^{n-1})+\ldots+R(T^{n-m+1}).$$

It follows that $y\in R(T^p)$ and so $R((T+N)^n)\subset R(T^p)$.

Now, let T_p be the restriction of T to $R(T^p)$ and N_p be the restriction of N to $R(T^p)$. We will show that T_p+N_p is a surjective relation from $R(T^p)$ to $R(T^p)$. Indeed, as N commutes with T^p, then $T_p+N_p\colon R(T^p)\longrightarrow R(T^p)$. Let $y\in R(T^p)$ and $n\ge m+p-1$, then $y\in T^{n+1}x$ for some $x\in X$. Proposition 1.2 entails that

$$T^{n+1}(x)-(-N)^{n+1}(x)=(T+N)\left(\sum_{t=0}^{m-1}(-1)^tT^{n-t}N^t\right)(x).$$

However, as N commutes with T, then, for all $0\le t\le m-1$, $T^{n-t}N^t\big(R(T^p)\big)\subset R(T^p)$. This implies that $y\in(T+N)\big(R(T^p)\big)=R(T_p+N_p)$. Thus, T_p+N_p is surjective. It follows that

$R(T^p) \subset R^\infty(T+N)$. We have proved that, for all $n \geq m+p-1$, $R\big((T+N)^n\big) = R(T^p)$. Hence, $des(T+N) \leq m+p-1$.

As an immediate consequence of Propositions 2.1 and 2.2, we obtain the main result of this section.

Theorem 2.1. *Let X be a Banach space, $T \in \mathcal{CR}(X)$ be everywhere defined and N be a nilpotent operator such that $TN = NT + T(0)$. Then*

$$\sigma_{asc}(T) = \sigma_{asc}(T+N) \quad \textit{and} \quad \sigma_{des}(T) = \sigma_{des}(T+N).$$

3. Regular relations. This section is devoted to study the stability of regular linear relations under perturbation by commuting nilpotent operators.

Lemma 3.1 [25, Lemma 2.7]**.** *Let X be a vector space and T be a linear relation in X. The following properties are equivalent*:

1) $N(T) \subset R(T^m)$ *for all* $m \in \mathbb{N}$,

2) $N(T^n) \subset R(T)$ *for all* $m \in \mathbb{N}$,

3) $N(T^n) \subset R(T^m)$ *for all* $n,\ m \in \mathbb{N}$.

Definition 3.1. *A linear relation $T \in \mathcal{LR}(X)$ is called **regular** if $R(T)$ is closed, and T verifies one of the equivalent conditions of Lemma* 3.1.

Proposition 3.1 [4, Proposition 12]**.** *Let X be a Banach space and $T \in \mathcal{LR}(X)$. If T is regular, then, for each $n \in \mathbb{N}$, we have*:

1) $R(T^{*n}) = N(T^n)^\perp$,

2) $R(T^n) = N(T^{*n})^\top$.

The following proposition yields the small perturbation result of regular linear relations.

Proposition 3.2 [4, Theorem 23]**.** *Let X be a Banach space and $T \in \mathcal{LR}(X)$. If T is a regular relation, then there exists $\epsilon > 0$ such that, if $|\lambda| < \epsilon$, then $T - \lambda$ is regular.*

Recall that a linear relation T, defined on a Banach space X is said to be **bounded below** if it is injective and open. Note that if T is closed, then T is bounded below if it is injective and $R(T)$ is closed.

Next, we exhibit the stability of regular linear relations by perturbation by a commuting nilpotent operator. For this, we need to prove the following characterization.

Proposition 3.3. *Let X be a Banach space and $T \in \mathcal{BCR}(X)$. Then a relation T is regular if and only if there exists M a closed subspace in X such that $TM = M$ and $\widetilde{T} \colon X/M \longrightarrow X/M$ is a bounded below operator.*

Proof. By virtue of Proposition 3.1, $R(T^n)$ is closed for every $n \in \mathbb{N}$ so that $R^\infty(T)$ is closed. Further we have from Lemma 1.7 that $T(R^\infty(T)) = R^\infty(T)$. Let $\widetilde{T} \colon X/R^\infty(T) \longrightarrow$ $\longrightarrow X/R^\infty(T)$. We claim that $\widetilde{T}$ is a bounded below operator. In fact, $\widetilde{T}(\overline{0}) = \{\overline{y} \colon y \in T(0)\} =$ $= \widetilde{0}$, since $T(0) \subset R^\infty(T)$. Furthermore, let $\overline{x} \in X/R^\infty(T)$ such that $\widetilde{T}\overline{x} = \overline{0}$. Then, for all $y \in Tx$, $\overline{y} = \overline{0}$ which is equivalent to $y \in R^\infty(T)$. As $x \in T^{-1}y$ and $T(R^\infty(T)) = R^\infty(T)$, then $x \in T^{-1}T(R^\infty(T)) = R^\infty(T) + N(T) = R^\infty(T)$ since T is regular. Hence, $\widetilde{T}$ is injective. However, $R(\widetilde{T}) = R(T)/R^\infty(T)$ and $R(T)$ and $R^\infty(T)$ are closed, then $R(\widetilde{T})$ is closed and as $\widetilde{T}$ is closed, then it is bounded below. Conversely, if $x \in N(T)$, then $0 \in Tx$, which implies that $\overline{x} \in N(\widetilde{T}) = \{\overline{0}\}$. So, $x \in M \subset R^\infty(T)$, since $TM = M$. Now, consider the canonical projection $P \colon X \longrightarrow X/M$. Then $R(T) = P^{-1}(R(\widetilde{T}))$. Lemma 1.5 entails that $R(T)$ is closed and hence the relation T is regular.

