DOI: 10.37863/umzh.v75i2.6602

UDC 517.5

A. B. Patil[1] (AISSMS College of Engineering, Pune, India),
T. G. Shaba (Landmark University, Omu-Aran, Nigeria)

SHARP INITIAL COEFFICIENT BOUNDS AND THE FEKETE – SZEGÖ PROBLEM FOR SOME CERTAIN SUBCLASSES OF ANALYTIC AND BI-UNIVALENT FUNCTIONS

ТОЧНІ ПОЧАТКОВІ КОЕФІЦІЄНТНІ МЕЖІ ТА ЗАДАЧА ФЕКЕТА – СЕГА ДЛЯ ДЕЯКИХ ПІДКЛАСІВ АНАЛІТИЧНИХ ТА БІОДНОВАЛЕНТНИХ ФУНКЦІЙ

We introduce two new subclasses $\mathcal{U}_\Sigma(\alpha,\lambda)$ and $\mathcal{B}_{1\Sigma}(\alpha)$ of analytic bi-univalent functions defined in the open unit disk $\mathbb{U}$, which are associated with the Bazilevich functions. In addition, for functions that belong to these subclasses, we obtain sharp bounds for the initial Taylor – Maclaurin coefficients a_2 and a_3, as well as the sharp estimate for the Fekete – Szegö functional $a_3-\mu a_2^2$.

Введено два нових підкласи $\mathcal{U}_\Sigma(\alpha,\lambda)$ і $\mathcal{B}_{1\Sigma}(\alpha)$ аналітичних біодновалентних функцій, що визначені у відкритому одиничному крузі $\mathbb{U}$ та асоціюються з функціями Базилевича. Крім того, для функцій, що належать до цих підкласів, отримано точні межі для початкових коефіцієнтів Тейлора – Маклорена a_2 і a_3, а також точну оцінку функціонала Фекета – Сега $a_3-\mu a_2^2$.

1. Introduction. After the famous Bieberbach conjecture (1916) (for more details see [6]) many researchers were attracted towards the study of univalent function theory and are studying the geometric and analytic properties of the analytic univalent functions. In continuation of this work, in this paper we introduce and study two new subclasses of analytic bi-univalent functions that are associated with the Bazilevich functions.

Let $\mathcal{A}$ denote the family of functions of the form

$$f(z)=z+a_2z^2+a_3z^3+\ldots+a_kz^k+\ldots,\quad z\in\mathbb{U},\tag{1}$$

which are analytic in the standard open unit disk $\mathbb{U}:=\{z\in\mathbb{C}\colon |z|<1\}$ and satisfies the normalization conditions $[f(z)]_{(z=0)}=0$ and $[f'(z)]_{(z=0)}=1$. $\mathcal{S}$ denote a well-known function class that contains all functions $f\in\mathcal{A}$ which are univalent in $\mathbb{U}$. Further, let $\mathcal{S}^*$ stands for all functions $f\in\mathcal{S}$, which are starlike in $\mathbb{U}$. Whereas, $f\in\mathcal{A}$ given by (1) is known as a starlike function, if it fulfil the condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>0,\quad z\in\mathbb{U}.$$

For a function $f\in\mathcal{A}$ given by (1), let g be an analytic univalent extension of the inverse function f^{-1} to $\mathbb{U}$ (see [6, 19]) given by

$$g(w)=w+b_2w^2+b_3w^3+b_4w^4+\ldots,\tag{2}$$

[1] Corresponding author, e-mail: amol223patil@yahoo.co.in.

where $b_2 = -a_2$, $b_3 = 2a_2^2 - a_3$ and $b_4 = 5a_2a_3 - 5a_2^3 - a_4$. A function $f \in \mathcal{A}$ given by (1), is said to be in the bi-univalent function class Σ if f and f^{-1} both are in the class $\mathcal{S}$.

Lewin [10] has introduced the family Σ of bi-univalent functions and proved that $|a_2|_{f\in\Sigma} < 1.51$. Further, Brannan and Clunie [4] claimed that $|a_2|_{f\in\Sigma} \leq \sqrt{2}$. Subsequently, Styer and Wright [24] proved that there are functions in Σ for which $|a_2| > 4/3$ and then Tan [25] showed that $|a_2|_{f\in\Sigma} \leq 1.485$. Since last fifty years several researchers have been taking efforts to correlate the coefficient estimates and geometrical behavior of these functions.

