DOI: 10.37863/umzh.v75i2.328

UDC 517.9

A. Benhassine[1] (Higher Institute of Science Computer and Mathematics Monastir University, Tunisia)

TWO DIFFERENT SEQUENCES OF INFINITELY MANY HOMOCLINIC SOLUTIONS FOR A CLASS OF FRACTIONAL HAMILTONIAN SYSTEMS

ДВІ РІЗНІ НЕСКІНЧЕННІ ПОСЛІДОВНОСТІ ГОМОКЛІНІЧНИХ РОЗВ'ЯЗКІВ ДЛЯ КЛАСУ ДРОБОВИХ ГАМІЛЬТОНОВИХ СИСТЕМ

We consider the problem of existence of infinitely many homoclinic solutions for the following fractional Hamiltonian systems:

$$
\begin{aligned}
&-{}_t D^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t x(t)) - L(t)x(t) + \nabla W(t, x(t)) = 0, \\
&x \in H^{\alpha}\Big(\mathbb{R}, \mathbb{R}^N\Big),
\end{aligned} \tag{FHS}
$$

where $\alpha \in \left(\dfrac{1}{2}, 1\right]$, $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, and ${}_{-\infty}D^{\alpha}_t$ and ${}_tD^{\alpha}_{\infty}$ are the left and right Liouville – Weyl fractional derivatives of order α on the whole axis $\mathbb{R}$, respectively. The novelty of our results is that, under the assumption that the nonlinearity $W \in C^1\big(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}\big)$ involves a combination of superquadratic and subquadratic terms, for the first time, we show that (FHS) possesses two different sequences of infinitely many homoclinic solutions via the Fountain theorem and the dual Fountain theorem such that the corresponding energy functional of (FHS) goes to infinity and zero, respectively. Some recent results available in the literature are generalized and significantly improved.

Розглянуто питання про існування нескінченної кількості гомоклінічних розв'язків для таких дробових гамільтонових систем:

$$
\begin{aligned}
&-{}_t D^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t x(t)) - L(t)x(t) + \nabla W(t, x(t)) = 0, \\
&x \in H^{\alpha}\Big(\mathbb{R}, \mathbb{R}^N\Big),
\end{aligned} \tag{FHS}
$$

де $\alpha \in \left(\dfrac{1}{2}, 1\right]$, $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, а ${}_{-\infty}D^{\alpha}_t$ і ${}_tD^{\alpha}_{\infty}$ — відповідно ліві та праві дробові похідні Ліувілля – Вейля порядку α на всій осі $\mathbb{R}$. Новизна отриманих результатів полягає в тому, що у випадку, коли нелінійність $W \in C^1\big(\mathbb{R}\times\mathbb{R}^N, \mathbb{R}\big)$ містить комбінацію суперквадратичних і субквадратичних членів, уперше показано за допомогою теореми Фонтена та дуальної теореми Фонтена, що (FHS) містить дві різні нескінченні послідовності гомоклінічних розв'язків такі, що відповідний енергетичний функціонал (FHS) прямує до нескінченності та нуля, відповідно. Деякі останні результати, відомі з літератури, узагальнено та значно покращено.

1. Introduction. In this work, we consider the following nonperiodic fractional Hamiltonian systems:

$$
\begin{aligned}
&-{}_t D^{\alpha}_{\infty}({}_{-\infty}D^{\alpha}_t x(t)) - L(t)x(t) + \nabla W(t, x(t)) = 0, \\
&x \in H^{\alpha}\big(\mathbb{R}, \mathbb{R}^N\big),
\end{aligned} \tag{FHS}
$$

[1] E-mail: ab.hassine@yahoo.com.

where $\alpha \in \left(\frac{1}{2}, 1\right]$, $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, ${}_{-\infty}D_t^{\alpha}$ and ${}_tD_{\infty}^{\alpha}$ are left and right Liouville–Weyl fractional derivatives of order α on the whole axis $\mathbb{R}$, respectively, $L \in C\big(\mathbb{R}, \mathbb{R}^{N^2}\big)$ is a symmetric matrix, $W \in C^1\big(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}\big)$ and $\nabla W(t,x)$ is the gradient of $W(t,x)$ at x.

Hamiltonian systems are a significant field of nonlinear functional analysis, since they arise in phenomena that are studied in several fields of applied science such as physics, astronomy, chemistry, biology, engineering and other fields of science. Since Newton wrote the differential equation describing the motion of the planet and derived the Kepler ellipse as its solution, the complex dynamical behavior of the Hamiltonian system has attracted a wide range of mathematicians and physicists. The variational methods to investigate the Hamiltonian system were first used by Poincaré, who used the minimal action principle of the Jacobi form to study the closed orbits of a conservative system with two degrees of freedom. Ambrosetti and Rabinowitz in [2] proved “Mountain pass theorem”, “Saddle point theorem”, “Linking theorem” and a series of a very important minimax form of the critical point theorem. The study of Hamiltonian systems makes a significant breakthrough, due to a critical point theory. Critical point theorem was first used by Rabinowitz [23] to obtain the existence of periodic solutions for first order Hamiltonian systems, while the first multiplicity result is due to Ambrosetti and Zelati [3]. Since then there exists large number of literatures on the use of the critical point theory and variational methods to prove the existence of solutions to Hamiltonian systems (see, for example, [4, 7, 8, 23, 24] and the references therein).

