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A NOTE ON THE WEIGHTED LOG CANONICAL THRESHOLD OF TORIC PLURISUBHARMONIC FUNCTIONS [2]

ПРО ЗВАЖЕНИЙ ЛОГ-КАНОНІЧНИЙ ПОРІГ ДЛЯ ТОРОЇДАЛЬНИХ ПЛЮРИСУБГАРМОНІЧНИХ ФУНКЦІЙ

We prove a semicontinuity theorem for a class of certain weighted log canonical threshold of toric plurisubharmonic functions.

Доведено теорему про напівнеперервність класу деякого зваженого лог-канонічного порогу для тороїдальних плюрисубгармонічних функцій.

1. Introduction and main result. Let Ω be a domain in $\mathbb{C}^n$ and let u be in the set PSH(Ω) of plurisubharmonic functions on Ω. Following Demailly and Kollár [4], we introduce the log canonical threshold of u at point $0 \in \Omega$:

$$c(u) = \sup\big\{c > 0 : e^{-2cu} \text{ is } L^1(dV_{2n}) \text{ on a neighborhood of } 0\big\} \in (0, +\infty],$$

where dV_{2n} denotes the Lebesgue measure in $\mathbb{C}^n$. It is an invariant of the singularity of u at 0. We refer to [2, 3, 5, 8, 10 – 12] for further information and applications to this number. In [4], the authors investigated the semicontinuity theorem of log canonical thresholds. This theorem is a fundamental result which have had many applications in complex geometry. For example, this theorem is precisely what is needed in order to construct Kähler – Einstein metric on Fano manifolds (see [4]).

For every nonnegative Radon measure μ on a neighborhood of $0 \in \mathbb{C}^n$, following Pham [7], we introduce the weighted log canonical threshold of u with the weight μ at 0:

$$c_\mu(u) = \sup\big\{c \geq 0 : e^{-2cu} \text{ is } L^1(\mu) \text{ on a neighborhood of } 0\big\} \in [0, +\infty].$$

In [7], Pham obtained the semicontinuity theorem of weighted log canonical thresholds with the weight $\mu = \|z\|^{2t} dV_{2n}$ for $t \in (-n, 1]$.

A function u defined on Ω is called a toric plurisubharmonic function ($u \in \mathrm{TPSH}(\Omega)$) if u is plurisubharmonic and $u(z)$ depends only on $|z_1|, \ldots, |z_n|$ for any $z \in \Omega$. For every $u \in \mathrm{PSH}^-(\Delta^n)$ with Δ^n is the unit polydisc in $\mathbb{C}^n$, we consider Kiselman's refined Lelong numbers of u at 0 (see [1, 10]):

$$\nu_u(x) = \lim_{t \to -\infty} \frac{\max\{u(z) : |z_1| = e^{tx_1}, \ldots, |z_n| = e^{tx_n}\}}{t}.$$

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This function is increasing in each variable x_j and concave on $\mathbb{R}^n_+ = [0, +\infty)^n$.

In this paper, we use ideas in [4] and [7] to study weighted log canonical thresholds of toric plurisubharmonic functions. The main result is contained in the following theorem.

Theorem 1.1. *Let* $\{u_j\}_{j\geq 1} \subset \mathrm{TPSH}^-(\Delta^n)$, $u \in \mathrm{TPSH}^-(\Delta^n)$ *and a nonnegative Radon measure* μ *on* Δ^n. *Assume that* $u_j \to u$ *in* $L^1_{\mathrm{loc}}(\Delta^n)$ *and*

$$\mu(\Delta_{r_1} \times \ldots \times \Delta_{r_n}) = h(r_1, \ldots, r_n) \sum\nolimits_{k=1}^{m} r_1^{2s_{k1}} \ldots r_n^{2s_{kn}}, \quad s_{k1}, \ldots, s_{kn} > 0, \quad 1 \leq k \leq m,$$

for all $r_1, \ldots, r_n > 0$, *where* $h(r_1, \ldots, r_n)$ *is a function that is bounded above and below by two positive constants and* Δ_r *is the disc of center* 0 *and radius* r. *Then*

$$\liminf_{j\to+\infty} c_\mu(u_j) \geq c_\mu(u).$$

Let $\mathcal{M}$ be the set of the measures that satisfy the conditions of the Theorem 1.1. It is easy to check that $\mathcal{M}$ is a convex cone set. The following we will give some models of the measure in $\mathcal{M}$.