In fact, after the research work of Brannan and Taha [5], the ground-breaking paper of Srivastava et al. [23] revived the concept of coefficient problem of functions in the bi-univalent function class Σ. Inspired by the investigation of Srivastava et al. [23], many researchers viz. [8, 9, 11, 17, 18, 20, 22] (including the references therein) investigated various subclasses of Σ and worked out estimates on initial coefficients for functions in them. However, still the coefficient estimation problem for $|a_n|$, $n = 3, 4, 5, \ldots$, is open.

In 1973, Singh [21] investigated the Bazilevich class $\mathcal{B}_1(\alpha)$, which consists of functions $f \in \mathcal{A}$ such that

$$\Re\left\{\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z)\right\} > 0, \quad z \in \mathbb{U}, \quad \alpha \geq 0.$$

It is well-known that for $\alpha \geq 0$, this class $\mathcal{B}_1(\alpha)$ consists of functions in $\mathcal{S}$ and, in particular, for $\alpha = 0$ it reduces to functions in $\mathcal{S}^*$. Moreover, for $\alpha = 1$ these functions satisfies the condition $\Re\left\{f'(z)\right\} > 0$, $z \in \mathbb{U}$, and reduces to the close-to-convex class of functions.

For the family $\mathcal{U}(\lambda)$ of all functions f of the form (1) such that

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| < \lambda \quad \text{for some} \quad 0 < \lambda \leq 1,$$

the problem of determining sharp bounds for the second coefficient of functions belong to this family was solved recently by Vasudevarao and Yanagihara [26] with a complicated proof whereas by Obradović et al. [16] with a simpler proof including sharp estimation for the Fekete – Szegö problem (also for more such results see paper [15]).

In 1998, Obradović [13] defined the class $\mathcal{U}(\alpha, \lambda)$, that consists of functions $f \in \mathcal{A}$ which satisfy the condition

$$\left|\left\{\left(\frac{z}{f(z)}\right)^{1+\alpha} f'(z) - 1\right\}\right| < \lambda, \quad z \in \mathbb{U}, \quad 0 < \alpha < 1, \quad 0 < \lambda \leq 1.$$

Observe that, for $\alpha < 0$, this class $\mathcal{U}(\alpha, \lambda)$ correlates to the class $\mathcal{B}_1(\alpha)$. Fournier and Ponnusamy [7] worked on the univalency problem for this class $\mathcal{U}(\alpha, \lambda)$ with α as a complex number.

We have used the following results to prove our theorems.

Lemma 1 [14]. *Let* $f \in \mathcal{U}(\alpha, \lambda)$, $0 < \alpha < 1$, $0 < \lambda \leq 1$. *Then*

$$\left(\frac{z}{f(z)}\right)^{\alpha} = 1 - \alpha\lambda z^{\alpha} \int_0^z \frac{u(x)}{x^{\alpha+1}}\, dx, \tag{3}$$

where $u \in \Omega$ *is of the form* $u(z) = c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ *with* $u(0) = 0,\ |u(z)| < 1$ *and is analytic in* $\mathbb{U}$.

Lemma 2 [12]. *Let* Ω *be the class of functions* $u(z)$ *which are univalent in* $\mathbb{U}$ *and satisfy the conditions* $u(0) = 0,\ |u(z)| < 1$ *for* $z \in \mathbb{U}$. *Then* $|u(z)| \le |z|$ *and if* $u(z) = c_1 z + c_2 z^2 + c_3 z^3 + \ldots \in$ $\in \mathbb{U}$ *then*

$$|c_1| \le 1 \quad \textit{and} \quad |c_2| \le 1 - |c_1|^2.$$

Zaprawa [27], Ali et al. [2], Altinkaya and Yalçin [3] etc. have investigated Fekete – Szegö problem for various subclasses of Σ. Motivated by their work along with the above mentioned results in [13 – 16, 21], in the present paper, for functions belong to the classes $\mathcal{U}_\Sigma(\alpha, \lambda)$ and $\mathcal{B}_{1\Sigma}(\alpha)$, we have obtained sharp bounds on the initial Taylor – Maclaurin coefficients and sharp estimate on the Fekete – Szegö functional.

