DOI: 10.37863/umzh.v75i2.6770

UDC 517.9

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ON A PROXIMAL POINT ALGORITHM FOR SOLVING MINIMIZATION PROBLEM AND COMMON FIXED POINT PROBLEM IN CAT(k) SPACES

ПРО ПРОКСИМАЛЬНИЙ ТОЧКОВИЙ АЛГОРИТМ ДЛЯ РОЗВ'ЯЗУВАННЯ ЗАДАЧІ МІНІМІЗАЦІЇ ТА СПІЛЬНОЇ ЗАДАЧІ ПРО НЕРУХОМУ ТОЧКУ В ПРОСТОРАХ CAT(k)

We propose a new modified proximal point algorithm in the setting of CAT(1) spaces, which can be used for solving the minimization problem and the common fixed-point problem. In addition, we prove several convergence results for the proposed algorithm under certain mild conditions. Further, we provide some applications for the convex minimization problem and the fixed point problem in the CAT(k) spaces with a bounded positive real number k. In the process, several relevant results available in the existing literature are generalized and improved.

Запропоновано новий модифікований проксимальний точковий алгоритм у постановці просторів CAT(1), який можна використовувати для розв'язування задачі мінімізації та спільної задачі про нерухому точку. Крім того, доведено кілька результатів про збіжність запропонованого алгоритму за деяких слабких умов. Далі, наведено деякі застосування до задачі опуклої мінімізації та задачі про нерухому точку в просторах CAT(k) з обмеженим додатним дійсним числом k. У процесі узагальнено та вдосконалено кілька відповідних результатів, відомих з літератури.

1. Introduction. Monotone operator theory holds an important place in nonlinear analysis. It plays a crucial role in convex analysis, optimization, variational inequalities, semigroup theory and evolution equations. Many nonlinear operator equations are of the following form:

$$0 \in A(x),$$

where A is a monotone operator in a Hilbert space X. A zero of a maximal monotone operator is a solution of the variational inequality problem associated to the monotone operator and also an equilibrium point of the evolution equation governed by the monotone operator as well as a solution of the minimization problem for a convex function when the monotone operator is a subdifferential of the convex function. Therefore, the existence and approximation of a zero of a maximal monotone operator is the center of consideration of many recent researchers.

The most popular method for approximation of a zero of a maximal monotone operator is the proximal point algorithm popularly known as the PPA. Its origin goes back to Martinet [1], Rockafellar [2] and Brézis and Lions [3]. Martinet introduced the PPA for variational inequality problem whereas Rockafellar showed the weak convergence of the sequence generated by the proximal point algorithm to a zero of the maximal monotone operator in Hilbert spaces. Güler's counterexample [4] showed that the sequence generated by the proximal point algorithm does not necessarily converge strongly even if the maximal monotone operator is the subdifferential of a convex, proper and lower semicontinuous function.

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Following this, Many mathematicians have tried to modify the PPA in such a way that the new iterative methods generate the sequences which converges strongly [5 – 8]. The literature on the subject has become too extensive to be even partially listed here. For some generalization in Hilbert spaces and Banach spaces, the reader can consult [9 – 13].

Recently, many convergence results for the PPA for solving the optimization problem have been extended from the classical linear spaces such as Euclidean spaces, Hilbert spaces and Banach spaces to the setting of manifolds [14 – 17].

Let X be a Hilbert space and $g\colon X \to (-\infty, \infty]$ be a proper and convex function. One of the major problems in optimization theory is to solve $x \in X$ such that

$$g(x) = \min_{y \in X} g(y).$$

We denote by

$$\arg\min_{y \in X} g(y),$$

the set of a minimizer of a convex function.

The minimizers of the objective convex functionals in the spaces with nonlinearity play a crucial role in the branch of analysis and geometry. Numerous applications in computer vision, machine learning, electronic structure computation, system balancing and robot manipulation can be considered as solving optimization problems on manifolds (see [18 – 21]).

Owing to the usefulness of the PPA, Bačák [22] introduced the proximal point algorithm in CAT(0) space in 2013. Bačák generalized Brézis and Lions [3] on the proximal point algorithm in Hilbert spaces to complete CAT(0) spaces. Inspired by this, numerous results have been obtained for the proximal point algorithm in the setting of CAT(0) spaces (see [23 – 27]).

In 2017, Kimura and Kohsaka [28] obtained the proximal point algorithm in a CAT(1) space (X, d) as follows:

$$\begin{gathered} x_1 \in X, \\ x_{n+1} = \arg\min_{y \in X} \left[g(y) + \frac{1}{\lambda_n} \tan\big(d(y, x_n)\big) \sin\big(d(y, x_n)\big) \right] \end{gathered} \tag{1}$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$. They showed that if g has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ Δ-converges to its minimizer. Following this, Pakkaranang et al. [29, 30], Wairojjana and Saipara [31] also introduced the modified proximal point algorithms in the setting of $\mathrm{CAT}(1)$ spaces.