We state now the main theorem of this section.

Theorem 3.1. *Let X be a Banach space, $T \in \mathcal{BCR}(X)$ and N be a nilpotent operator such that $TN = NT + T(0)$. If T is a regular linear relation, then $T + N$ is regular.*

Proof. As T is regular, then by Proposition 3.3, there exists a closed subspace which is $R^\infty(T)$ such that $T(R^\infty(T)) = R^\infty(T), \widetilde{T}\colon X/R^\infty(T) \longrightarrow X/R^\infty(T)$ is a bounded below operator. However, we have $N\big(R^\infty(T)\big) \subset R^\infty(T)$ by Lemma 1.10. Given, the operator induced by N, $\widetilde{N}\colon X/R^\infty(T) \longrightarrow X/R^\infty(T)$. Then $\widetilde{N}$ is a nilpotent operator and so, using [24, Theorem 6], we get $\widetilde{T+N} = \widetilde{T} + \widetilde{N}$ is a bounded below operator. Now, we claim that $(T+N)\big(R^\infty(T)\big) = R^\infty(T)$. Indeed, the first inclusion is satisfied. For the second, let $y \in R^\infty(T)$. Then, for $n > m$, where m be the degree of nilpotency of N, we have $y \in T^n x_n$ for some $x_n \in R^\infty(T)$. Proposition 1.10 ensures that

$$T^n(x_n) - (-N)^n(x_n) = (T+N)\left(\sum_{t=0}^{n-1}(-1)^t T^{n-1-t}N^t\right)(x_n).$$

However, since, for all $k, p \in \mathbb{N}$, $T^k N^p\big(R^\infty(T)\big) \subset T^k\big(R^\infty(T)\big) = R^\infty(T)$, then $y \in (T + N)\big(R^\infty(T)\big)$. We have shown that $(T+N)\big(R^\infty(T)\big) = R^\infty(T)$. Thus, using Proposition 3.3, $T + N$ is a regular relation.

Here, we give a perturbation result for bounded below linear relations by commuting nilpotent operator.

Corollary **3.1.** *Let X be a Banach space, $T \in \mathcal{BCR}(X)$ and N be a nilpotent operator such that $TN = NT + T(0)$. If T is a bounded below relation, then $T + N$ is bounded below too.*

Proof. Using Theorem 3.1, $T + N$ is regular, in particular $R(T+N)$ is closed. Now, reasoning as in the proof of Proposition 2.1, we will show that $T + N$ is injective. Indeed, let $x \in N(T+N)$, then $(T+N)(x) = T(0)$. Applying N^{m-1}, where m is the degree of nilpotency of the operator N, we obtain $N^{m-1}(T+N)(x) = N^{m-1}T(0) \subset T(0)$. So, $TN^{m-1}(x) = T(0)$, which implies that $N^{m-1}(x) = 0$. Similarly, we can prove that, for all $1 \le k \le m-1$, $N^{m-k}(x) = 0$. However, as $(T+N)(x) = T(0)$, it follows that $T(x) = T(0)$. Thus, $x = 0$ and hence $T + N$ is bounded below.

4. Properties of left and right Drazin invertible relations. The this section deals with left and right Drazin invertible linear relations introduced by A. Ghorbel and M. Mnif in [21]. We focus on the study of some of their properties. We begin by recalling the definitions.

Definition 4.1. *Let X be a Banach space and $T \in \mathcal{BCR}(X)$. We say that T is* ***left Drazin invertible*** *if there exists $p \in \mathbb{N}$ such that $asc(T) = p < \infty$ and $R(T^{p+1})$ is closed,* ***right Drazin invertible*** *if there exists $q \in \mathbb{N}$ such that $des(T) = q < \infty$ and $R(T^q)$ is closed. The corresponding left Drazin and right Drazin spectra of T, are defined respectively by*

$$\sigma_{ld}(T) = \big\{\lambda \in \sigma_a(T), T - \lambda \textit{ is not left Drazin invertible}\big\}$$

and

$$\sigma_{rd}(T) = \big\{\lambda \in \sigma_s(T), T - \lambda \textit{ is not right Drazin invertible}\big\},$$

where $\sigma_a(T)$ and $\sigma_s(T)$ denote the approximate point spectrum and the surjective spectrum of T, respectively,

$$\sigma_a(T) = \big\{\lambda \in \mathbb{C}, T - \lambda \textit{ is not bounded below }\big\}$$

and

$$\sigma_s(T) = \big\{\lambda \in \mathbb{C}, T - \lambda \textit{ is not surjective }\big\}.$$

Here, we introduce the notions of quasinilpotent part and analytical core of linear relations and collect some of their basic properties which are important for what will follow.