Also, fractional calculus has received an increased popularity and importance in the past decades to describe long-memory processes. For more details, we refer the reader to the monographs [1, 16, 20], the articles [5, 15, 17] and the references therein. Recently, the critical point theory has become an effective tool in studying the existence of solutions to fractional differential equations by constructing fractional variational structures. In [18], Jiao and Zhou were the first who used the critical point theory to study the existence of solutions to this fractional boundary-value problem

$$\begin{aligned} &{}_tD_T^{\alpha}({}_0D_t^{\alpha}x(t)) = \nabla W(t, x(t)), \quad \text{a.e.} \quad t \in [0,T], \\ &x(0) = x(T), \end{aligned}$$

where $\alpha \in \left(\frac{1}{2}, 1\right)$, $x \in \mathbb{R}^N$, $W \in C^1\big([0,T] \times \mathbb{R}^N, \mathbb{R}\big)$ and obtained the existence of at least one nontrivial solution. Next, Jiao and Zhou in [19] considered a class of fractional boundary-value problems

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}\,{}_0D_t^{-\beta}(x'(t)) +_t D_T^{-\beta}(x'(t))\right) + F(t, x(t)) = 0, \quad \text{a.e.} \quad t \in [0,T], \\ &x(0) = x(T) = 0. \end{aligned}$$

They established the variational structure and obtained various criteria on the existence of solutions. Motivated by the above works, more and more authors began considering fractional Hamiltonian systems (see [6, 7, 13, 25, 26]). For example, Torres in [25] showed that (FHS) possesses at least one nontrivial solution via Mountain pass theorem, by assuming that L and W satisfy the following hypotheses:

(L) $L \in C\big(\mathbb{R}, \mathbb{R}^{N^2}\big)$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists an $l \in C(\mathbb{R}, (0, \infty))$ such that $l(t) \to \infty$ as $|t| \to \infty$ and $L(t)xx \geq l(t)|x|^2$ for all $t \in \mathbb{R},\ x \in \mathbb{R}^N$;

(W1) $\big|\nabla W(t, x)\big| = o(|x|)$ as $|x| \to 0$ uniformly in $t \in \mathbb{R}$;

(W2) there exists $\overline{W} \in C\big(\mathbb{R}^N, \mathbb{R}\big)$ such that

$$|W(t, x)| + |\nabla W(t, x)| \leq \big|\overline{W}(x)\big| \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^N;$$

(W3) there exists a constant $\mu > 2$ such that

$$0 < \mu W(t, x) \leq \nabla W(t, x)x \quad \forall t \in \mathbb{R}, \quad x \in \mathbb{R}^N \backslash \{0\}.$$

A strong motivation for investigating (FHS) comes from fractional advection-dispersion equation (ADE). This is a generalization of the classical ADE in which the second-order derivative is replaced with a fractional-order derivative. In contrast to the classical ADE, the fractional ADE has solutions that resemble the highly skewed and heavy-tailed breakthrough curves observed in field and laboratory studies (see [10, 11]). In particular, in contaminant transport of ground-water flow see [11]. Benson et al. stated that solutes moving through a highly heterogeneous aquifer violations violates the basic assumptions of the local second-order theories because of large deviations from the stochastic process of Brownian motion.

Definition 1.1. *We call that a solution x of systems* (FHS) *is homoclinic* (*to* 0) *if $x(t) \to 0$ as $|t| \to \infty$. In addition, if $x \not\equiv 0$ then x is called a nontrivial homoclinic solution.*

In (FHS), if $\alpha = 1$, then it reduces to the following second-order Hamiltonian system:

$$\ddot{x} - L(t)x + \nabla W(t, x) = 0. \tag{HS}$$

It is well-known that the existence of homoclinic solutions for Hamiltonian systems and their importance in the study of the behavior of dynamical systems has been recognized from Poincaré [22]. They may be “organizing centers” for the dynamics in their neighborhood. From their existence one may, under certain conditions, infer the existence of chaos nearby or the bifurcation behavior of periodic orbits. During the past two decades, with the works of [21] and [23] variational methods and critical point theory have been successfully applied for the search of the existence and multiplicity of homoclinic solutions of (HS).

Assuming that $L(t)$ and $W(t, x)$ are independent of t or periodic in t, many authors have studied the existence of homoclinic solutions for (HS) (see, for example, [3, 14, 23] and the references therein). In this case, the existence of homoclinic solutions can be obtained by going to the limit of periodic solutions of approximating problems. If $L(t)$ and $W(t, x)$ are neither autonomous nor periodic in t, the problem of existence of homoclinc solutions of (HS) is quite different from the ones just described, because the lack of compactness of Sobolev embedding, such as [14, 21, 24] and the references mentioned there.

Assumption (W3) is the so-called global Ambrosetti – Rabinowitz condition, which implies that $W(t, x)$ is of superquadratic growth as $|x| \to \infty$. Motivated by [25], in [13], Chen et al. gave some more general superquadratic conditions on $W(t, x)$ and obtained that (FHS) possesses infinitely many nontrivial solutions. For the case that $W(t, x)$ is subquadratic as $|x| \to \infty$, Zhang and Yuan in [26] established some new criterion to guarantee the existence of infinitely many solutions of (FHS). Moreover, in [27], Zhang and Yuan investigate the case that the nonlinearity $W(t, x)$ involves a combination of superquadratic and subquadratic terms but $W(t,x)$ does not change in sign. That is,

$$W(t,x) = W_1(t,x) + W_2(t,x),$$

where $W_1(t,x),\ W_2(t,x) \geq 0$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}^N$, $W_1(t,x)$ is superquadratic as $|x| \to \infty$, $W_2(t,x)$ is of subquadratic growth at infinity and, under some others reasonable assumptions on $L(t)$, $W_1(t,x)$ and $W_2(t,x)$ they proved the existence of at least two nontrivial solutions of (FHS) but the existence of infinitely many solutions of (FHS) remains an open question (see [27, p. 5]).