Fascinated by the ongoing research, in this paper, we introduce a new modified proximal point algorithm for finding a common element of the set of common fixed points of three nonexpansive mappings in $\mathrm{CAT}(1)$ spaces and the minimizers of convex function. Also, we obtain some Δ and strong convergence results of the proposed algorithm under some mild conditions.

2. Preliminaries. In this section, we will mention some basic concepts, definitions, notations and few lemmas for use in the next section.

Let (X, d) be a metric space and $x_1, x_2 \in X$ such that $d(x_1, x_2) = r$. A *geodesic path* from x_1 to x_2 is an isometry $\gamma\colon [0, r] \to X$ such that $\gamma(0) = x_1$ and $\gamma(r) = x_2$. The image of a geodesic path is called the *geodesic segment*. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic. (X, d) is called a *uniquely geodesic space* if every two points of X are joined by exactly one geodesic segment and this unique geodesic segment is denoted by $[x_1, x_2]$. For all $x_1, x_2 \in X$ and $t \in [0, 1]$, there exists a unique $x_3 \in [x_1, x_2]$ such that

$$d(x_1, x_3) = td(x_1, x_2) \quad \text{and} \quad d(x_2, x_3) = (1 - t)d(x_1, x_2).$$

We use the notation $(1 - t)x_1 \oplus tx_2$ for the above mentioned unique point x_3.

A subset C of X is said to be *convex* if it contains every geodesic segment joining any two of its points. The set C is said to be *bounded* if

$$\operatorname{diam}(C) = \sup \big\{d(x_1, x_2) \colon x_1, x_2 \in C\big\} < \infty.$$

Definition 1. *For any $k \in \mathbb{R}$, we use M_k^n to denote the following metric spaces*:

(i) *If $k = 0$, then M_0^n is the Euclidean space E^n.*

(ii) *If $k > 0$, then M_k^n is obtained from the spherical space $\mathbb{S}^n$ by multiplying the distance function by the constant $\dfrac{1}{\sqrt{k}}$.*

(iii) *If $k < 0$, then M_k^n is obtained from the hyperbolic space $\mathbb{H}^n$ by multiplying the distance function by the constant $\dfrac{1}{\sqrt{-k}}$.*

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points $x_1, x_2, x_3 \in \in X$ and three geodesic segments between each pair of vertices. A *comparison triangle* for a geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in M_k^2 such that

$$d(x_i, x_j) = d_{M_k^2}(x_i, x_j) \quad \text{for each} \quad i, j = 1, 2, 3.$$

Also, if $k \leq 0$, then such a comparison triangle always exists in M_k^2 and if $k < 0$, then such a triangle exists whenever $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2D_k$, where $D_k = \dfrac{\pi}{\sqrt{k}}$.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in X is said to satisfy the *CAT*(k) *inequality* if, for any $p, q \in \Delta(x_1, x_2, x_3)$ and for their comparison points $\bar{p}, \bar{q} \in \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$, we have

$$d(p, q) \leq d_{M_k^2}(\bar{p}, \bar{q}).$$

A metric space (X, d) is known as D-geodesic space if any two points of X with distance less than D (where $D > 0$) are joined by a geodesic.

Definition 2. *A metric space (X, d) is called a CAT(k) space if it is D_k-geodesic and any geodesic triangle $\Delta(x_1, x_2, x_3)$ in X with $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2D_k$ satisfies the CAT(k) inequality.*

Let (X, d) be a CAT(1) space such that $d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_1) < 2D_1$ for all $x_1,\ x_2,\ x_3 \in X$. Then the following holds for any $\alpha \in [0, 1]$:

$$\cos d(\alpha x_1 \oplus (1 - \alpha)x_2, x_3) \geq \alpha \cos d(x_1, x_3) + (1 - \alpha) \cos d(x_2, x_3). \tag{2}$$

Let $\{x_n\}$ be a bounded sequence in a complete CAT(1) space X. For all $x \in X$, we denote

$$r\big(x, \{x_n\}\big) = \limsup_{n \to \infty} d(x, x_n).$$

The *asymptotic radius* $r\big(\{x_n\}\big)$ is given by

$$r\big(\{x_n\}\big) = \inf\big\{r(x, x_n) : x \in X\big\}$$

and the *asymptotic center* $A\big(\{x_n\}\big)$ of $\{x_n\}$ is defined as

$$A\big(\{x_n\}\big) = \big\{x \in X : r(x, x_n) = r\big(\{x_n\}\big)\big\}.$$

Definition 3. *Let* (X, d) *be a CAT*(1) *space. A sequence* $\{x_n\}$ *in* X *is said to be* Δ*-convergent to a point* $x \in X$ *if* x *is the unique asymptotic center of every subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$*. In this case, we write* $\Delta\text{-}\lim_{n\to\infty} x_n = x$.