2. Sharp inequalities for the class $\mathcal{U}_\Sigma(\alpha, \lambda)$.

Definition 1. *A function* $f \in \Sigma$ *of the form* (1) *is said to be in the class* $\mathcal{U}_\Sigma(\alpha, \lambda),\ 0 < \alpha < 1,$ $0 < \lambda \le 1,$ *if the following two conditions are fulfilled:*

$$\left|\left\{\left(\frac{z}{f(z)}\right)^{1+\alpha} f'(z) - 1\right\}\right| < \lambda, \quad z \in \mathbb{U},$$

and

$$\left|\left\{\left(\frac{w}{g(w)}\right)^{1+\alpha} g'(w) - 1\right\}\right| < \lambda, \quad w \in \mathbb{U},$$

where g *is of the form* (2), *be an extension of* f^{-1} *to* $\mathbb{U}$ *and hence* $g \in \mathcal{S}$.

***Remark* 1.** The subclass $\mathcal{U}_\Sigma(\alpha, \lambda)$ forms a branch of bi-univalent functions that belong to the analytic and univalent function class $\mathcal{U}(\alpha, \lambda)$ defined by Obradović [13].

***Remark* 2.** For $\alpha = 1$, the subclass $\mathcal{U}_\Sigma(\alpha, \lambda)$ forms a branch of bi-univalent functions that belong to the analytic and univalent function class $\mathcal{U}(\lambda)$ studied by Obradović et al. [16] (see also the references therein).

Theorem 1. *Let* $f(z) \in \mathcal{U}_\Sigma(\alpha, \lambda),\ 0 < \alpha < 1,\ 0 < \lambda \le 1,$ *is given by* (1). *Then the following sharp estimates hold:*

$$|a_2| \le \begin{cases} \sqrt{\dfrac{\lambda}{2-\alpha}}, & 0 < \lambda \le \dfrac{2(1-\alpha)^2}{(1+\alpha)(2-\alpha)}, \\ \dfrac{\lambda}{1-\alpha}\sqrt{\dfrac{1+\alpha}{2}}, & \dfrac{2(1-\alpha)^2}{(1+\alpha)(2-\alpha)} \le \lambda \le 1, \end{cases}$$

$$|a_3| \le \begin{cases} \dfrac{\lambda}{2-\alpha}, & 0 < \lambda \le \dfrac{2(1-\alpha)^2}{(1+\alpha)(2-\alpha)}, \\ \dfrac{(1+\alpha)\lambda^2}{2(1-\alpha)^2}, & \dfrac{2(1-\alpha)^2}{(1+\alpha)(2-\alpha)} \le \lambda \le 1, \end{cases}$$

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{\lambda}{2-\alpha}, & 0 < \lambda \le \dfrac{2(1-\alpha)^2}{(1+\alpha-2\mu)(2-\alpha)}, \\ \dfrac{(1+\alpha-2\mu)\lambda^2}{2(1-\alpha)^2}, & \dfrac{2(1-\alpha)^2}{(1+\alpha-2\mu)(2-\alpha)} \le \lambda \le 1, \end{cases}$$

where $a_3 - \mu a_2^2$ is the Fekete – Szegö functional with μ as a real number.

Proof. Let $u(z),\ v(w) \in \Omega$ as in Lemma 1 are of the form

$$u(z) = c_1 z + c_2 z^2 + c_3 z^3 + \ldots \tag{4}$$

and

$$v(w) = d_1 w + d_2 w^2 + d_3 w^3 + \ldots . \tag{5}$$

By using equation (3) to the pairs of functions (f, u) and (g, v), we get

$$\left(\frac{z}{f(z)}\right)^{\alpha} = 1 - \alpha\lambda \sum_{j=1}^{\infty} \frac{c_j}{j-\alpha}\, z^j$$

and

$$\left(\frac{w}{g(w)}\right)^{\alpha} = 1 - \alpha\lambda \sum_{j=1}^{\infty} \frac{d_j}{j-\alpha}\, w^j,$$

where $j \in \mathbb{N}$ and f and g are as given in (1) and (2), respectively. For principal values, these equations are equivalent to

$$\frac{f(z)}{z} = \left(1 - \alpha\lambda \sum_{j=1}^{\infty} \frac{c_j}{j-\alpha}\, z^j\right)^{-1/\alpha}$$

and

$$\frac{g(w)}{w} = \left(1 - \alpha\lambda \sum_{j=1}^{\infty} \frac{d_j}{j-\alpha}\, w^j\right)^{-1/\alpha}.$$