***Example* 1.1.** Let $f_1, \ldots, f_k$ be some holomorphic functions on some neighborhood of the origin and $a_1, \ldots, a_k \geq 0$. Then $\mu = (|f_1|^{a_1} + \ldots + |f_k|^{a_k})dV_{2n}$ is in $\mathcal{M}$. Indeed, since $\mathcal{M}$ is convex cone, we only need to show that $\mu = |f|^a dV_{2n} \in \mathcal{M}$, where f is a holomorphic function on some neighborhood of the origin and $a \geq 0$. This infers from the proof of the corollary in [9].

***Example* 1.2.** Let $f_1, \ldots, f_k$ be some real analytic functions on the real part $(x_1, \ldots, x_n)$ of $z = (x_1 + iy_1, \ldots, x_n + iy_n) \in \Delta_{r_1} \times \ldots \times \Delta_{r_n}$. We set $\mu = (|f_1|^2 + \ldots + |f_k|^2)dV_{\mathbb{R}^n}$ and $\rho(x + iy) = x$ is the real projection. Then $\rho^*\mu$ is in $\mathcal{M}$. Indeed, since $\mathcal{M}$ is convex cone, we only have to prove that $\rho^*\mu \in \mathcal{M}$ for the case $\mu = |f|^2 dV_{\mathbb{R}^n}$, where f is a real analytic function on the real part $(x_1, \ldots, x_n)$. Then $\rho^*\mu = |f \circ \rho|^2 dV_{\mathbb{R}^n} \circ \rho$. Set

$$f(x) = \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n} c_\alpha x_1^{\alpha_1} \ldots x_n^{\alpha_n}.$$

We define $\mathcal{I}_f$ to be the ideal generated by $\{x^\alpha : c_\alpha \neq 0\}$. From the Noetherian property of the polynomial ring, $\mathcal{I}_f$ is generated by finite elements $\{x^{\alpha^1}, \ldots, x^{\alpha^m}\}$. We will prove the following:

$$\int\limits_{\Delta_{r_1}\times\ldots\times\Delta_{r_n}} |f \circ \rho|^2 dV_{\mathbb{R}^n} \circ \rho = O(1) \sum_{k=1}^{m} r_1^{\alpha_1^k+1} \ldots r_n^{\alpha_n^k+1},$$

where $O(1)$ is a bounded positive quantity. First, we have

$$|(f \circ \rho)(z)| = |f(x)| \leq \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n} |c_\alpha||x_1|^{\alpha_1} \ldots |x_n|^{\alpha_n} \leq$$

$$\leq \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n} |c_\alpha| r_1^{\alpha_1} \ldots r_n^{\alpha_n} \leq C(f) \sum_{j=1}^{k} r_1^{\alpha_1^j} \ldots r_n^{\alpha_n^j}$$

for every $|x_1| \leq r_1, \ldots, |x_n| \leq r_n$, where $C(f)$ is a positive constant which only depends on f. So we infer

$$\int\limits_{\Delta_{r_1}\times\ldots\times\Delta_{r_n}} |f \circ \rho|^2 dV_{\mathbb{R}^n} \circ \rho \leq C(f)^2 2^n \sum_{k=1}^{m} r_1^{2\alpha_1^k+1} \ldots r_n^{2\alpha_n^k+1}.$$