Definition 4. *A mapping* $T : X \to X$ *is said to be demi-compact if, for any sequence* $\{x_n\}$ *in* C *such that* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, $\{x_n\}$ *has a convergent subsequence.*

Definition 5. *Let* (X, d) *be a geodesic metric space.*

(i) *Let* $x_1, x_2, x_3 \in P$, *where* P *is an open set in* X. *Then, for all* $R \in [0, 2]$, P *is said to be a* C_R*-domain if, for any minimal geodesic* $\gamma : [0, 1] \to X$ *between* x_2 *and* x_3 *with* $\alpha \in [0, 1]$, *we have the following*:

$$d^2(x_1, (1-\alpha)x_2 \oplus \alpha x_3) \le (1-\alpha)d^2(x_1, x_2) + \alpha d^2(x_1, x_3) - \frac{R}{2}(1-\alpha)\alpha d^2(x_2, x_3). \quad (3)$$

(ii) *A geodesic metric space* (X, d) *is known as* R*-convex if* X *is itself a* C_R*-domain for any* $R \in [0, 2]$.

A CAT(1) space X is said to be *admissible* if $d(x_1, x_2) < \dfrac{\pi}{2}$ for all $x_1, x_2 \in X$. Further, the sequence $\{x_n\}$ is said to be *spherically bounded* in X if

$$\inf_{y\in X} \limsup_{n\to\infty} d(y, x_n) < \frac{\pi}{2}.$$

A function $g : X \to (-\infty, \infty]$ is said to be *proper* if

$$\mathrm{Dom}(g) = \big\{x \in X : g(x) \in \mathbb{R}\big\} \ne \varnothing.$$

Also, g is said to be *lower semicontinuous* if the set $K = \{x \in X : g(x) \le \beta\}$ is closed in X for all $\beta \in \mathbb{R}$.

For all $\lambda > 0$, define the *resolvent* of a proper lower semicontinuous function g in admissible CAT(1) spaces as follows:

$$R_\lambda(x) = \arg\min_{y\in X}\left[g(y) + \frac{1}{\lambda}\tan d(x, y)\sin d(x, y)\right] \quad \text{for all} \quad x \in X.$$

The mapping R_λ is well defined and the set of fixed points of the resolvent associated with g coincides with the set of minimizers of g.

Next, we have the following important lemmas.

Lemma 1 [28]**.** *Let* (X, d) *be a admissible complete CAT*(1) *space and* $g : X \to (-\infty, \infty]$ *be a proper lower semicontinuous function. If* $\lambda > 0$, $x \in X$ *and* $u \in \arg\min_X g$, *then the following inequalities hold*:

$$\frac{\pi}{2}\left(\frac{1}{\cos^2 d(R_\lambda x, x)} + 1\right)\big(\cos d(R_\lambda x, x)\cos d(u, R_\lambda x) - \cos d(u, x)\big) \ge \lambda\big(g(R_\lambda x) - g(u)\big) \quad (4)$$

and

$$\cos d(R_\lambda x, x)\cos d(u, R_\lambda x) \ge \cos d(u, x). \quad (5)$$

Lemma 2 [33]**.** *Let (X,d) be a admissible complete CAT(1) space and $g\colon X\to(-\infty,\infty]$ be a proper lower semicontinuous function. Then g is Δ-lower semicontinuous.*

Lemma 3 [32]**.** *Let (X,d) be a admissible complete CAT(1) space and $\{x_n\}$ be a spherical bounded sequence in X. If $\big\{d(x_n,p)\big\}$ is convergent for all $p\in W_\Delta\big(\{x_n\}\big)$, then the sequence $\{x_n\}$ is Δ-convergent.*

In 2014, Panyanak [34] obtained the demiclosedness principle for a total asymptotically mapping in CAT(k) spaces. Since every nonexpansive mapping is a total asymptotically mapping, we have the following result for nonexpansive mappings.

Lemma 4. *Let $T\colon C\to C$ be a nonexpansive mapping defined on a nonempty closed convex subset C of a complete CAT(1) space (X,d). If $\{x_n\}$ is a bounded sequence with $\lim_{n\to\infty}d(x_n,Tx_n)=0$ and Δ-$\lim_{n\to\infty}x_n=y$, then $y\in C$ and $Ty=y$.*

3. Main results. In this section, we state our main results. We begin with a crucial lemma.

Lemma 5. *Let (X,d) be an admissible complete CAT(1) space and $g\colon X\to(-\infty,\infty]$ be a proper lower semicontinuous convex function. Let T_1, T_2 and T_3 be three nonexpansive mappings on X such that $\omega=F(T_1)\cap F(T_2)\cap F(T_3)\cap\arg\min_{x\in X}g(x)\neq\varnothing$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a,b]$ for some $a,b\in(0,1)$ for all $n\geq1$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n\geq\lambda>0$ for all $n\geq1$ and for some λ. Suppose that the sequence $\{x_n\}$ is generated in the following manner for $x_1\in X$:*

$$\begin{gathered}
w_n=\arg\min_{y\in X}\left[g(y)+\frac{1}{\lambda_n}\tan\big(d(y,x_n)\big)\sin\big(d(y,x_n)\big)\right],\\
z_n=(1-\alpha_n)x_n\oplus\alpha_nT_1w_n,\\
y_n=(1-\beta_n)x_n\oplus\beta_nT_2z_n,\\
x_{n+1}=(1-\gamma_n)T_2y_n\oplus\gamma_nT_3y_n
\end{gathered}\tag{6}$$

for all $n\geq1$. Then we have the following:

(i) $\lim_{n\to\infty}d(x_n,p)$ *exists for all* $p\in\omega$,

(ii) $\lim_{n\to\infty}d(x_n,w_n)=0$,

(iii) $\lim_{n\to\infty}d(x_n,T_1x_n)=\lim_{n\to\infty}d(x_n,T_2x_n)=\lim_{n\to\infty}d(x_n,T_3x_n)=0$.

Proof. First, we will show that $\{x_n\}$ is spherically bounded. Note that $w_n=R_{\lambda_n}x_n$ for all $n\geq1$. Let $p\in\omega$. Then, from Lemma 1, we have

$$\min\big(\cos d(p,w_n),\cos d(w_n,x_n)\big)\geq\cos d(p,w_n)\cos d(w_n,x_n)\geq\cos d(p,x_n),\tag{7}$$

which implies that

$$\max\big\{d(p,w_n),d(w_n,x_n)\big\}\leq d(p,x_n).\tag{8}$$

Since the mappings T_1, T_2 and T_3 are nonexpansive mappings and X is admissible, using (2), we get

$$\begin{gathered}
\cos d(p,z_n)=\cos d(p,(1-\alpha_n)x_n\oplus\alpha_nT_1w_n)\geq\\
\geq(1-\alpha_n)\cos d(p,x_n)+\alpha_n\cos d(p,T_1w_n)\geq\\
\geq(1-\alpha_n)\cos d(p,x_n)+\alpha_n\cos d(p,w_n)\geq
\end{gathered}$$

$$\geq (1-\alpha_n)\cos d(p,x_n) + \alpha_n \cos d(p,x_n) = \cos d(p,x_n), \tag{9}$$

$$\cos d(p,y_n) = \cos d(p,(1-\beta_n)x_n \oplus \beta_n T_2 z_n) \geq$$

$$\geq (1-\beta_n)\cos d(p,x_n) + \beta_n \cos d(p,T_2 z_n) \geq$$

$$\geq (1-\beta_n)\cos d(p,x_n) + \beta_n \cos d(p,z_n) \geq$$

$$\geq (1-\beta_n)\cos d(p,x_n) + \beta_n \cos d(p,x_n) = \cos d(p,x_n) \tag{10}$$

and

$$\cos d(p,x_{n+1}) = \cos d(p,(1-\gamma_n)T_2 y_n \oplus \gamma_n T_3 y_n) \geq$$

$$\geq (1-\gamma_n)\cos d(p,T_2 y_n) + \gamma_n \cos d(p,T_3 y_n) \geq$$

$$\geq (1-\gamma_n)\cos d(p,x_n) + \gamma_n \cos d(p,x_n) = \cos d(p,x_n), \tag{11}$$

which yields

$$d(p,x_{n+1}) \leq d(p,x_n) \leq d(p,x_1) < \frac{\pi}{2}. \tag{12}$$

It follows from (8) and (12) that

$$\limsup_{n\to\infty} d(p,w_n) \leq \limsup_{n\to\infty} d(p,x_n) < \frac{\pi}{2}.$$

Therefore, the sequences $\{w_n\}$ and $\{x_n\}$ are spherically bounded. Also, $\sup_{n\geq 1} d(x_n,w_n) < \frac{\pi}{2}$ and $\lim_{n\to\infty} d(p,x_n) < \frac{\pi}{2}$ exists for all $p \in \omega$. Let

$$\lim_{n\to\infty} d(p,x_n) = r \geq 0. \tag{13}$$

Now, we show that $\lim_{n\to\infty} d(x_n,w_n) = 0$. Consider

$$\cos d(p,x_{n+1}) = \cos d(p,(1-\gamma_n)T_2 y_n \oplus \gamma_n T_3 y_n) \geq$$

$$\geq (1-\gamma_n)\cos d(p,T_2 y_n) + \gamma_n \cos d(p,T_3 y_n) \geq$$

$$\geq \cos d(p,x_n) - \gamma_n \cos d(p,x_n) + \gamma_n \cos d(p,y_n),$$

which implies that

$$\gamma_n \cos d(p,x_n) \geq \cos d(p,x_n) - \cos d(p,x_{n+1}) + \gamma_n \cos d(p,y_n),$$

i.e.,

$$\cos d(p,x_n) \geq \frac{1}{\gamma_n}\big(\cos d(p,x_n) - \cos d(p,x_{n+1})\big) + \cos d(p,y_n).$$