Definition 4.2. *Let X be a Banach and $T \in \mathcal{LR}(X)$. The* ***quasinilpotent part*** *of T, denoted by $H_0(T)$, is defined as the set of all $x \in X$, for which there exists a sequence $(u_n)_n \subset X$, satisfying*

$$x = x_0, \qquad x_{n+1} \in Tx_n \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \lim_{n \longrightarrow \infty} \|x_n\|^{\frac{1}{n}} = 0.$$

The ***algebraic core*** *of T, denoted by $C(T)$, is defined to be the greatest subspace M of X for which $T(M) = M$. It is clear that $C(T) \subseteq R^{\infty}(T)$, and the* ***analytical core*** *of T, denoted by $K(T)$, is defined as the set of all $x \in X$, for which there exists $a > 0$ and a sequence $(u_n)_n \subset X$ satisfying*

$$x = u_0, u_n \in Tu_{n+1} \text{ and } dis(u_n, T(0) \cap N(T)) \leq a^n dis(x, T(0) \cap N(T)) \text{ for all } n \in \mathbb{N}.$$

Lemma 4.1 [28, Lemmas 2.3, 2.8, 3.2 and Proposition 3.3], [26, Theorem 2.8]**.** *Let X be a Banach space and $T \in \mathcal{BCR}(X)$. Then*

1. *If there exists $m \in \mathbb{N}$ such that $N(T) \cap R(T^m) = N(T) \cap R(T^{m+k})$ $\forall k \in \mathbb{N}^*$, then $C(T) = R^{\infty}(T)$.*
2. *If $C(T)$ is closed, then $C(T) = K(T)$.*
3. *$N(T^j) \subset H_0(T)$ $\forall j \in \mathbb{N}$.*
4. *If $\rho(T) \neq \varnothing$, then $H_0(T) \subseteq K(T^*)^{\top}$.*

Definition 4.3. *If T is a bounded linear relation on a Banach space X, then, for each nonnegative integer n, T induces a linear transformation from the vector space $R(T^n)/R(T^{n+1})$ to the space $R(T^{n+1})/R(T^{n+2})$. Let $k_n(T)$ be the dimension of the null space of the induced map and $k_{-1}(T) = \infty$.*

Lemma 4.2 [7, Proposition 3.1]**.** *Let X be a Banach space and $T \in \mathcal{LR}(X)$ be bounded. Then, for each nonnegative integer n,*

$$k_n(T) = \dim(N(T) \cap R(T^n))/(N(T) \cap R(T^{n+1})).$$

Equivalently,

$$k_n(T) = \dim(R(T) + N(T^{n+1}))/(R(T) + N(T^n)).$$

The following lemma is useful to the proof of the coming propositions.

Lemma 4.3 [7, Lemma 4.1]**.** *Let X be a Banach space, $T \in \mathcal{BCR}(X)$ and $d \in \mathbb{N}$ such that $\rho(T) \neq \varnothing$ and $k_n(T) < \infty$ for every $n \geq d$. Then the following statements are equivalent*:

1) *there exists $n_0 \geq d + 1$ such that $R(T^{n_0})$ is closed*,
2) *$R(T^n)$ is closed for every $n \geq d$*,
3) *$R(T^n) + N(T^m)$ is closed for all n, m with $n + m \geq d$.*

Remark **4.1.** According to Lemma 4.3, the hypothesis $\rho(T) \neq \varnothing$ is necessary to ensure that the relation T^n is closed for all $n \in \mathbb{N}$. Therefore, if the closure of T^n is verified for all n, then this hypothesis can be omitted.

Now, we are ready to state the first result of this section.

Proposition 4.1. *Let X be a Banach space and $T \in \mathcal{BCR}(X)$ with $\rho(T) \neq \varnothing$.*

1. *If T is left Drazin invertible, then there exists $d \in \mathbb{N}$ such that $H_0(T) = N(T^d)$ and hence $H_0(T)$ is closed.*
2. *If T is right Drazin invertible, then there exists $d \in \mathbb{N}$ such that $K(T) = R(T^d)$.*

Proof. 1. Since T is left Drazin invertible, then there exists d such that $asc(T) = d$ and $R(T^{d+1})$ is closed. The inclusion $N(T^d) \subset H_0(T)$ is given by Lemma 4.1. To get the reverse one, we will prove that $R^\infty(T^*) = K(T^*)$. In fact, as $asc(T) = d$, then $k_n(T) = 0$ for all $n \geq d$ and $N(T^d) = N(T^n)$ for all $n \geq d$. As T is closed and $\rho(T) \neq \varnothing$, then for all $n \geq d$, T^n is closed, which implies that $R(T^{d*})^\top = R(T^{n*})^\top$. However, since $R(T^{d+1})$ is closed, then Lemma 4.3 leads to $R(T^n)$ is closed for all $n \geq d$. Hence, using Lemma 1.4, we get that, for all $n \geq d$, $R(T^{*d}) = R(T^{d*}) = R(T^{n*}) = R(T^{*n})$. Thus, for all $n \geq d$, $N(T^*) \cap R(T^{*d}) = N(T^*) \cap R(T^{n*})$ and so $C(T^*) = R^\infty(T^*) = R(T^{*d})$ which is closed. This implies by Lemma 4.1, that $C(T^*) = K(T^*) = R^\infty(T^*)$. According to Lemma 4.1, we have

$$H_0(T) \subset K(T^*)^\perp = R^\infty(T^*)^\perp = R(T^{*d})^\perp = N(T^d).$$

2. As T is right Drazin invertible, then there exists d such that $R(T^d) = R(T^n)$ for all $n \geq d$ and $R(T^d)$ is closed. Lemma 4.1 ensures that $C(T) = R^\infty(T) = R(T^d)$ which is closed and hence $K(T) = C(T) = R(T^d)$.