Motivated by the works mentioned above, in this paper we investigate the case that the nonlinearity $W(t,x)$ involves a combination of superquadratic and subquadratic terms and we prove the existence of two different sequence of infinitely many homoclinic solutions (x_k) and (y_k) via the Fountain theorem and the dual Fountain theorem such that the corresponding energy functional of (FHS) goes to infinity and zero, respectively. More precisely, we consider the following assumptions:

(H1) $W(t,x) = W_1(t,x) + W_2(t,x)$, where $W_1, W_2 \in C^1\big(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}\big)$ are even in x;

(H2) there exists a constant $\mu > 2$ such that

$$0 < \mu W_1(t,x) \leq \nabla W_1(t,x)x \quad \forall t \in \mathbb{R}, \quad x \in \mathbb{R}^N \backslash \{0\};$$

(H3) $\big|\nabla W_1(t,x)\big| = o(|x|)$ as $|x| \to 0$ uniformly with respect to $t \in \mathbb{R}$, and for all $(t,x) \in \in \mathbb{R} \times \mathbb{R}^N$, there exists some constant $C > 0$ such that

$$|\nabla W_1(t,x)| \leq C\big(|x| + |x|^{\mu-1}\big);$$

(H4) $W_2(t,0) = 0$ for all $t \in \mathbb{R}$ and there exists a positive continuous function $a\colon \mathbb{R} \to \mathbb{R}$ such that

$$W_2(t,x) \geq a(t)|x|^{\vartheta} \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N,$$

where $1 < \vartheta < 2$ is a constant;

(H5) for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$,

$$|\nabla W_2(t,x)| \leq b(t)|x|^{\vartheta-1},$$

where $b\colon \mathbb{R} \to \mathbb{R}$ is a positive continuous function such that

$$\lim_{|t| \to +\infty} b(t) = 0;$$

(H6) there exists a bounded continuous function $d\colon \mathbb{R} \to \mathbb{R}^+$ such that

$$\mu W_2(t,x) - \nabla W_2(t,x)x \leq d(t)|x|^{\vartheta} \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that* (L) *and* (H1) – (H6) *are satisfied, then* (FHS) *possesses two different sequences of homoclinic solutions* (x_k) *and* (y_k) *satisfying*

$$\int\limits_{\mathbb{R}} \left(\frac{1}{2}|{}_{-\infty}D_t^{\alpha} x_k(t)|^2 + \frac{1}{2}L(t)x_k(t)x_k(t) - W(t,x_k(t))\right) dt \to +\infty$$

and

$$\int_{\mathbb{R}} \left(\frac{1}{2} |_{-\infty} D_t^{\alpha} y_k(t)|^2 + \frac{1}{2} L(t) y_k(t) y_k(t) - W(t, y_k(t)) \right) dt \to 0^-$$

as $k \to +\infty$, *respectively.*

***Remark* 1.1.** In view of (H2) and (H3), it deduces that $W_1(t,x) = o\big(|x|^2\big)$ as $|x| \to 0$ uniformly with respect to $t \in \mathbb{R}$, which yields that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 \le W_1(t,x) \le \varepsilon |x|^2 \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad |x| \le \delta.$$

From (H1), (H2) and (H4), it is obvious that

$$W(t,x) \ge W_2(t,x) \ge a(t)|x|^{\vartheta} \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N.$$

On the other hand, it is easy to check that $W_2(t,x)$ is subquadratic as $|x| \to +\infty$. In fact, in view of (H4) and (H5), we obtain

$$W_2(t,x) = \int_0^1 \nabla W_2(t,sx)x\, ds \le \frac{b(t)}{\vartheta} |x|^{\vartheta} \le \frac{\bar{b}}{\vartheta} |x|^{\vartheta}, \tag{1.1}$$

where $\bar{b} = \max_{t\in\mathbb{R}} b(t)$, which implies that $W_2(t,x)$ is of subquadratic growth as $|x| \to +\infty$. Moreover, by (H3), we have

$$W_1(t,x) = \int_0^1 \nabla W_1(t,sx)xds \le c\left(\frac{1}{2} |x|^2 + \frac{1}{\mu} |x|^{\mu} \right),$$

which combining with (1.1) yields that

$$W(t,x) \le c\Big(|x|^{\vartheta} + |x|^{\mu} \Big) \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.2}$$

In addition, according to (H2), it is easy to verify that, for $t \in \mathbb{R}$ and $|x| \ge 1$,

$$W_1(t,x) \ge W_1\left(t, \frac{x}{|x|}\right)|x|^{\mu} \ge \min_{|x|=1} W_1(t,x)|x|^{\mu} = e(t)|x|^{\mu}. \tag{1.3}$$

Meanwhile, in view of (1.2), it is easy to check that $e(t)$ is bounded.

Here and in the following $x.y$ denotes the inner product of $x, y \in \mathbb{R}^N$ and $|.|$ denotes the associated norm. Throughout the paper we denote by c, c_i the various positive constants which may vary from line to line and are not essential to the problem.

The remaining part of this paper is organized as follows. Some preliminary results are presented in Section 2 and Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminary results. ***2.1. Liouville – Weyl fractional calculus.***

Definition 2.1. *The left and right Liouville – Weyl fractional integrals of order* $0 < \alpha < 1$ *on the whole axis* $\mathbb{R}$ *are defined by*

$$_{-\infty}I_t^{\alpha} x(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-\xi)^{\alpha-1} x(\xi) d\xi,$$

$$_{t}I_{\infty}^{\alpha} x(t) := \frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} (\xi - t)^{\alpha-1} x(\xi)\, d\xi,$$

respectively, where $t \in \mathbb{R}$.

Definition 2.2. *The left and right Liouville–Weyl fractional derivatives of order* $0 < \alpha < 1$ *on the whole axis* $\mathbb{R}$ *are defined by*

$$ {}_{-\infty}D_t^{\alpha}x(t) := \frac{d}{dt}\,{}_{-\infty}I_t^{1-\alpha}x(t), \tag{2.1}$$

$$ {}_{t}D_{\infty}^{\alpha}x(t) := -\frac{d}{dt}\,{}_{t}I_{\infty}^{1-\alpha}x(t), \tag{2.2}$$

respectively, where $t \in \mathbb{R}$.