From $|(f\circ\rho)(z)| = |f(x)|^2 = \sum_{\alpha,\beta\in\mathbb{N}^n} c_\alpha c_\beta x_1^{\alpha_1+\beta_1}\ldots x_n^{\alpha_n+\beta_n}$ we obtain

$$\int\limits_{\Delta_{r_1}\times\ldots\times\Delta_{r_n}} |f\circ\rho|^2 dV_{\mathbb{R}^n}\circ\rho = \int\limits_{\{-r_1\le x_1\le r_1\}\times\ldots\times\{-r_n\le x_n\le r_n\}} |f(x)|^2 dx_1\ldots dx_n =$$

$$= \sum_{\alpha,\beta\in\mathbb{N}^n} c_\alpha c_\beta \int\limits_{\{-r_1\le x_1\le r_1\}\times\ldots\times\{-r_n\le x_n\le r_n\}} x^{\alpha+\beta} dx_1\ldots dx_n.$$

By the Fubini theorem we get

$$\int\limits_{\{-r_1\le x_1\le r_1\}\times\ldots\times\{-r_n\le x_n\le r_n\}} x^{\alpha+\beta} dx_1\ldots dx_n =$$

$$= \int\limits_{\{-r_1\le x_1\le r_1\}} x_1^{\alpha_1+\beta_1} dx_1 \ldots \int\limits_{\{-r_n\le x_n\le r_n\}} x_n^{\alpha_n+\beta_n} dx_n =$$

$$= \begin{cases} 0 & \text{if there exist } j \text{ such that } \alpha_j+\beta_j \text{ is odd,} \\ \dfrac{2^n r_1^{\alpha_1+\beta_1+1}\ldots r_n^{\alpha_n+\beta_n+1}}{(\alpha_1+\beta_1+1)\ldots(\alpha_n+\beta_n+1)} & \text{if } \alpha_j+\beta_j \text{ are even } j=1,2,\ldots,n. \end{cases}$$

Moreover, we have the following matrix is strict positive definite symmetric:

$$\left[\frac{1}{(\alpha_1+\beta_1+1)\ldots(\alpha_n+\beta_n+1)}\right]_{\alpha,\beta\in E},$$

where E is the finite subset of $\mathbb{N}^n$. This implies

$$\int\limits_{\{-r_1\le x_1\le r_1\}\times\ldots\times\{-r_n\le x_n\le r_n\}} x^{\alpha+\beta} dx_1\ldots dx_n \ge D(f)\sum_{k=1}^m r_1^{2\alpha_1^k+1}\ldots r_n^{2\alpha_n^k+1},$$

where $D(f)$ is a positive constant which only depends on f.

***Remark* 1.1.** The semicontinuity theorem for the weighted log canonical thresholds is no longer true in case of the measure $\mu = |z_1|^2 dV_{2n}$ without the condition toric function. Indeed, as in Remark 1.3 [6], we have $\varphi_j = \ln\left|z_1 + \dfrac{z_2}{j}\right| \to \varphi = \ln|z_1|$ in $L^1_{\text{loc}}(\mathbb{C}^n)$. However $c_\mu(\varphi_j) = 1$ and $c_\mu(\varphi) = 2$ do not satisfy Theorem 1.1.

2. Proof of the main result. First, we need the following lemma.

Lemma 2.1. *Let $u, v\in \mathrm{TPSH}^-(\Delta^n)$ and a nonnegative Radon measure μ on Δ^n. Assume that*

$$\mu(\Delta_{r_1}\times\ldots\times\Delta_{r_n}) = h(r_1,\ldots,r_n)\sum_{k=1}^m r_1^{2s_{k1}}\ldots r_n^{2s_{kn}} \quad (\forall r_1,\ldots,r_n>0),$$

where $s_{k1},\ldots,s_{kn}>0$, $1\le k\le m$ and $h(r_1,\ldots,r_n)$ is a function, that is, bounded above and below by two positive constants and Δ_r is the disc of center 0 and radius r. Then

$$c_\mu(\max(u,v)) \le c_\mu(u) + \max_{1\le k\le m}\left(\max\left\{\nu_v(x) : x\in\mathbb{R}^n_+, \sum_{j=1}^n s_{kj}x_j = 1\right\}\right)^{-1}.$$