In the second part of this section, we will prove that if T is left or right Drazin invertible then there exists $\epsilon > 0$ such that, for all $\mid \lambda \mid < \epsilon$, $R(T - \lambda)$ is closed.

Definition 4.4. *Let X be a Banach space and $T \in \mathcal{BCR}(X)$. We say that T is **strictly quasi-Fredholm of degree** $d \in \mathbb{N}$, if $k_n(T) = 0$ for all $n \geq d$, $k_{d-1}(T) \neq 0$ and $R(T^{d+1})$ is closed.*

***Remark* 4.2.** Clearly every left or right Drazin invertible relation is strictly quasi-Fredholm relation.

Proposition 4.2. *Let X be a Banach space and $T \in \mathcal{BCR}(X)$. If T is strictly quasi-Fredholm of degree d, then there exists $\epsilon > 0$ such that, for $0 <\mid \lambda \mid < \epsilon$, $R(T - \lambda)$ is closed.*

Proof. Let T_d be the restriction of T to $R(T^d)$. We claim that T_d is a regular linear relation. First, note that $R(T_d) = R(T^{d+1})$ which is closed. Furthermore, we can show by induction on j that, for all $j \geq 1$, $N(T^j) \cap R(T^d) \subset R^\infty(T)$. In fact, since, for all $n \geq d$, $k_n(T) = 0$, then for $j = 1$ the inclusion is obvious by Lemma 4.2. Suppose that the inclusion holds for $j \geq 1$ and let $x \in N(T^{j+1}) \cap R(T^d)$ and $n \geq d$. Then, for some $y \in Tx$, $y \in N(T^j) \cap R(T^d) \subset R(T^{n+1})$. Thus, $y \in T^{n+1}z$ for some $z \in X$ and as $x \in T^{-1}y$, then $x \in T^{-1}T^{n+1}z \subset R(T^n) + N(T) \subset R(T^d) + N(T) \subset R(T^n)$. Hence, the desired inclusion is verified and so $N^\infty(T_d) \subset R(T^{d+1}) = R(T_d)$. Thus, T_d is regular. The use of Proposition 3.2 implies that there exists $\epsilon > 0$, $0 < |\lambda| < \epsilon$, $R(T_d - \lambda_d) = R((T-\lambda)T^d)$ is closed. As $T^d(0) \subset R((T-\lambda)T^d))$ then, by Lemma 1.5, $T^{-d}\big(R((T-\lambda)T^d)\big)$ is closed. Finally, we verify that $R(T-\lambda) = T^{-d}\big(R((T-\lambda)T^d)\big)$. Indeed, $T^{-d}\big(R((T-\lambda)T^d)\big) = T^{-d}T^d(R(T-\lambda)) = R(T-\lambda) + N(T^d)$. However, Lemma 1.7 leads to $N(T^d) \subset R(T-\lambda)$. Hence, $R(T-\lambda) = T^{-d}\big(R((T-\lambda)T^d)\big)$ which is closed.

***Corollary* 4.1.** *Let X be a Banach space and $T \in \mathcal{BCR}(X)$ with $\rho(T) \neq \varnothing$. If T is left or right Drazin invertible, then there exists $\epsilon > 0$ such that, for $\mid \lambda \mid < \epsilon$, $R(T - \lambda)$ is closed.*

Let $\operatorname{iso} M$ denotes the isolated points of the subset M. The last corollary allows us to get the following result.

Theorem 4.1. *Let X be a Banach space and $T \in \mathcal{BCR}(X)$ with $\rho(T) \neq \varnothing$.*

1. *If T is left Drazin invertible, then $0 \in \operatorname{iso} \sigma_a(T)$.*
2. *If T is right Drazin invertible, then $0 \in \operatorname{iso} \sigma_s(T)$.*

Proof. 1. Suppose that T is left Drazin invertible and $0 \notin \operatorname{iso} \sigma_a(T)$. Then there exists $\lambda_n \in \sigma_a(T)$ such that $\lim_{n\to\infty} \lambda_n = 0$. Hence, Proposition 4.2 leads to $R(T - \lambda_n)$ is closed and hence $\lambda_n \in \sigma_p(T)$ where $\sigma_p(T)$ denotes the injective spectrum of T. Let $T_1 = T_{R^\infty(T)}$ be the restriction of T to $R^\infty(T)$ and λ_{n1} be the restriction of $\lambda_n I$ to $R^\infty(T)$. Then, from Lemma 2.1, T_1 is injective. Further, combining Lemma 4.2 with Lemma 1.7, we get that T_1 is surjective and so invertible. However, $N(T_1-\lambda_{n1}) = N(T-\lambda_n)\cap R^\infty(T) = N(T-\lambda_n)$. So, $\lambda_{n1}\in\sigma_a(T_1)\subset\sigma(T_1)$ which is closed. This implies that $0\in\sigma(T_1)$ which contradict the fact that T_1 is invertible. Hence, $0\in \operatorname{iso}\sigma_a(T)$.