This, on some calculations shows that

$$\sum_{j=1}^{\infty} a_{j+1} z^j = \sum_{j=1}^{\infty} \frac{\lambda c_j}{j-\alpha}\, z^j + \frac{1+\alpha}{2}\left(\sum_{j=1}^{\infty} \frac{\lambda c_j}{j-\alpha}\, z^j\right)^2 +$$

$$+ \frac{(1+\alpha)(1+2\alpha)}{6}\left(\sum_{j=1}^{\infty} \frac{\lambda c_j}{j-\alpha} z^j\right)^3 + \ldots \tag{6}$$

and

$$\sum_{j=1}^{\infty} b_{j+1} w^j = \sum_{j=1}^{\infty} \frac{\lambda d_j}{j-\alpha}\, w^j + \frac{1+\alpha}{2}\left(\sum_{j=1}^{\infty} \frac{\lambda d_j}{j-\alpha} w^j\right)^2 +$$

$$+ \frac{(1+\alpha)(1+2\alpha)}{6}\left(\sum_{j=1}^{\infty} \frac{\lambda d_j}{j-\alpha}\, w^j\right)^3 + \ldots, \tag{7}$$

where $b_2 = -a_2,\ b_3 = 2a_2^2 - a_3$ and $b_4 = -\big(5a_2^3 - 5a_2 a_3 + a_4\big)$.

By comparing coefficients in equation (6) and (7), we get

$$a_2 = \frac{\lambda}{1-\alpha}\, c_1, \tag{8}$$

$$a_3 = \frac{\lambda}{2-\alpha}\, c_2 + \frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2}\, c_1^2, \tag{9}$$

$$b_2 = -a_2 = \frac{\lambda}{1-\alpha}\, d_1, \tag{10}$$

$$b_3 = 2a_2^2 - a_3 = \frac{\lambda}{2-\alpha}\, d_2 + \frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2}\, d_1^2. \tag{11}$$

Also, we have $|c_1| \leq 1,\ |d_1| \leq 1,\ |c_2| \leq 1 - |c_1|^2$ and $|d_2| \leq 1 - |d_1|^2$ (see Lemma 2).

Hence, equations (8) and (10) together yields

$$c_1 = -d_1 \quad \text{and} \quad |a_2| \leq \frac{\lambda}{1-\alpha}.$$

Adding (9) and (11), we get

$$2a_2^2 = \frac{\lambda}{2-\alpha}\,(c_2 + d_2) + \frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2}\,\big(c_1^2 + d_1^2\big).$$

This, by using Lemma 2, we obtain

$$|a_2|^2 \leq \frac{\lambda}{2-\alpha}\big(1 - |c_1|^2\big) + \frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2}\,|c_1|^2 =$$

$$= \frac{\lambda}{2-\alpha} + \left[\frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2} - \frac{\lambda}{2-\alpha}\right]|c_1|^2,$$

which gives the desired estimate on a_2 according to the cases when the coefficient of $|c_1|^2$ is either positive or negative along with $|c_1| \leq 1$.

Further, we have

$$a_3 = \frac{\lambda}{2-\alpha}\, c_2 + \frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2}\, c_1^2. \tag{12}$$

It shows that

$$|a_3| \leq \frac{\lambda}{2-\alpha}\,|c_2| + \frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2}\,|c_1|^2 \leq$$

$$\leq \frac{\lambda}{2-\alpha}\big[1 - |c_1|^2\big] + \frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2}\,|c_1|^2 =$$

$$= \frac{\lambda}{2-\alpha} + \left[\frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2} - \frac{\lambda}{2-\alpha}\right]|c_1|^2,$$

which gives the desired estimate on a_3 similarly as in the case of a_2.