Proof. From the definition of Kiselman's refined Lelong numbers, we have

$$\nu_{\max(u,v)} = \min(\nu_u, \nu_v).$$

Moreover, by the main theorem in [9], we obtain

$$c_\mu(\max(u,v)) = \min_{1\le k\le m}\left(\max\left\{\min(\nu_u(x),\nu_v(x)) : x\in\mathbb{R}^n_+,\ \sum_{j=1}^n s_{kj}x_j = 1\right\}\right)^{-1}$$

and

$$c_\mu(u) = \min_{1\le k\le m}\left(\max\left\{\nu_u(x) : x\in\mathbb{R}^n_+,\ \sum_{j=1}^n s_{kj}x_j = 1\right\}\right)^{-1}.$$

Take $k\in\{1,\ldots,m\}$ and $x^0, y^0 \in \left\{x\in\mathbb{R}^n_+ :\ \sum_{j=1}^n s_{kj}x_j = 1\right\}$ such that $c_\mu(u) = \dfrac{1}{\nu_u(x^0)}$ and

$$\max\left\{\nu_v(x) : x\in\mathbb{R}^n_+,\ \sum_{j=1}^n s_{kj}x_j = 1\right\} = \nu_v(y^0).$$

Take $t = \dfrac{\nu_v(y^0)}{\nu_u(x^0)+\nu_v(y^0)}$ and $z^0 = tx^0 + (1-t)y^0 \in \left\{x\in\mathbb{R}^n_+ :\ \sum_{j=1}^n s_{kj}x_j = 1\right\}$. Since ν_u, ν_v are concave functions on $\mathbb{R}^n_+$, we have

$$\nu_u(z^0) \ge t\nu_u(x^0) + (1-t)\nu_u(y^0) \ge t\nu_u(x^0) = \frac{\nu_u(x^0)\nu_v(y^0)}{\nu_u(x^0)+\nu_v(y^0)}$$

and

$$\nu_v(z^0) \ge t\nu_v(x^0) + (1-t)\nu_v(y^0) \ge (1-t)\nu_v(y^0) = \frac{\nu_u(x^0)\nu_v(y^0)}{\nu_u(x^0)+\nu_v(y^0)}.$$

Hence

$$\nu_{\max(u,v)}(z^0) = \min(\nu_u(z^0), \nu_v(z^0)) \ge \frac{\nu_u(x^0)\nu_v(y^0)}{\nu_u(x^0)+\nu_v(y^0)}.$$

This implies that

$$c_\mu(\max(u,v)) \le \frac{1}{\nu_{\max(u,v)}(z^0)} \le \frac{\nu_u(x^0)+\nu_v(y^0)}{\nu_u(x^0)\nu_v(y^0)} = \frac{1}{\nu_u(x^0)} + \frac{1}{\nu_v(y^0)} =$$

$$= c_\mu(u) + \max_{1\le k\le m}\left(\max\left\{\nu_v(x) : x\in\mathbb{R}^n_+,\ \sum_{j=1}^n s_{kj}x_j = 1\right\}\right)^{-1}.$$

Lemma 2.1 is proved.

Proof of the main result. First, we consider the case $u_j, u \geq C \max(\ln|z_1|, \ldots, \ln|z_n|)$ for all $j \geq 1$ $(C > 0)$. We have

$$\nu_{u_j}(x) \leq C\nu_{\max(\ln|z_1|,\ldots,\ln|z_n|)}(x) = C\min(x_1, \ldots, x_n) \quad \forall x \in \mathbb{R}^n_+.$$

By Proposition 3.12 and Example 4.11 in [1], we obtain

$$\overline{\lim_{j\to+\infty}}\, \nu_{u_j}(x) \leq \nu_u(x) \quad \forall x \in \mathbb{R}^n_+.$$