2. Suppose that $0\notin \operatorname{iso}\sigma_s(T) = \operatorname{iso}\sigma_a(T^*)$, then there exists $\lambda_n\in\sigma_a(T^*)$ such that $\lim_{n\to\infty}\lambda_n = 0$. As T is right Drazin invertible then we have, from Corollary 4.1, that $R(T-\lambda_n)$ is closed and so $R(T^*-\lambda_n)$ is closed. Hence, $\lambda_n\in\sigma_p(T^*)$. Let $S = T^*_{R^\infty(T^*)\cap D(T^*)}\longrightarrow R^\infty(T^*)$. We claim that S is an invertible operator. Evidently, $N(S) = N(T^*)\cap R^\infty(T^*) = \{0\}$. Thus, S is injective. Now the fact that T is right Drazin invertible implies that $asc(T^*) = d$. Furthermore, since $R_c(T^*) = \{0\}$, then [30, Lemma 4.4] leads to $N(T^*)\cap R(T^{*n+d})\simeq N(T^{*n+d+1})/N(T^{*n+d})$ and $N(T^*)\cap R(T^{*d})\simeq N(T^{*d+1})/N(T^{*d})$. Thus, $N(T^*)\cap R(T^{*d}) = N(T^*)\cap R(T^{*n+d})$. Lemma 1.7 ensures that $R(S) = T^*(D(T^*)\cap R^\infty(T^*)) = R^\infty(T^*)$ and so S is invertible. On the other hand, let λ_{n_1} be the restriction of $\lambda_n I$ to $R^\infty(T^*)$. Then $N(S-\lambda_{n_1}) = N(T^*-\lambda_n)\cap R^\infty(T^*) = N(T^*-\lambda_n)$ by Lemma 1.7. So, $\lambda_{n1}\in\sigma_a(S)\subset\sigma(S)$. As $\sigma(S)$ is closed, then $0\in\sigma(S)$ which is absurd.

5. Perturbation results for left and right Drazin invertible linear relations. In the remainder of this paper, we intend to set up a perturbation theorem of Drazin invertible linear relations under commuting nilpotent operators. For this, we begin by recalling the definition of such relations.

Definition 5.1. *Let X be a Banach space and $T\in\mathcal{LR}(X)$. The relation T is said to be* ***Drazin invertible of degree*** $k\in\mathbb{N}$ *if T is everywhere defined and there exists a bounded operator $T^D\in\mathcal{L}(\mathcal{X})$ such that*:

(i) $TT^D = T^DT + T(0)$,

(ii) $T^DTT^D = T^D$,

(iii) $T^{k+1}T^D = T^k + T^{k+1}(0)$.

T^D is called a Drazin inverse of T.

The Drazin spectrum of an everywhere defined linear relation T is defined by

$$\sigma_d(T) = \{\lambda\in\mathbb{C}, T-\lambda I \text{ is not Drazin invertible}\}.$$

Note that, if $\rho(T)\neq\varnothing$, then $T\in\mathcal{BCR}(X)$ is Drazin invertible if T is both left and right Drazin invertible which is equivalent to the fact that $asc(T)$ and $des(T)$ are finite (see [21, Proposition 3.5 and Theorem 3.3]). Hence,

$$\sigma_{ld}(T)\cup\sigma_{rd}(T) = \sigma_d(T).$$

In [21, Theorem 3.3], it was shown that, for $T\in\mathcal{BCR}(X)$, if there exist two closed subspaces M and N of X such that $X = M\oplus N$ and $T = T_M\oplus T_N$, where T_M is an invertible linear relation and T_N is a bounded nilpotent operator, then T is Drazin invertible. Through this, we may give an example of such relation. Consider the separable Hilbert space $l^2(\mathbb{N})$ and let $k\geq 2$. Define the following linear relation in $l^2(\mathbb{N})$:

$$T\colon (x_1,x_2,\ldots)\longmapsto (x_2+\ldots+x_k, x_3+\ldots+x_k,\ldots,x_k,0,0,x_{k+1},x_{k+2},\ldots)+\operatorname{span}(e_{k+1}).$$

Let $N = \operatorname{span}(e_i)_{i=1}^k$ and T_N is the following matrix with k zeros on its diagonal, that is, $T_N =$
$= \begin{pmatrix} 0 & 1 & \ldots & 1\\ 0 & 0 & \ldots & 1\\ \vdots & \vdots & \ldots & 1\\ \vdots & \vdots & \ldots & 0\end{pmatrix}$. Then T_N is a bounded nilpotent operator of degree k. Now, let $M =$

$= \operatorname{span}(e_i)_{i=k+1}^{\infty}$ and $T_M = L^{-1}$, where L is the left shift operator in M defined by

$$L\colon (x_{k+1}, x_{k+2}, \ldots) \longmapsto (x_{k+2}, x_{k+3}, \ldots).$$

Then $T_M \in \mathcal{BCR}(M)$ is an invertible linear relation and hence $T = T_M \oplus T_N$ is Drazin invertible of degree k.

Recently, the concept of semi-B-Browder and B-Browder linear relations was introduced by Mnif and Ghorbel [21] as follows. For a bounded and closed linear relation T defined on a Banach space X, we say that T is **upper** (resp., **lower**) **semi-B-Browder** if there exists $d \in \mathbb{N}$ such that $R(T^d)$ is closed and the restriction $T_d = T_{R(T^d)}$ is upper (resp., lower) semi-Fredholm with finite ascent (resp., finite descent). And T is called **B-Browder** if T is both upper and lower semi-B-Browder. For $T \in \mathcal{BCR}(X)$, let $\sigma_{ubb}(T), \sigma_{lbb}(T)$ and $\sigma_{bb}(T)$ denote respectively the upper semi-B-Browder, the lower semi-B-Browder and the B-Browder spectrum of T.