Finally, for some real μ, we have

$$a_3 - \mu a_2^2 = \frac{\lambda}{2-\alpha} c_2 + \left[\frac{(1+\alpha)\lambda^2}{2(1-\alpha)^2} - \frac{\mu\lambda^2}{(1-\alpha)^2}\right] c_1^2 =$$

$$= \frac{\lambda}{2-\alpha} c_2 + \frac{(1+\alpha-2\mu)\lambda^2}{2(1-\alpha)^2} c_1^2,$$

which then by following the steps as in a_3 starting from (12) proves the last inequality for the Fekete – Szegö functional $a_3 - \mu a_2^2$.

Theorem 1 is proved.

In this theorem, sharpness for the first part of all the three inequalities are ensure by setting

$$u(z) = z^2 \quad \text{and} \quad v(w) = w^2;$$

whereas sharpness for the second part of all the three inequalities are ensure by setting

$$u(z) = z \quad \text{and} \quad v(w) = -w.$$

Moreover, the sharp coefficient bounds on a_2 and a_3 for the class $\mathcal{U}(\alpha, \lambda)$ given by Ali et al. [1] ensures the sharpness of these results.

3. Sharp inequalities for the class $\mathcal{B}_{1\Sigma}(\alpha)$. For function f of the form (1) belonging to the Bazilevich function class $\mathcal{B}_1(\alpha)$, in 1973 Ram Singh [21] obtained sharp bounds on initial three coefficients a_2, a_3 and a_4. Also, in 2020 Ali et al. [1] gave an alternating proof for the same in their paper. In this section, we define the corresponding analytic and bi-univalent function class $\mathcal{B}_{1\Sigma}(\alpha)$ and using the approach of Ali et al. [1], obtain sharp bounds on the initial Taylor – Maclaurin coefficients a_2 and a_3 and sharp estimate on the Fekete – Szegö functional $a_3 - \mu a_2^2$.

Definition 2. *A function $f \in \Sigma$ of the form* (1) *is said to be in the class $\mathcal{B}_{1\Sigma}(\alpha)$, $\alpha > 0$, if the following two conditions are fulfilled*:

$$\Re\left\{\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z)\right\} > 0, \quad z \in \mathbb{U},$$

and

$$\Re\left\{\left(\frac{g(w)}{w}\right)^{\alpha-1} g'(w)\right\} > 0, \quad w \in \mathbb{U},$$

where g is of the form (2), *be an extension of f^{-1} to $\mathbb{U}$ and hence $g \in \mathcal{S}$.*

***Remark* 3.** The subclass $\mathcal{B}_{1\Sigma}(\alpha)$ forms a branch of bi-univalent functions that belong to the Bazilevich function class $\mathcal{B}_1(\alpha)$ defined by Ram Singh [21].

Theorem 2. *Let $f(z) \in \mathcal{B}_{1\Sigma}(\alpha)$, $\alpha > 0$, is given by* (1). *Then the following sharp estimates hold:*

$$|a_2| \le \begin{cases} \sqrt{\dfrac{2(3+\alpha)}{(2+\alpha)(1+\alpha)^2}}, & 0 < \alpha \le 1, \\ \sqrt{\dfrac{2}{2+\alpha}}, & \alpha \ge 1, \end{cases}$$

$$|a_3| \leq \begin{cases} \dfrac{2(3+\alpha)}{(2+\alpha)(1+\alpha)^2}, & 0 < \alpha \leq 1, \\ \dfrac{2}{2+\alpha}, & \alpha \geq 1, \end{cases}$$

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{2[(2+\alpha)(1-2\mu)+1]}{(2+\alpha)(1+\alpha)^2}, & 0 < \alpha \leq 1-2\mu \quad or \quad 0 < \mu \leq \dfrac{1-\alpha}{2}, \\ \dfrac{2}{2+\alpha}, & \alpha \geq 1-2\mu \quad or \quad \mu \geq \dfrac{1-\alpha}{2}, \end{cases}$$

where $a_3 - \mu a_2^2$ is the Fekete–Szegö functional with μ as a real number.