***Remark* 2.1.** The definitions (2.1) and (2.2) may be written in an alternative form:

$$ {}_{-\infty}D_t^{\alpha}x(t) = \frac{\alpha}{\Gamma(1-\alpha)}\int\limits_0^{\infty}\frac{x(t)-x(t-\xi)}{\xi^{\alpha+1}}\,d\xi,$$

$$ {}_{t}D_{\infty}^{\alpha}x(t) = \frac{\alpha}{\Gamma(1-\alpha)}\int\limits_0^{\infty}\frac{x(t)-x(t+\xi)}{\xi^{\alpha+1}}\,d\xi.$$

Recall that the Fourier transform $\widehat{x}(z)$ of $x(t)$ is defined by

$$\widehat{x}(z) = \int\limits_{-\infty}^{\infty} e^{-itz}x(t)\,dt.$$

We establish the Fourier transform properties of the fractional integral and fractional operators as follows:

$$\widehat{{}_{-\infty}I_t^{\alpha}x(t)}(z) := (iz)^{-\alpha}\widehat{x}(z),$$

$$\widehat{{}_{t}I_{\infty}^{\alpha}x(t)}(z) := (-iz)^{-\alpha}\widehat{x}(z),$$

$$\widehat{{}_{-\infty}D_t^{\alpha}x(t)}(z) := (iz)^{\alpha}\widehat{x}(z),$$

$$\widehat{{}_{t}D_{\infty}^{\alpha}x(t)}(z) := (-iz)^{\alpha}\widehat{x}(z).$$

2.2. Fractional derivative spaces. Let us recall that for any $\alpha > 0$, the seminorm

$$|x|_{I_{-\infty}^{\alpha}} := \|{}_{-\infty}D_t^{\alpha}x\|_{L^2},$$

and norm

$$\|x\|_{I_{-\infty}^{\alpha}} := \left(\|x\|_{L^2}^2 + |x|_{I_{-\infty}^{\alpha}}^2\right)^{1/2},$$

and let the space $I_{-\infty}^{\alpha}(\mathbb{R})$ denote the completion of $C_0^{\infty}(\mathbb{R})$ with respect to the norm $\|.\|_{I_{-\infty}^{\alpha}}$, i.e.,

$$I_{-\infty}^{\alpha}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R})}^{\|\cdot\|_{I_{-\infty}^{\alpha}}}.$$

Next, we define the fractional Sobolev space $H^{\alpha}(\mathbb{R})$ in terms of the Fourier transform. For $0 < \alpha < 1$, define the seminorm

$$|x|_{\alpha} = \||z|^{\alpha}\widehat{x}\|_{L^2},$$

the norm

$$\|x\|_\alpha = \left(\|x\|_{L^2}^2 + |x|_\alpha^2\right)^{1/2},$$

and let

$$H^\alpha(\mathbb{R}) := \overline{C_0^\infty(\mathbb{R})}^{\|\cdot\|_\alpha}.$$

We note that a function $x \in L^2(\mathbb{R})$ belongs to $I_{-\infty}^\alpha(\mathbb{R})$ if and only if $|z|^\alpha \widehat{x} \in L^2(\mathbb{R})$. In particular, $|x|_{I_{-\infty}^\alpha} = \||z|^\alpha \widehat{x}\|_{L^2(\mathbb{R})}$.

Therefore, $H^\alpha(\mathbb{R})$ and $I_{-\infty}^\alpha(\mathbb{R})$ are equivalent, with equivalent seminorm and norm (see [25]).

Analogous to $I_{-\infty}^\alpha(\mathbb{R})$, we introduce $I_\infty^\alpha(\mathbb{R})$. Let the seminorm

$$|x|_{I_\infty^\alpha} := \|{}_tD_\infty^\alpha\|_{L^2(\mathbb{R})},$$

the norm

$$\|x\|_{I_\infty^\alpha} := \left(\|x\|_{L^2}^2 + |x|_{I_\infty^\alpha}^2\right)^{1/2},$$

and let

$$I_{-\infty}^\alpha(\mathbb{R}) = \overline{C_0^\infty(\mathbb{R})}^{\|\cdot\|_{I_{-\infty}^\alpha}}.$$

Moreover, $I_\infty^\alpha(\mathbb{R})$ and $I_{-\infty}^\alpha(\mathbb{R})$ are equivalent, with equivalent seminorm and norm.

Lemma 2.1 [25]. *If $\alpha > \frac{1}{2}$, then $H^\alpha(\mathbb{R}) \subset C(\mathbb{R})$ and there exists a constant $c = c_\alpha$ such that*

$$\|x\|_{L^\infty} = \sup_{t\in\mathbb{R}} |x(t)| \le c\|x\|_\alpha,$$

where $C(\mathbb{R})$ denote the space of continuous functions from $\mathbb{R}$.

***Remark* 2.2.** If $x \in H^\alpha(\mathbb{R})$, then $x \in L^q(\mathbb{R})$ for all $q \in [2,\infty]$, since

$$\int_{\mathbb{R}} |x(t)|^q\,dt \le \|x\|_{L^\infty}^{q-2}\|x\|_{L^2}^2.$$

Now we introduce a new fractional space. Set

$$X^\alpha = \left\{x \in H^\alpha\left(\mathbb{R},\mathbb{R}^N\right):\ \int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha x(t)|^2 + L(t)x(t)x(t)\,dt < \infty\right\}.$$

The space X^α is a Hilbert space with the inner product

$$(x,y)_{X^\alpha} = \int_{\mathbb{R}} \left(\left({}_{-\infty}D_t^\alpha x(t)\,{}_{-\infty}D_t^\alpha y(t)\right) + L(t)x(t)y(t)\right)dt$$

and the corresponding norm

$$\|x\|_{X^\alpha} = \sqrt{(x,x)_{X^\alpha}}.$$

Lemma 2.2 [25]. *If L satisfies* (L), *then X^α is continuously embedded in $H^\alpha\left(\mathbb{R},\mathbb{R}^N\right)$. Moreover, for all $p \in [2,+\infty]$, there exists $\tau_p > 0$ such that*

$$\|x\|_{L^p} \le \tau_p\|x\|_{X^\alpha} \quad \forall x \in X^\alpha. \tag{2.3}$$