We will prove that $\{\nu_{u_j}\}_{j\geq1}$ is a sequence of uniformly continuous functions on $\Big\{x \in \mathbb{R}^n_+ : \sum_{j=1}^{n} x_j \leq D\Big\}$ for all $D > 0$. Let $\epsilon > 0$. Since $\{\nu_{u_j}\}_{j\geq1}$ is a sequence of uniformly bounded, concave functions on $\Big\{x \in \mathbb{R}^n_+ : \sum_{j=1}^{n} x_j \leq D + 1\Big\}$, we can find $K > 0$ such that

$$\big|\nu_{u_j}(x) - \nu_{u_j}(y)\big| \leq K\|x - y\|$$

for all $x, y \in \Big\{x \in \mathbb{R}^n_+ : \sum_{j=1}^{n} x_j \leq D,\ \min(x_1, \ldots, x_n) \geq \dfrac{\epsilon}{4C}\Big\}$. We will show that $\big|\nu_{u_j}(x) - \nu_{u_j}(y)\big| < \epsilon$ for all $x, y \in \Big\{x \in \mathbb{R}^n_+ : \sum_{j=1}^{n} x_j \leq D\Big\}$, $\|x - y\| < \delta = \min\Big(\dfrac{\epsilon}{K}, \dfrac{\epsilon}{4C}\Big)$. Indeed, if $\min(x_1, \ldots, x_n) < \dfrac{\epsilon}{4C}$ or $\min(y_1, \ldots, y_n) < \dfrac{\epsilon}{4C}$, then

$$\big|\nu_{u_j}(x) - \nu_{u_j}(y)\big| \leq \nu_{u_j}(x) + \nu_{u_j}(y) \leq C(\min(x_1, \ldots, x_n) + \min(y_1, \ldots, y_n)) < \epsilon.$$

Otherwise, if $\min(x_1, \ldots, x_n) \geq \dfrac{\epsilon}{4C}$ and $\min(y_1, \ldots, y_n) \geq \dfrac{\epsilon}{4C}$, we have

$$\big|\nu_{u_j}(x) - \nu_{u_j}(y)\big| \leq K\|x - y\| < K\delta \leq \epsilon.$$

By the Arzelà – Ascoli theorem we can assume that $\{\nu_{u_j}\}_{j\geq1}$ uniformly converges to $\phi \leq \nu_u$ on $\Big\{x \in \mathbb{R}^n_+ : \sum_{j=1}^{n} x_j \leq D\Big\}$ for all $D > 0$. This implies that

$$\overline{\lim_{j\to+\infty}} \max\left\{\nu_{u_j}(x) : x \in \mathbb{R}^n_+,\ \exists k = 1, \ldots, m,\ \sum_{j=1}^{n} s_{kj}x_j = 1\right\} \leq$$

$$\leq \max\left\{\nu_u(x) : x \in \mathbb{R}^n_+,\ \exists k = 1, \ldots, m,\ \sum_{j=1}^{n} s_{kj}x_j = 1\right\}.$$

Moreover, by the main theorem in [9], we obtain

$$\liminf_{j\to+\infty} c_\mu(u_j) \geq c_\mu(u).$$

In general case, we set

$$u_j^l = \max\big(u_j,\ l\max(\log|z_1|, \ldots, \log|z_n|)\big),$$

$$u_l = \max\big(u,\ l\max(\log|z_1|, \ldots, \log|z_n|)\big).$$

By the first case, we get

$$\liminf_{j\to+\infty} c_\mu(u_j^l) \geq c_\mu(u^l) \geq c_\mu(u).$$

On the other hand, by Lemma 2.1 we have

$$c_\mu(u_j^l) \leq c_\mu(u_j) + \frac{1}{l} \max_{1\leq k\leq m} \frac{1}{\sum_{j=1}^{n} s_{kj}}.$$

Therefore,

$$\liminf_{j\to+\infty} c_\mu(u_j) \geq c_\mu(u).$$

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