The following lemma gives another characterization of upper and lower semi-B-Browder linear relations.

Lemma 5.1. *Let X be a Banach space, $T \in \mathcal{BR}(X)$ and $d \in \mathbb{N}^*$. Let T_d denotes the restriction of T on $R(T^d)$, then*:

1) *$des(T)$ is finite if and only if $des(T_d)$ is finite,*

2) *if furthermore $R_c(T) = \{0\}$, then $asc(T)$ is finite if and only if $asc(T_d)$ is finite.*

Proof. 1. If $des(T_d) = q$, then $R(T^{d+q}) = R(T^{d+q+1})$ and so $des(T)$ is finite. Suppose now that $des(T) = q$. If $d \geq q$, then $R(T_d) = R(T^{d+1}) = R(T^{d+2}) = R(T_d^2)$ and if $d < q$, then $R(T_d^{q-d}) = R(T^q) = R(T^{q+1}) = R(T_d^{q-d+1})$. In both cases, $des(T_d)$ is finite.

2. The first implication is obvious. For the second one, suppose that $asc(T_d) = p$. Since $R_c(T) = \{0\}$, then according to [30, Lemma 4.4], $N(T^{d+p}) = N(T^{d+p+1})$ and so $asc(T)$ is finite.

***Remark* 5.1.** Thanks to Lemma 5.1, a linear relation $T \in \mathcal{BCR}(X)$ with $R_c(T) = \{0\}$ is upper (resp., lower) semi-B-Browder, if $asc(T) < \infty$ (resp., $des(T) < \infty$) and there exists $d \in \mathbb{N}$ such that $R(T^d)$ is closed and the restriction $T_d = T_{R(T^d)}$ is upper (resp., lower) semi-Fredholm.

The connection between Drazin invertible linear relations and semi-B-Browder linear relations is establised in the following proposition.

Proposition 5.1 [21, Corollary 3.15]**.** *Let X be a Banach space and $T \in \mathcal{BCR}(X)$ with $\rho(T) \neq \varnothing$. Then*:

1) $\sigma_{ld}(T) = \sigma_{ubb}(T)$,

2) $\sigma_{rd}(T) = \sigma_{lbb}(T)$,

3) $\sigma_d(T) = \sigma_{bb}(T)$.

***Remark* 5.2.** The hypothesis $\rho(T) \neq \varnothing$ can be replaced by the following conditions $R_c(T) = \{0\}$ and T^n is closed for all $n \in \mathbb{N}$. So in the sequel, according to Remarks 1.2 and 1.3, if T is a bounded linear relation with $\rho(T) \neq \varnothing$ and N a bounded operator which verifies $TN = NT + T(0)$, then the results of Proposition 5.1, remains true for $T + N$, without the assumption $\rho(T+N) \neq \varnothing$.

In [17, Theorem 2.6], B. P. Duggal showed that, if T and N are two bounded mutually commuting operators such that N is nilpotent, a sufficient condition for $\sigma_x(T) = \sigma_x(T+N)$, where $\sigma_x = \sigma_{ubb}$ or σ_{lbb}, is that T and $T+N$ satisfy a specified property, henceforth referred to as property (**P**):

$$\text{If for every} \quad \lambda \in \operatorname{iso}\sigma_a(T) \quad \text{such that} \quad asc(T-\lambda) = d < \infty,$$

$$\text{the subspace} \quad R(T-\lambda) + N((T-\lambda)^d) \quad \text{is closed, then it is complemented.}$$

However, in the proof he used the fact that, if T and N are two commuting operators, defined on a decomposable space $X = E_1 \oplus E_2$, where E_1 and E_2 are two T-invariant closed subspaces of X and such that $T = T_{E_1} \oplus T_{E_2}$, then $N = N_{E_1} \oplus N_{E_2}$, which is not the case in general.

In what follows, adopting different proofs from those given by Duggal in [17], we will extend the above invariance results, in one hand for the more general case of left Drazin and right Drazin linear relations spectra, on another hand without using the conditions T and $T + N$ verify the property (**P**).

To achieve this, we need the following Neubauer lemma.

Lemma 5.2. *Let X be a Banach space, M and N be two paracomplete subspaces of X. If $M \cap N$ and $M + N$ are closed, then both M and N are closed.*

Now, we are ready to state our first main result.