Since $\gamma_n \geq a > 0$ for each $n \geq 1$, we get

$$\cos d(p,x_n) \geq \frac{1}{a}\big(\cos d(p,x_n) - \cos d(p,x_{n+1})\big) + \cos d(p,y_n), \tag{14}$$

which on using (13) yields

$$r = \liminf_{n\to\infty} \cos d(p, x_n) \geq \liminf_{n\to\infty} \cos d(p, y_n). \tag{15}$$

Also, from (10), we have

$$\limsup_{n\to\infty} \cos d(p, y_n) \geq \limsup_{n\to\infty} \cos d(p, x_n) = r. \tag{16}$$

Thus, (15) and (16) results into

$$\lim_{n\to\infty} \cos d(p, y_n) = r. \tag{17}$$

Next, consider

$$\cos d(p, y_n) = \cos d(p, (1-\beta_n)x_n \oplus \beta_n T_2 z_n) \geq$$

$$\geq (1-\beta_n)\cos d(p, x_n) + \beta_n \cos d(p, z_n) \geq$$

$$\geq \cos d(p, x_n) - \beta_n \cos d(p, x_n) + \beta_n \cos d(p, z_n),$$

which on using the fact that $\beta_n \geq a > 0$ for all $n \geq 1$ gives

$$\cos d(p, x_n) \geq \frac{1}{a}\big(\cos d(p, x_n) - \cos d(p, y_n)\big) + \cos d(p, z_n), \tag{18}$$

which on using (13) and (17) yields

$$r = \liminf_{n\to\infty} \cos d(p, x_n) \geq \liminf_{n\to\infty} \cos d(p, z_n). \tag{19}$$

Also, from (9), we have

$$\limsup_{n\to\infty} \cos d(p, z_n) \geq \limsup_{n\to\infty} \cos d(p, x_n) = r. \tag{20}$$

Thus, (19) and (20) results into

$$\lim_{n\to\infty} \cos d(p, z_n) = r. \tag{21}$$

From (8) and (9), we get

$$\cos d(p, z_n) \geq (1-\alpha_n)\cos d(p, x_n) + \alpha_n \cos d(p, w_n) \geq$$

$$\geq (1-\alpha_n)\cos d(p, x_n) + \alpha_n \frac{\cos d(p, x_n)}{\cos d(w_n, x_n)} =$$

$$= \cos d(p, x_n) + \alpha_n \cos d(p, x_n)\left[\frac{1}{\cos d(w_n, x_n)} - 1\right],$$

i.e.,

$$\frac{\cos d(p, z_n)}{\cos d(p, x_n)} - 1 \geq \alpha_n \left[\frac{1}{\cos d(w_n, x_n)} - 1\right].$$

Since $\alpha_n \geq a > 0$ for each $n \geq 1$, from (13) and (21), it follows that

$$\lim_{n\to\infty} d(w_n, x_n) = 0, \tag{22}$$

which is same as

$$\lim_{n\to\infty} d(R_{\lambda_n}x_n, x_n) = 0.$$

Also, as $\lambda_n \geq \lambda > 0$ for each $n \geq 1$, we obtain

$$\lim_{n\to\infty} d(R_{\lambda}x_n, x_n) = 0.$$

Next, we prove that $\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = \lim_{n\to\infty} d(x_n, T_3x_n) = 0.$ From (3), we have

$$d^2(p, z_n) = d^2(p, (1-\alpha_n)x_n \oplus \alpha_n T_1 w_n) \leq$$

$$\leq (1-\alpha_n)d^2(p, x_n) + \alpha_n d^2(p, T_1w_n) - \frac{R}{2}(1-\alpha_n)\alpha_n d^2(x_n, T_1w_n) \leq$$

$$\leq (1-\alpha_n)d^2(p, x_n) + \alpha_n d^2(p, x_n) - \frac{R}{2}ab d^2(x_n, T_1w_n) =$$

$$= d^2(p, x_n) - \frac{R}{2}ab d^2(x_n, T_1w_n),$$

which gives

$$d^2(x_n, T_1w_n) \leq \frac{2}{Rab}\big[d^2(p, x_n) - d^2(p, z_n)\big].$$

From here we get

$$\lim_{n\to\infty} d(x_n, T_1w_n) = 0. \tag{23}$$

On using triangle inequality along with (22) and (23), we obtain

$$d(x_n, T_1x_n) \leq d(x_n, T_1w_n) + d(T_1w_n, T_1x_n) \leq$$

$$\leq d(x_n, T_1w_n) + d(w_n, x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Next, consider

$$d^2(p, y_n) = d^2(p, (1-\beta_n)x_n \oplus \beta_n T_2 z_n) \leq$$

$$\leq (1-\beta_n)d^2(p, x_n) + \beta_n d^2(p, T_2z_n) - \frac{R}{2}(1-\beta_n)\beta_n d^2(x_n, T_2z_n) \leq$$