Proof. Given $f(z) \in \mathcal{B}_{1\Sigma}(\alpha),\ \alpha > 0$, we have

$$\left[\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z)\right] = \frac{1+u(z)}{1-u(z)}$$

and

$$\left[\left(\frac{g(w)}{w}\right)^{\alpha-1} g'(w)\right] = \frac{1+v(w)}{1-v(w)},$$

where $u, v \in \Omega$ are as in (4) and (5). Thus, we obtain

$$\left(\frac{f(z)}{z}\right)^{\alpha} = 1 + \frac{2\alpha}{z^\alpha} \int\limits_0^z x^{\alpha-1} \frac{u(x)}{1-u(x)}\, dx$$

and

$$\left(\frac{g(w)}{w}\right)^{\alpha} = 1 + \frac{2\alpha}{w^\alpha} \int\limits_0^w x^{\alpha-1} \frac{v(x)}{1-v(x)}\, dx,$$

which are equivalent to

$$\frac{f(z)}{z} = \left[1 + \frac{2\alpha}{z^\alpha} \int\limits_0^z x^{\alpha-1} \big(u(x) + u^2(x) + \ldots\big) dx\right]^{1/\alpha}$$

and

$$\frac{g(w)}{w} = \left[1 + \frac{2\alpha}{w^\alpha} \int\limits_0^w x^{\alpha-1} \big(v(x) + v^2(x) + \ldots\big) dx\right]^{1/\alpha}.$$

Calculating and comparing initial coefficients in these equations, we get

$$a_2 = \frac{2}{1+\alpha}\, c_1,$$

$$a_3 = \frac{2}{2+\alpha}\, c_2 + \left[\frac{2}{2+\alpha} + \frac{2(1-\alpha)}{(1+\alpha)^2}\right] c_1^2,$$

$$-a_2 = \frac{2}{1+\alpha}\, d_1,$$

$$2a_2^2 - a_3 = \frac{2}{2+\alpha}\, d_2 + \left[\frac{2}{2+\alpha} + \frac{2(1-\alpha)}{(1+\alpha)^2}\right] d_1^2.$$

These equations after some simple computations, yields

$$2a_2^2 = \frac{2}{2+\alpha}\,(c_2+d_2) + \left[\frac{2}{2+\alpha} + \frac{2(1-\alpha)}{(1+\alpha)^2}\right]\left(c_1^2+d_1^2\right), \tag{13}$$

$$a_3 = \frac{2}{2+\alpha} c_2 + \left[\frac{2}{2+\alpha} + \frac{2(1-\alpha)}{(1+\alpha)^2}\right] c_1^2 \tag{14}$$

and

$$a_3 - \mu a_2^2 = \frac{2}{2+\alpha}\, c_2 + \left[\frac{2}{2+\alpha} + \frac{2(1-\alpha-2\mu)}{(1+\alpha)^2}\right] c_1^2. \tag{15}$$

Now, by using the inequalities $|c_1| \le 1$ and $|c_2| \le 1 - |c_1|^2$ in the equations (13) to (15), we can complete the further proof as in the proof of Theorem 1.

In the above theorem, the first parts of all the three inequalities are sharp for

$$\left[\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z)\right] = \left(\frac{1+z}{1-z}\right) \quad \text{and} \quad \left[\left(\frac{g(w)}{w}\right)^{\alpha-1} g'(w)\right] = \left(\frac{1-w}{1+w}\right),$$

whereas the second parts of all the three inequalities are sharp for

$$\left[\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z)\right] = \left(\frac{1+z^2}{1-z^2}\right) \quad \text{and} \quad \left[\left(\frac{g(w)}{w}\right)^{\alpha-1} g'(w)\right] = \left(\frac{1+w^2}{1-w^2}\right).$$

Also, to ensure the sharpness of the inequalities in this theorem we may refer the results given by Singh [21] and Ali et al. [1] for the class $\mathcal{B}_1(\alpha)$.

4. Conclusion. In the present paper, we introduce the subclasses $\mathcal{U}_\Sigma(\alpha,\lambda)$ and $\mathcal{B}_{1\Sigma}(\alpha)$ of analytic and bi-univalent functions that are defined on the unit disk and are associated with the Bazilevich functions. Moreover, to study the interrelationship between geometric behaviour and analytic characterization of the functions belong to these subclasses, we have obtain sharp coefficient bounds on the initial Taylor – Maclaurin coefficients and sharp estimation on the Fekete – Szegö functional. In the present investigation the research work by Obradović [13, 14], Obradović et al. [15, 16] and Singh [21] played an important role.