Theorem 5.1. *Let X be a Banach space, $T \in \mathcal{BCR}(X)$ with $\rho(T) \neq \varnothing$ and N be a nilpotent operator which verifies $TN = NT + T(0)$. Then*

$$\sigma_{ld}(T + N) = \sigma_{ld}(T).$$

Proof. Let $\lambda \notin \sigma_{ld}(T)$. Then there exists d such that $asc(T - \lambda) = d < \infty$ and $R((T - \lambda)^{d+1})$ is closed. As $\rho(T - \lambda) \neq \varnothing$, then Lemma 4.3 entails that $R((T - \lambda)^n)$ is closed for all $n \geq d$. Let m be the degree of nilpotency of N. We claim that, for every $n \geq m + d$, $R((T - \lambda + N)^n)$ is closed. Indeed, let $(T - \lambda)_1$ and N_1 be the restrictions of $T - \lambda$ and N respectively to $R\big((T - \lambda)^n\big)$. Since N commutes with T, then $N_1 : R\big((T - \lambda)^n\big) \to R\big((T - \lambda)^n\big)$. We obtain that $R((T - \lambda)_1) = R((T - \lambda)^{n+1})$ is closed and $N((T - \lambda)_1) = N(T - \lambda) \cap R((T - \lambda)^n) = \{0\}$ by Lemma 2.1. Hence, $(T - \lambda)_1$ is a bounded below relation in $R((T - \lambda)^n)$. Corollary 3.1 leads to $(T - \lambda + N)_1$ is bounded below too and hence regular. It follows, by Proposition 3.1, that $R\big((T - \lambda + N)_1\big)$ is closed in $R\big((T - \lambda)^n\big)$ which is closed in X. Hence, $R\big((T - \lambda + N)_1\big)$ is closed in X. Moreover, the commutativity of $(T - \lambda + N)^n$ and $(T - \lambda)^n$ implies that

$$R\big((T - \lambda + N)_1\big) = (T - \lambda + N)^n R\big((T - \lambda)^n\big) =$$

$$= R((T - \lambda + N)^n (T - \lambda)^n)) =$$

$$= R((T - \lambda)^n (T - \lambda + N)^n) = (T - \lambda)^n R((T - \lambda + N)^n).$$

Thus, $(T - \lambda)^n R((T - \lambda + N)^n)$ is a closed subspace, and since $(T - \lambda)^n$ is bounded and closed, $(T - \lambda)^n(0) \subset (T - \lambda)^n R((T - \lambda + N)^n)$, then, by Lemma 1.5, $(T - \lambda)^{-n}(T - \lambda)^n R((T - \lambda + N)^n) = R((T - \lambda + N)^n) + N\big((T - \lambda)^n\big)$ is closed. Further, similarly to the proof of Proposition 2.2, we can see that $R((T - \lambda + N)^n) \cap N\big((T - \lambda)^n\big) \subset R((T - \lambda)^d) \cap N\big((T - \lambda)^n\big) = \{0\}$. Hence, using the Neubauer lemma, we get $R((T - \lambda + N)^n)$ is closed. Now, let $n_1 = m + d$. Then, by Proposition 2.1, $asc(T + N - \lambda) < n_1$. On the other hand, if $(T - \lambda + N)_{n_1}$ denotes the restriction of $T - \lambda + N$ to $R((T - \lambda + N)^{n_1})$, then $R((T - \lambda + N)_{n_1}) = R((T - \lambda + N)^{n_1+1})$ which is closed and $N((T - \lambda + N)_{n_1})) = N((T - \lambda + N)) \cap R((T - \lambda + N)^{n_1}) = \{0\}$. So, $(T - \lambda + N)_{n_1})$ is bounded below, in particular is upper semi-Fredholm. Hence, $\lambda \notin \sigma_{ubb}(T + N) = \sigma_{ld}(T + N)$. By the same way, the reverse inclusion can be verified.

The same invariance result for right Drazin spectrum of linear relations under perturbation by commuting nilpotent operators, is given in the following theorem.

Theorem 5.2. *Let X be a Banach space, $T \in \mathcal{BCR}(X)$ and N be a nilpotent operator which verifies $TN = NT + T(0)$. Then*

$$\sigma_{rd}(T + N) = \sigma_{rd}(T).$$

Proof. Let $\lambda \notin \sigma_{rd}(T)$. Then $des(T-\lambda I) = d < \infty$ and $R((T-\lambda)^d)$ is closed. If m is the degree of nilpotency of N, then, by Proposition 2.2, $des(T+N-\lambda) \leq m+d-1$ for all $n \geq m+d-1$, $R((T+N-\lambda)^n) = R((T-\lambda)^d)$ and the relation $(T-\lambda I)_{R((T-\lambda)^d)} + N_{R((T-\lambda)^d)}$ is surjective. Let $n_1 = m+d$. Then $R((T+N-\lambda)^{n_1}) = R((T-\lambda)^d)$ is closed. Now, if $(T+N-\lambda)_{n_1}$ denotes the restriction of $T+N-\lambda$ to $R((T+N-\lambda)^{n_1})$, then

$$R\big((T+N-\lambda)_{n_1}\big) = R\Big((T-\lambda I)_{R((T-\lambda)^d)} + N_{R((T-\lambda)^d)}\Big) = R((T-\lambda)^d).$$

It follows that $(T+N-\lambda)_{n_1}$ is a surjective linear relation. Thus $\lambda \notin \sigma_{lbb}(T+N) = \sigma_{rd}(T+N)$.

Let $T \in \mathcal{BCR}(X)$ with $\rho(T) \neq \varnothing$, as

$$\sigma_{ld}(T) \cup \sigma_{des}(T) = \sigma_{rd}(T) \cup \sigma_{asc}(T) = \sigma_{ld}(T) \cup \sigma_{rd}(T) = \sigma_d(T),$$

then a straightforward consequence of Proposition 2.2 and Theorem 5.1 or Proposition 2.1 and Theorem 5.2 or Theorems 5.1 and 5.2, which provides the stability of Drazin invertible linear relations under perturbations by commuting nilpotent operators, is stating below.