Lemma 2.3 [25]. *If L satisfies* (L), *then X^α is compactly embedded in $L^p\big(\mathbb{R},\mathbb{R}^N\big)$ for all $p\in \in[2,\infty)$.*

Lemma 2.4 [25]. *Under* (L) *and* (H_3)*, if $x_j \rightharpoonup x$ in X^α, then there exists one subsequence still denoted by (x_j) such that $\nabla W_1(t,x_j)\to\nabla W_1(t,x)$ in $L^2\big(\mathbb{R},\mathbb{R}^N\big)$.*

Lemma 2.5. *Under* (L) *and* (H5)*, if $x_j \rightharpoonup x$ in X^α, then $\nabla W_2(t,x_j)\to\nabla W_2(t,x)$ in $L^\rho\big(\mathbb{R},\mathbb{R}^N\big)$, where ρ satisfies $\rho(\vartheta-1)\geq 1$.*

Proof. Assume that $x_j \rightharpoonup x$ in X^α, then, by Banach – Steinhaus theorem, there exists a constant $M>0$ such that

$$\sup_{j\in\mathbb{N}}\|x_j\|_{X^\alpha}\leq M \quad\text{and}\quad \|x\|_{X^\alpha}\leq M. \tag{2.4}$$

Moreover, in view of (H5), for the case $\infty>\rho(\vartheta-1)\geq 1$, it deduces that

$$|\nabla W_2(t,x_j(t))-\nabla W_2(t,x(t))|^\rho \leq b^\rho(t)\Big(|x_j(t)|^{\vartheta-1}+|x(t)|^{\vartheta-1}\Big)^\rho\leq$$
$$\leq 2^{\rho-1}b^\rho(t)\Big(|x_j(t)|^{\rho(\vartheta-1)}+|x(t)|^{\rho(\vartheta-1)}\Big), \tag{2.5}$$

and from (H5), for any $\varepsilon>0$ sufficiently small, there exists some $T>0$ such that $b(t)<\varepsilon$ for $|t|> >T$. Therefore, in view of (2.3) – (2.5) and since X^α is compactly embedded in $L^\infty\big((-T,T),\mathbb{R}^N\big)$, we have

$$\int_{\mathbb{R}}|\nabla W_2(t,x_j(t))-\nabla W_2(t,x(t))|^\rho\,dt\leq$$
$$\leq \int_{|t|\leq T}|\nabla W_2(t,x_j(t))-\nabla W_2(t,x(t))|^\rho\,dt+$$
$$+\int_{|t|>T}2^{\rho-1}b^\rho(t)\Big(|x_j(t)|^{\rho(\vartheta-1)}+|x(t)|^{\rho(\vartheta-1)}\Big)\,dt\leq$$
$$\leq \varepsilon+\varepsilon^\rho 2^{\rho-1}\big(\tau_{\rho(\vartheta-1)}M\big)^{\rho(\vartheta-1)}\leq \varepsilon\Big(1+2^\rho\big(\tau_{\rho(\vartheta-1)}M\big)^{\rho(\vartheta-1)}\Big). \tag{2.6}$$

For the case that $\rho=\infty$, on account of (2.3) and (2.4), we obtain

$$|\nabla W_2(t,x_j(t))-\nabla W_2(t,x(t))|^\rho\,dt\leq \max_{|t|\leq T}|\nabla W_2(t,x_j(t))-\nabla W_2(t,x(t))|+$$
$$+\sup_{|t|>T}b(t)\Big(|x_j(t)|^{(\vartheta-1)}+|x(t)|^{\vartheta-1}\Big)\leq\varepsilon\Big(1+2(\tau_\infty M)^{\vartheta-1}\Big),$$

which combining with (2.6) yields that the proof is complete.

Remark **2.3.** From Lemma 2.5, it is obvious that if $x\to x_0$ in X^α, then $\nabla W_2(t,x)\to \to\nabla W_2(t,x_0)$ in $L^\rho\big(\mathbb{R},\mathbb{R}^N\big)$.

Define the functional $I\colon X^\alpha\to\mathbb{R}$ by

$$I(x)=\int_{\mathbb{R}}\left(\frac12\,|{}_{-\infty}D_t^\alpha x(t)|^2+\frac12\,L(t)x(t)x(t)-W(t,x(t))\right)dt=$$

$$= \frac{1}{2}\|x\|_{X^\alpha}^2 - \int_{\mathbb{R}} W(t, x(t))\, dt.$$

Under the conditions of Theorem 1.1, we see that I is a continuously Fréchet-differentiable functional defined on X^α, i.e., $I \in C^1(X^\alpha, \mathbb{R})$. Moreover, we have

$$I'(x)y = \int_{\mathbb{R}} \left(({}_{-\infty}D_t^\alpha x(t)\,{}_{-\infty}D_t^\alpha y(t)) + L(t)x(t)y(t) - \nabla W(t, x(t))y(t)\right) dt$$

for all $x, y \in X^\alpha$, which yields

$$I'(x)x = \|x\|_{X^\alpha}^2 - \int_{\mathbb{R}} \nabla W(t, x(t))x(t)\, dt.$$

According to [25], we know that in order to find homoclinic solutions of (FHS), it is sufficient to obtain the critical points of I. For this purpose we recall the following definitions and results (see [28]).