$$\leq (1-\beta_n)d^2(p, x_n) + \beta_n d^2(p, x_n) - \frac{R}{2}ab d^2(x_n, T_2z_n) =$$

$$= d^2(p, x_n) - \frac{R}{2}ab d^2(x_n, T_2z_n),$$

which is equivalent to

$$d^2(x_n, T_2z_n) \leq \frac{2}{Rab}\big[d^2(p, x_n) - d^2(p, y_n)\big].$$

This gives

$$\lim_{n\to\infty} d(x_n, T_2z_n) = 0. \tag{24}$$

Also,

$$d(z_n, x_n) = d((1-\alpha_n)x_n \oplus \alpha_n T_1w_n, x_n) \leq \alpha_n d(T_1w_n, x_n),$$

which on using (23) gives

$$\lim_{n\to\infty} d(z_n, x_n) = 0. \tag{25}$$

On using triangle inequality along with (24) and (25), we get

$$d(x_n, T_2x_n) \leq d(x_n, T_2z_n) + d(T_2z_n, T_2x_n) \leq$$

$$\leq d(x_n, T_2z_n) + d(z_n, x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Now, we have

$$d^2(p, x_{n+1}) = d^2(p, (1-\gamma_n)T_2y_n \oplus \gamma_n T_3y_n) \leq$$

$$\leq (1-\gamma_n)d^2(p, T_2y_n) + \gamma_n d^2(p, T_3y_n) - \frac{R}{2}(1-\gamma_n)\gamma_n d^2(T_2y_n, T_3y_n) \leq$$

$$\leq (1-\gamma_n)d^2(p, x_n) + \gamma_n d^2(p, x_n) - \frac{R}{2}abd^2(T_2y_n, T_3y_n) =$$

$$= d^2(p, x_n) - \frac{R}{2}abd^2(T_2y_n, T_3y_n),$$

which results into

$$d^2(T_2y_n, T_3y_n) \leq \frac{2}{Rab}\big[d^2(p, x_n) - d^2(p, x_{n+1})\big].$$

We obtain

$$\lim_{n\to\infty} d(T_2y_n, T_3y_n) = 0. \tag{26}$$

Consider,

$$d(y_n, x_n) = d((1-\beta_n)x_n \oplus \beta_n T_2z_n, x_n) \leq \beta_n d(T_2z_n, x_n),$$

which on using (24) gives

$$\lim_{n\to\infty} d(y_n, x_n) = 0. \tag{27}$$

Now, triangle inequality along with (24), (25), (26) and (27) yields

$$d(x_n, T_3x_n) \leq d(x_n, T_2z_n) + d(T_2z_n, T_2y_n) + d(T_2y_n, T_3y_n) + d(T_3y_n, T_3x_n) \leq$$

$$\leq d(x_n, T_2z_n) + d(z_n, x_n) + d(x_n, y_n) + d(T_2y_n, T_3y_n) + d(y_n, x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, the assertion (iii) follows.

Lemma 5 is proved.

Theorem 1. *Let* (X, d) *be an admissible CAT*(1) *space and* $g\colon X \to (-\infty, \infty]$ *be a proper lower semicontinuous convex function. Then the sequence* $\{x_n\}$ *generated by* (6) Δ*-converges to an element of* ω.

Proof. Let $p \in \omega$, then $g(p) \leq g(w_n)$ for each $n \geq 1$. Now, from Lemma 1, we get

$$\lambda_n\big(g(w_n) - g(p)\big) \leq \frac{\pi}{2}\Big(\frac{1}{\cos^2 d(w_n, x_n)} + 1\Big)\big(\cos d(w_n, x_n)\cos d(p, w_n) - \cos d(p, x_n)\big),$$

which yields

$$0 \le \lambda_n\big(g(w_n) - g(p)\big) \le \frac{\pi}{2}\Big(\frac{1}{\cos^2 d(w_n, x_n)} + 1\Big)\big(\cos d(w_n, x_n)\cos d(p, w_n) - \cos d(p, x_n)\big). \quad (28)$$

Since $\lambda_n > \lambda > 0$ for each $n \ge 1$, from Lemma 5, we obtain that

$$\lim_{n\to\infty} d(w_n, x_n) = 0, \qquad \lim_{n\to\infty} d(p, x_n) \quad \text{and} \quad \lim_{n\to\infty} d(p, w_n) \quad \text{exist.} \quad (29)$$

From (28) and (29), we have

$$\lim_{n\to\infty} g(w_n) = \inf g(X). \quad (30)$$

Now, we claim that $W_\Delta\big(\{x_n\}\big) \subset \omega$. Let $w \in W_\Delta\big(\{x_n\}\big)$, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which Δ-converges to the point w. Using the fact that $\lim_{n\to\infty} d(w_n, x_n) = 0$, we can say that the subsequence $\{w_{n_i}\}$ of $\{w_n\}$ also Δ-converges to the point w. From Lemma 2 and (30), we have

$$g(w) \le \liminf_{i\to\infty} g(w_{n_i}) \le \lim_{n\to\infty} g(w_n) = \inf g(X).$$