***Corollary* 5.1.** *Let X be a Banach space, $T \in \mathcal{BCR}(X)$ with $\rho(T) \neq \varnothing$ and N be a nilpotent operator which verifies $TN = NT + T(0)$. Then $\sigma_d(T+N) = \sigma_d(T)$.*

References

1. J. V. Neumann, *Functional operators, vol. 2, The geometry of orthogonal spaces*, Ann. Math. Stud., Princeton Univ. Press, Princeton (1950).
2. P. Aiena, T. M. Biondi, C. Carpintero, *On Drazin invertibility*, Proc. Amer. Math. Soc., **136**, 2839 – 2848 (2008).
3. T. Alvarez, *Small perturbation of normally solvable relations*, Publ. Math. Debrecen, **80**, 155 – 168 (2012).
4. T. Alvarez, *On regular linear relations*, Acta Math. Sin. (Engl. Ser.), **28**, 183 – 194 (2012).
5. T. Alvarez, *Left-right Browder linear relations and Riesz perturbations*, Acta Math. Sci., **37**, 1437 – 1452 (2017).
6. T. Alvarez, F. Fakhfakh, M. Mnif, *Left-right Fredholm and left-right Browder linear relations*, Filomat, **31**, № 2, 255 – 271 (2017).
7. H. Bouaniza, M. Mnif, *On strictly quasi-Fredholm linear relations and semi-B-Fredholm linear relation perturbations*, Filomat, **31**, № 20, 6337 – 6355 (2017).
8. E. A. Coddington, *Multivalued operators and boundary value problems*, Lect. Notes in Math., **183**, Springer-Verlag, Berlin (1971).
9. S. R. Caradus, *Generalized inverses and operator theory*, Queen's Papers in Pure and Appl. Math., Queen's Univ., Kingston, Ontario, **50** (1978).
10. R. W. Cross, *Multivalued linear operator*, Marcel Dekker Inc. (1998).
11. E. Chafai, M. Mnif, *Descent and essential descent spectrum of linear relations*, Extracta Math., **29**, 117 – 139 (2014).
12. E. Chafai, M. Mnif, *Ascent and essential ascent spectrum of linear relations*, Extracta Math., **31**, 145 – 167 (2016).
13. Y. Chamkha, M. Mnif, *The class of B-Fredholm linear relations*, Complex Anal. and Oper. Theory, **9**, 1681 – 1699 (2015).
14. M. P. Drazin, *Pseudo-inverses in associative rings and semigroups*, Amer. Math. Monthly, **65**, 506 – 514 (1958).
15. M. P. Drazin, *Extremal definitions of generalized inverses*, Linear Algebra and Appl., **165**, 185 – 196 (1992).
16. H. Du, K. Yang, *Perturbations of Drazin invertible operators*, Front. Math. China, **10**, 199 – 208 (2015).
17. B. P. Duggal, *B-Browder operators and perturbations*, Funct. Anal. Approx. and Comput., **4**, 71 – 75 (2012).
18. T. Kaczynski, *Multivalued maps as a tool in modeling and rigorous Numerics*, J. Fixed Point Theory and Appl., **2**, 151 – 176 (2008).
19. A. Favini, A. Yagi, *Multivalued linear operators and degenerate evolution equations*, Ann. Mat. Pura Appl., **163**, 353 – 384 (1993).
20. F. Fakhfakh, M. Mnif, *Perturbation theory of lower semi-Browder multivalued linear operators*, Publ. Math. Debrecen., **78**, 595 – 606 (2011).

21. A. Ghorbel, M. Mnif, *Drazin inverse of multivalued operators and its application*, Monatsh. Math., **189**, 273 – 293 (2019).
22. N. C. Gonzalez, J. J. Koliha, *Perturbation of the Drazin inverse for closed linear operators*, Integral Equat. and Oper. Theory, **36**, 92 – 106 (2000).
23. C. F. King, *A note on Drazin inverses*, Pacific J. Math., **70**, 383 – 390 (1977).
24. V. Kordula, V. Muller, *The distance from the Apostol spectrum*, Proc. Amer. Math. Soc., **124**, 2055 – 3061 (1996).
25. L.-Ph. Labrousse, A. Sandovici, H. S. V. de Snoo, H. Winkler, *The Kato decomposition of quasi-Fredholm relations*, Oper. Matrices, **4**, № 1, 1 – 51 (2010).
26. M. Mnif, A. A. Ouled-Hmed, *Local spectral theory and surjective spectrum of linear relations*, Ukr. Mat. Zh., **73**, № 2, 222 – 237 (2021).
27. M. Z. Nashed, *Generalized inverses and applications*, Acad. Press, New York (1976).
28. A. Ouled Hmed, M. Mnif, *Analytic core and quasi-nilpotent part of linear relations in Banach spaces*, Filomat, 2499 – 2515 (2018).
29. A. Sandovici, H. de Snoo, *An index formula for the product of linear relations*, Linear Algebra and Appl., **431**, 2160 – 2171 (2009).
30. A. Sandovici, H. Snoo, H. Winkler, *Ascent, descent, nullity, defect and related notions for linear relations in linear spaces*, Linear Algebra and Appl., **423**, 456 – 497 (2007).

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