Let E be a Hilbert space with inner product $\langle ., . \rangle$ and the corresponding norm. Then there exists orthogonal basis $\{e_k\}_{k\in\mathbb{N}}$. For every $k \in \mathbb{N}$, denote by

$$X_k = \operatorname{span}\{e_k\}, \quad Y_k = \oplus_{j=0}^{k} X_j, \quad Z_k = \overline{\oplus_{j=k}^{\infty} X_j}.$$

Lemma 2.6. *Assume that*

(A1) $I \in C^1(E, \mathbb{R})$ *is an even functional.*

If, for every $k \in \mathbb{N}$, *there exist* $\rho_k > r_k > 0$ *such that*

(A2) $a_k := \max_{x\in Y_k, \|x\|=\rho_k} I(x) \leq 0$;

(A3) $b_k := \inf_{x\in Z_k, \|x\|=r_k} I(x) \to +\infty$ *as* $k \to +\infty$;

(A4) I *satisfies the* $(PS)_c$*-condition for every* $c > 0$, *i.e., for any sequence* $(x_j) \subset E$ *such that* $I(x_j) \to c$ *and* $I'(x_j) \to 0$ *has a convergent subsequence,*
then I *has an unbounded sequence of critical values.*

Definition 2.3. *Let* $I \in C^1(E, \mathbb{R})$ *and* $c \in \mathbb{R}$. *The functional* I *satisfies the* $(PS)_c^*$*-condition (with respect to* Y_n*) if any sequence* $(x_{n_j}) \subset E$ *such that*

$$n_j \to +\infty, \quad x_{n_j} \in Y_{n_j}, \quad I(x_{n_j}) \to c, \quad I|_{Y_{n_j}}' \to 0$$

contains a subsequence converging to a critical point of I.

Lemma 2.7. *Let* $I \in C^1(E, \mathbb{R})$ *is an even functional. If there exists a* $k_0 > 0$ *such that, for every* $k \geq k_0$, *there exist* $\rho_k > r_k > 0$ *such that*

(B1) $a_k := \inf_{x\in Z_k, \|x\|=\rho_k} I(x) \geq 0$;

(B2) $b_k := \max_{x\in Y_k, \|x\|=r_k} I(x) < 0$;

(B3) $d_k := \inf_{x\in Z_k, \|x\|\leq\rho_k} I(x) \to 0$ *as* $k \to +\infty$;

(B4) I *satisfies the* $(PS)_c^*$*-condition for every* $c \in [d_{k_0}, 0)$,
then I *has a sequence of negative critical values converging to* 0.

3. Proof of Theorem 1.1.

Lemma 3.1. *Suppose that the hypothesis of Theorem* 1.1 *are satisfied, then* I *satisfies* $(PS)_c^*$-*condition.*

Proof. Assume that $(x_{n_j}) \subset X^\alpha$ is a sequence such that $x_{n_j} \in Y_{n_j}, I(x_{n_j}) \to c$ and $I|'_{Y_{n_j}} \to 0$. Firstly, we show that (x_{n_j}) is bounded. From (1.2) and (2.3), we have

$$2c \geq I(x_{n_j}) = \frac{1}{2}\|x_{n_j}\|^2_{X^\alpha} - \int\limits_{\mathbb{R}} W(t, x_{n_j}(t))\,dt \geq$$

$$\geq \frac{1}{2}\|x_{n_j}\|^2_{X^\alpha} - C\|x_{n_j}\|^\vartheta_{L^\vartheta} - C\|x_{n_j}\|^\mu_{L^\mu} \geq$$

$$\geq \frac{1}{2}\|x_{n_j}\|^2_{X^\alpha} - C\tau_\vartheta^\vartheta\|x_{n_j}\|^\vartheta - C\tau_\mu^\mu\|x_{n_j}\|^\mu,$$

which yields that (x_{n_j}) is bounded. Therefore, there exists a subsequence, still denoted by (x_{n_j}), such that $x_{n_j} \rightharpoonup x$. On the other hand, since $X^\alpha = \overline{\cup_{n_j} Y_{n_j}}$, we can choose $y_{n_j} \in Y_{n_j}$ such that $y_{n_j} \to x$. Therefore,

$$\lim_{n_j\to+\infty} I'(x_{n_j})(x_{n_j} - x) = \lim_{n_j\to+\infty} I'(x_{n_j})(x_{n_j} - y_{n_j}) + \lim_{n_j\to+\infty} I'(x_{n_j})(y_{n_j} - x) =$$

$$= \lim_{n_j\to+\infty} I|'_{Y_{n_j}}(x_{n_j})(x_{n_j} - y_{n_j}) = 0.$$

In addition, since $x_{n_j} \rightharpoonup x$, we have $I'(x_{n_j})(x_{n_j} - x) \to 0$, and according to Lemmas 2.4 and 2.5 and the Hölder inequality, we obtain

$$\int\limits_{\mathbb{R}} \big(\nabla W(t, x_{n_j}(t)) - \nabla W(t, x(t))\big)\big(x_{n_j}(t) - x(t)\big)\,dt =$$

$$= \int\limits_{\mathbb{R}} \big(\nabla W_1(t, x_{n_j}(t)) - \nabla W_1(t, x(t))\big)\big(x_{n_j}(t) - x(t)\big)\,dt +$$

$$+ \int\limits_{\mathbb{R}} \big(\nabla W_2(t, x_{n_j}(t)) - \nabla W_2(t, x(t))\big)\big(x_{n_j}(t) - x(t)\big)\,dt \to 0.$$

On the other hand, an easy computations shows that

$$\big(I'(x_{n_j}) - I'(x)\big)(x_{n_j} - x) = \|x_{n_j} - x\|^2_{X^\alpha} -$$

$$- \int\limits_{\mathbb{R}} \big(\nabla W(t, x_{n_j}(t)) - \nabla W(t, x(t)))(x_{n_j}(t) - x(t)\big)\,dt,$$

which yields that $\|x_{n_j} - x\|_{X^\alpha} \to 0$. Futhermore, we have $I'(x_{n_j}) \to I'(x)$.

In what follows, we show that $I'(x) = 0$. For any given $z_k \in Y_k$, if $n_j \geq k$, then $z_k \in Y_{n_j}$. Since

$$I'(x)z_k = (I'(x) - I'(x_{n_j}))z_k + I'(x_{n_k})z_k = \big(I'(x) - I'(x_{n_j})\big)z_k + I|'_{Y_{n_j}}(x_{n_k})z_k,$$

we have

$$I'(x)z_k = 0 \quad \text{for all} \quad z_k \in Y_k \subset Y_{n_j},$$

that is, $I'(x) = 0$, which implies that, for any $c \in \mathbb{R}$, the functional I satisfies the $(PS)_c^*$-condition.