Thus, $w \in \arg\min_{x\in X}$ which gives $W_\Delta\big(\{x_n\}\big) \subset \arg\min_{x\in X} g(x)$. Also,

$$\lim_{n\to\infty} d(x_n, T_1x_n) = \lim_{n\to\infty} d(x_n, T_2x_n) = \lim_{n\to\infty} d(x_n, T_3x_n) = 0$$

and $\{x_n\}$ Δ-converges to w, so it follows from Lemma 4 that $w \in F(T_1) \cap F(T_2) \cap F(T_3)$, which gives $W_\Delta\big(\{x_n\}\big) \subset \omega$. Now, from (29) and $W_\Delta\big(\{x_n\}\big) \subset \omega$, we can observe that $d(w, x_n)$ is convergent for all $w \in W_\Delta\big(\{x_n\}\big)$. On using Lemma 3, we get that $\{x_n\}$ Δ-converges to an element of ω.

Theorem 1 is proved.

Theorem 2. *Let (X, d) be an admissible complete CAT(1) space and $g\colon X \to (-\infty, \infty]$ be a proper lower semicontinuous function. Then the sequence $\{x_n\}$ generated by (6) converges strongly to an element of ω if and only if $\liminf_{n\to\infty} d(x_n, \omega) = 0$, where $d(x_n, \omega) = \big\{\inf(d(x, p)\colon p \in \omega\big\}$.*

Proof. It is obvious that $\liminf_{n\to\infty} d(x_n, \omega) = 0$ if the sequence $\{x_n\}$ converges to a point $p \in \omega$.

For the converse part, let $\liminf_{n\to\infty} d(x_n, \omega) = 0$. For all $p \in \omega$, we have

$$d(x_{n+1}, p) \le d(x_n, p),$$

which gives

$$d(x_{n+1}, \omega) \le d(x_n, \omega).$$

So, $\lim_{n\to\infty} d(x_n, \omega) = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence in X. Let ϵ >0 be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, \omega) = 0$, there exists n_0 such that, for all $n \ge n_0$, we get

$$d(x_n, \omega) < \frac{\epsilon}{4}.$$

In particular, we obtain

$$\inf\big\{d(x_{n_0}, p) : p \in \omega\big\} < \frac{\epsilon}{4},$$

so there must exists a $p^* \in \omega$ such that

$$d(x_{n_0}, p^*) < \frac{\epsilon}{2}.$$

Thus, for $m, n \geq n_0$, we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p^*) + d(x_n, p^*) < 2d(x_{n_0}, p^*) < 2\frac{\epsilon}{2} = \epsilon,$$

which shows that $\{x_n\}$ is a Cauchy sequence in X. Thus, $\{x_n\}$ converges to a point x^* in X and so $d(x^*, \omega) = 0$. Also, $x^* \in \omega$ as ω is closed.

Theorem 2 is proved.

A family $\{P, Q, R, S\}$ of mappings is said to satisfy condition (Ω) if there exists a nondecreasing function $f\colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$d(x, Px) \geq f\big(d(x, F)\big),$$

or

$$d(x, Qx) \geq f\big(d(x, F)\big),$$

or

$$d(x, Rx) \geq f\big(d(x, F)\big),$$

or

$$d(x, Sx) \geq f\big(d(x, F)\big)$$

for all $x \in X$, where $F = F(P) \cap F(Q) \cap F(R) \cap F(S)$.

Theorem 3. *Let (X, d) be an admissible complete CAT(1) space and $g\colon X \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. If the mappings R_λ, T_1, T_2 and T_3 satisfy the condition (Ω), then the sequence $\{x_n\}$ generated by* (6) *converges strongly to an element of ω.*

Proof. From Lemma 5, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in \omega$. So $\lim_{n\to\infty} d(x_n, \omega)$ exists. Now, by using condition (Ω), we get

$$\lim_{n\to\infty} f\big(d(x_n, \omega)\big) \leq \lim_{n\to\infty} d(x_n, R_\lambda x_n) = 0,$$

or

$$\lim_{n\to\infty} f\big(d(x_n, \omega)\big) \leq \lim_{n\to\infty} d(x_n, T_1 x_n) = 0,$$

or

$$\lim_{n\to\infty} f\big(d(x_n, \omega)\big) \leq \lim_{n\to\infty} d(x_n, T_2 x_n) = 0,$$

or

$$\lim_{n\to\infty} f\big(d(x_n, \omega)\big) \leq \lim_{n\to\infty} d(x_n, T_3 x_n) = 0.$$

Therefore, $\lim_{n\to\infty} f\big(d(x_n, \omega)\big) = 0$ which on using property of f gives $\lim_{n\to\infty} d(x_n, \omega) = 0$. Thus, the proof follows from Theorem 2.

Theorem 3 is proved.