Lemma 3.1 is proved.

***3.1. Proof of Theorem* 1.1.** According to (H1) and Remark 1.1, it is obvious that I is even and $I(0) = 0$. Next, we divide the proof into two steps.

***Step* 1.** To obtain the existence of (x_k), define

$$\eta_k = \sup_{y\in Z_k,\ \|y\|_{X^\alpha}=1} \int_{\mathbb{R}} |y(t)|^\vartheta\, dt, \quad \xi_k = \sup_{y\in Z_k,\ \|y\|_{X^\alpha}=1} \int_{\mathbb{R}} |y(t)|^\mu\, dt, \tag{3.1}$$

it is easy to check that $\eta_k,\ \xi_k > 0$ and $\eta_k,\ \xi_k \to 0$. For any $x \in Z_k$ with $\|x\|_{X^\alpha} = \lambda \ge 1$, define $y = \dfrac{x}{\|x\|_{X^\alpha}}$. Then $y \in Z_k$ and $\|y\|_{X^\alpha} = 1$. Thus, in view of (1.2), we have

$$\int_{\mathbb{R}} W(t, x(t))\, dt \le C \int_{\mathbb{R}} \Big(\lambda^\vartheta |y(t)|^\vartheta + \lambda^\mu |y(t)|^\mu\Big)\, dt \le C\Big(\lambda^\vartheta \eta_k + \lambda^\mu \xi_k\Big).$$

As a result, we obtain

$$I(x) \ge \frac{1}{2}\, \|x\|^2_{X^\alpha} - C\eta_k \|x\|^\vartheta_{X^\alpha} - C\xi_k \|x\|^\mu_{X^\alpha}.$$

On the other hand, $\eta_k \to 0$, then, for k large enough, $C\eta_k < \dfrac{1}{4}$. Therefore, we get

$$I(x) \ge \frac{1}{4}\, \|x\|^2_{X^\alpha} - C\xi_k \|x\|^\mu_{X^\alpha}, \quad \|x\|_{X^\alpha} \ge 1.$$

Set

$$r_k = \left(\frac{1}{8C\xi_k}\right)^{\frac{1}{\mu-2}},$$

then, for any given $x \in Z_k$ with $\|x\|_{X^\alpha} = r_k$, one deduces that

$$I(x) \ge \frac{1}{8} \left(\frac{1}{8C\xi_k}\right)^{\frac{2}{\mu-2}}.$$

Moreover, $\xi_k \to 0$. Consequently,

$$\inf_{x\in Z_k, \|x\|_{X^\alpha}=r_k} I(x) \to +\infty \quad \text{as} \quad k \to +\infty,$$

which implies that (A3) of Lemma 2.6 is verified. On the other hand, for any $y \in Y_k$ with $\|y\|_{X^\alpha} = 1$, there exists some compact subset $\Omega \subset \mathbb{R}$ such that $y(t) \ne 0$ for all $t \in \Omega$. Taking $s > 0$ large enough such that $s|v(t)| \ge 1$ for all $t \in \Omega$, then, in view of (1.3), we have

$$I(sy) = \frac{s^2}{2}\, \|v\|^2_{X^\alpha} - \int_{\mathbb{R}} W(t, sy(t))\, dt \le \frac{s^2}{2} - \int_{\Omega} e(t)|sv(t)|^\mu\, dt \le \frac{s^2}{2} - \underline{e} s^\mu \int_{\Omega} |y(t)|^\mu\, dt,$$

where $\underline{e} = \min_{t\in\mathbb{R}} e(t)$. Since I is even and $\mu > 2$, there exists $s_y > 0$ such that

$$I(sv) < 0 \quad \forall |s| \ge s_v > 0.$$

Since $Y_k \subset X^\alpha$ is a finite dimentional subspace, there exists $\rho_k > r_k$ such that $I(\rho_k y) < 0$ for any $y \in Y_k$. Set $x = \rho_k y$, then it indicates that

$$\max_{x\in Y_k, \|x\|^2_{X^\alpha}=\rho_k} I(x) \le 0,$$

which yields that (A_2) of Lemma 2.6 holds. Up to now, all the conditions of Lemma 2.6 are verified.

Therefore, (FHS) possesses infintitely many homoclinic solutions (x_k) such that

$$\int_{\mathbb{R}} \left(\frac{1}{2} \left| {}_{-\infty}D_t^{\alpha} x_k(t) \right|^2 + \frac{1}{2} L(t) x_k(t) x_k(t) - W(t, x_k(t)) \right) dt \to +\infty \quad \text{as} \quad k \to +\infty.$$

***Step* 2.** We apply Lemma 2.7 to obtain the existence of (y_k). For any $y \in Z_k$ with $\|y\|_{X^\alpha} = 1$ and $0 < s < 1$, we obtain

$$I(sy) \ge \frac{s^2}{2} - C\eta_k s^{\vartheta} - C\xi_k s^{\mu},$$

where η_k, ξ_k are defined in (3.1). Since $\xi_k \to 0$, so, for k large enough, it deduces that $C\xi_k < \dfrac{1}{4}$. Moreover, since $\mu > 2$ and $0 < s < 1$, we get

$$I(sy) \ge \frac{1}{4} s^2 - C\eta_k s^{\vartheta}. \tag{3.2}$$

Choose $\rho_k = (4C\eta_k)^{\frac{1}{2-\vartheta}}$, then, for k large enough, $\rho_k < 1$. Let $x = \rho_k y$, then $x \in Z_k$, $\|x\|_{X^\alpha} = \rho_k$ and $I(x) \ge 0$. Therefore, for k large enough, one deduces that

$$\max_{x \in Z_k, \|x\|_{X^\alpha} = \rho_k} I(x) \ge 0,$$

which implies that (B1) holds. Define

$$\delta_k = \inf_{y \in Y_k, \|y\|_{X^\alpha} = 1} \int_{\mathbb{R}} a(t) |y(t)|^{\vartheta} \, dt,$$

then $\delta_k > 0$. For any $y \in Y_k$ with $\|y\|_{X^\alpha} = 1$ and $0 < s < 1$, by (H2) and (H4), we have

$$I(sy) \le \frac{1}{2} s^2 - \delta_k s^{\vartheta}.$$

Since $\vartheta < 2$, there exists $r_k \in (0, \rho_k)$ such that $I(r_k y) < 0$. Let $x = r_k y$, then $x \in Y_k$ with $\|x\|_{X^\alpha} = r_k$ and $I(x) < 0$, which indicates that (B2) is satisfied.

In view of $Y_k \cap Z_k \neq \varnothing$ and $r_k < \rho_k$, it is easy to verify that

$$d_k = \inf_{x \in Z_k, \|x\|_{X^\alpha} \le \rho_k} I(x) \le \inf_{x \in Y_k, \|x\|_{X^\alpha} = r_k} I(x) < 0.$$

On the other hand, for $y \in Z_k$ with $\|y\|_{X^\alpha} = 1$ and $0 < s \le \rho_k$, let $x = sy$, then $x \in Z_k$ with $\|x\|_{X^\alpha} \le \rho_k < 1$. Consequently, according to (3.2) and $\rho < 1$, we obtain that

$$I(x) = I(sy) \ge \frac{1}{4} s^2 - C\eta_k s^{\vartheta} \ge -C\eta_k s^{\vartheta} \ge -C\eta_k,$$

which yields that $d_k \to 0$, i.e., (B3) holds.

References

1. O. Agrawal, J. Tenreiro Machado, J. Sabatier, *Fractional derivatives and their application*, Nonlinear Dynam., Springer-Verlag, Berlin (2004).

2. A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **14**, № 4, 349 – 381 (1973).
3. A. Ambrosetti, V. C. Zelati, *Multiple homoclinic orbits for a class of conservative systems*, Rend. Semin. Mat. Univ. Padova, **89**, 177 – 194 (1993).
4. A. Bahri, *Critical points at infinity in some variational problems*, Pitman Res. Notes in Math. Ser., London (1988).
5. Z. Bai, H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. and Appl., **311**, № 2, 495 – 505 (2005).
6. A. Benhassine, *Multiplicity of solutions for nonperiodic perturbed fractional Hamiltonian equations*, Electron. J. Different. Equat., **93**, 1 – 15 (2017).
7. A. Benhassine, *Multiple of homoclinic solutions for a perturbed dynamical systems with combined nonlinearities*, Mediterr. J. Math., **14**, № 3, 1 – 20 (2017).
8. A. Benhassine, *Existence and multiplicity of periodic solutions for a class of the second order Hamiltonian systems*, Nonlinear Dyn. and Syst. Theory, **14**, № 3, 257 – 264 (2014).
9. A. Benhassine, *Existence and infinitely of many solutions for a nonperiodic fractional Hamiltonian systems*, Different. and Integral Equat. (to appear).
10. D. Benson, S. Wheatcraft, M. Meerschaert, *Application of a fractional advection-dispersion equation*, Water Resour. Res., **36**, № 6, 1403 – 1412 (2000).
11. D. Benson, S. Wheatcraft, M. Meerschaert, *The fractional-order governing equation of lvy motion*, Water Resour. Res., **36**, № 6, 1413 – 1423 (2000).
12. P. C. Carriao, O. H. Miyagaki, *Existence of homoclinic solutions for a class of time-dependent Hamiltonian systems*, J. Math. Anal. and Appl., **230**, 157 – 172 (1999).
13. P. Chen, X. He, X. H. Tang, *Infinitely many solutions for a class of fractional Hamiltonian systems via critical point theory*, Math. Methods Appl. Sci., **39**, № 5, 1005 – 1019 (2016).
14. Y. Ding, *Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems*, Nonlinear Anal., **25**, № 11, 1095 – 1113 (1995).
15. V. Ervin, J. Roop, *Variational formulation for the stationary fractional advection dispersion equation*, Numer. Methods Partial Different. Equat., **22**, 58 – 76 (2006).
16. R. Hilfer, *Applications of fractional calculus in physics*, World Sci., Singapore (2000).
17. W. Jang, *The existence of solutions for boundary-value problems of fractional differential equations at resonance*, Nonlinear Anal., **74**, № 5, 1987 – 1994 (2011).
18. F. Jiao, Y. Zhou, *Existence results for fractional boundary-value problem via critical point theory*, Internat. J. Bifurcation and Chaos, **22**, № 4, 1 – 17 (2012).
19. F. Jiao, Y. Zhou, *Existence of solutions for a class of fractional boundary value problem via critical point theory*, Comput. Math. Appl., **62**, № 3, 1181 – 1199 (2011).
20. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Math. Studies, Singapore (2006).
21. W. Omana, M. Willem, *Homoclinic orbits for a class of Hamiltonian systems*, Different. Integral Equat., **5**, № 5, 1115 – 1120 (1992).
22. H. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Gauthier-Villars, Paris (1897 – 1899).
23. P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, Amer. Math. Soc., Providence, RI, **65**, 45 – 60 (1986).
24. P. H. Rabinowitz, K. Tanaka, *Some results on connecting orbits for a class of Hamiltonian systems*, Math. Z., **206**, № 3, 473 – 499 (1991).
25. C. Torres, *Existence of solution for a class of fractional Hamiltonian systems*, Electron. J. Different. Equat., **2013**, № 259, 1 – 12 (2013).
26. Z. Zhang, R. Yuan, *Variational approach to solutions for a class of fractional Hamiltonian systems*, Math. Methods Appl. Sci., **37**, № 13, 1873 – 1883 (2014).
27. Z. Zhang, R. Yuan, *Existence of solutions to fractional Hamiltonian systems with combined nonlinearities*, Electron. J. Different. Equat., **2016**, № 40, 1 – 13 (2016).
28. W. Zou, *Variant fountain theorems and their applications*, Manuscripta Math., **104**, 343 – 358 (2001).

Received 14.09.18