Theorem 4. *Let (X, d) be an admissible complete CAT(1) space and $g\colon X \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. If the mappings R_λ, or T_1, or T_2, or T_3 is demicompact, then the sequence $\{x_n\}$ generated by* (6) *converges strongly to an element of ω.*

Proof. From Lemma 5, we have

$$\lim_{n\to\infty} d(x_n, R_\lambda x_n) = \lim_{n\to\infty} d(x_n, T_1 x_n) = \lim_{n\to\infty} d(x_n, T_2 x_n) = \lim_{n\to\infty} d(x_n, T_3 x_n) = 0. \tag{31}$$

Without loss of generality, we may assume that R_λ or T_1 or T_2 or T_3 is demi-compact, then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to $p^* \in X$. Using (31) and the nonexpansiveness of the mappings R_λ, T_1, T_2, T_3, we obtain

$$d(p^*, R_\lambda p^*) = d(p^*, T_1 p^*) = d(p^*, T_2 p^*) = d(p^*, T_3 p^*) = 0,$$

which gives $p^* \in \omega$. Further, we can prove the strong convergence of $\{x_n\}$ to an element of ω.

Theorem 4 is proved.

4. Applications. In this section, we obtain some applications to the convex minimization problem and the common fixed point problem in CAT(k) space, where k is a bounded positive real number.

Throughout this section, we assume that the following assertions hold:

(I) X is a complete CAT(k) space with $d(x_1, x_2) < \dfrac{D_k}{2}$ for all $x_1, x_2 \in X$,

(II) k is a positive real number and $D_k = \dfrac{\pi}{\sqrt{k}}$,

(III) $g\colon X \to (-\infty, \infty]$ is a proper lower semicontinuous convex function,

(IV) $\hat{R}_\lambda$ is a resolvent mapping on X defined as

$$\hat{R}_\lambda(x) = \arg\min_{y\in X}\left[g(y) + \frac{1}{\lambda}\tan(\sqrt{k}d(y,x))\sin(\sqrt{k}d(y,x))\right]$$

for all $\lambda > 0$ and $x \in X$.

Now, $(X, \sqrt{k}d)$ is an admissible complete CAT(1) space and the mapping $\hat{R}_\lambda$ is well defined [33].

So, we have the following results corresponding to Theorems 1, 2, 3, and 4, respectively.

***Corollary* 1.** *Let T_1, T_2 and T_3 be three nonexpansive mappings on X such that $\omega = F(T_1) \cap \cap F(T_2) \cap F(T_3) \cap \arg\min_{x\in X} g(x) \neq \varnothing$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[a,b]$ for some $a, b \in (0,1)$ for all $n \geq 1$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and for some λ. Suppose that the sequence $\{x_n\}$ is generated in the following manner for $x_1 \in X$:*

$$\begin{gathered} w_n = \arg\min_{y\in X}\left[g(y) + \frac{1}{\lambda_n}\tan\left(\sqrt{k}d(y,x_n)\right)\sin\left(\sqrt{k}d(y,x_n)\right)\right], \\ z_n = (1-\alpha_n)x_n \oplus \alpha_n T_1 w_n, \\ y_n = (1-\beta_n)x_n \oplus \beta_n T_2 z_n, \\ x_{n+1} = (1-\gamma_n)T_2 y_n \oplus \gamma_n T_3 y_n \end{gathered} \tag{32}$$

for each $n \geq 1$. If the assumptions (I)–(IV) *hold, then sequence $\{x_n\}$ Δ-converges to an element of ω.*

***Corollary* 2.** *If the assumptions* (I)–(IV) *hold, then the sequence $\{x_n\}$ generated by* (32) *converges strongly to an element of ω if and only if* $\liminf_{n\to\infty} d(x_n, \omega) = 0$, *where* $d(x_n, \omega) = = \inf\{d(x,p) : p \in \omega\}$.

***Corollary* 3.** *If the mappings* R_λ, T_1, T_2 *and* T_3 *satisfy the condition* (Ω) *and the assumptions* (I) – (IV) *are true, then the sequence* $\{x_n\}$ *generated by* (32) *converges strongly to an element of* ω.

***Corollary* 4.** *If the mappings* R_λ, *or* T_1, *or* T_2, *or* T_3 *is demi-compact and the assumptions* (I) – (IV) *are true, then the sequence* $\{x_n\}$ *generated by* (32) *converges strongly to an element of* ω.

5. Conclusion. In this paper, we proposed a new modified proximal point algorithm involving three nonexpansive mappings in the setting of CAT(1) spaces for solving convex minimization problem and common fixed point problem. We proved some strong and Δ-convergence results under mild conditions. Also, we presented an application on convex minimization and common fixed-point problem over CAT(k) spaces with the bounded positive real number k. In the process, we extended the results of Pakkaranang et al. [29, 30] and Wairojjana and Saipara [31] to three nonexpansive mappings using the technique of proximal point algorithm.

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Received 02.06.21