After the Biberbach conjecture, study of the univalent/multivalent/bi-univalent functions by means of various interesting problems concerning the relationship between geometric behavior and analytic characterization actually became the main key of interest of the researchers in this field. During the development of the univalent function theory, many researchers have obtained non-sharp/sharp estimations on initial coefficients of univalent/multivalent/bi-univalent functions. The results in the present paper can motivate researchers to obtain more interesting results in this field.

References

1. R. M. Ali, S. K. Lee, M. Obradović, *Sharp bounds for initial coefficients and the second Hankel determinant*, Bull. Korean Math. Soc., **57**, № 4, 839 – 850 (2020).
2. R. M. Ali, V. Ravichandran, N. Seenivasagan, *Coefficient bounds for p-valent functions*, Appl. Math. and Comput., **187**, № 1, 35 – 46 (2007).

3. Ş. Altinkaya, S. Yalçin, *Fekete – Szegö inequalities for certain classes of bi-univalent functions*, Int. Sch. Res. Notices, Article ID 327962, 1 – 6 (2014).
4. D. A. Brannan, J. G. Clunie (Eds), *Aspects of contemporary complex analysis*, Proc. of the NATO Advanced Study Institute Held at the University of Durham, Durham, July 1 – 20, 1979, Acad. Press, New York, London (1980).
5. D. A. Brannan, T. S. Taha, *On some classes of bi-univalent functions*, Stud. Univ. Babeş-Bolyai Math., **31**, № 2, 70 – 77 (1986).
6. P. L. Duren, *Univalent functions*, Grundlehren Math. Wiss., **259** (1983).
7. R. Fournier, S. Ponnusamy, *A class of locally univalent functions defined by a differential inequality*, Complex Var. and Elliptic Equat., **52**, № 1, 1 – 8 (2007).
8. B. A. Frasin, M. K. Aouf, *New subclasses of bi-univalent functions*, Appl. Math. Lett., **24**, 1569 – 1573 (2011).
9. S. B. Joshi, S. S. Joshi, H. Pawar, *On some subclasses of bi-univalent functions associated with pseudo-starlike functions*, J. Egyptian Math. Soc., **24**, № 4, 522 – 525 (2016).
10. M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc., **18**, 63 – 68 (1967).
11. U. H. Naik, A. B. Patil, *On initial coefficient inequalities for certain new subclasses of bi-univalent functions*, J. Egyptian Math. Soc., **25**, № 3, 291 – 293 (2017).
12. Z. Nehari, *Conformal mapping*, McGraw-Hill, New York (1952).
13. M. Obradović, *A class of univalent functions*, Hokkaido Math. J., **27**, № 2, 329 – 335 (1998).
14. M. Obradović, *A class of univalent functions II*, Hokkaido Math. J., **28**, № 3, 557 – 562 (1999).
15. M. Obradović, S. Ponnusamy, K.-J. Wirths, *Coefficient characterizations and sections for some univalent functions*, Sib. Math. J., **54**, 679 – 696 (2013).
16. M. Obradović, S. Ponnusamy, K.-J. Wirths, *Geometric studies on the class* $U(\lambda)$, Bull. Malays. Math. Sci. Soc., **39**, 1259 – 1284 (2016).
17. A. B. Patil, U. H. Naik, *Bounds on initial coefficients for a new subclass of bi-univalent functions*, New Trends Math. Sci., **6**, № 1, 85 – 90 (2018).
18. A. B. Patil, T. G. Shaba, *On sharp Chebyshev polynomial bounds for a general subclass of bi-univalent functions*, Appl. Sci., **23**, 109 – 117 (2021).
19. C. Pommerenke, *Univalent functions*, Vandenhoeck and Rupercht, Göttingen (1975).
20. S. Porwal, M. Darus, *On a new subclass of bi-univalent functions*, J. Egyptian Math. Soc., **21**, № 3, 190 – 193 (2013).
21. R. Singh, *On Bazilevič functions*, Proc. Amer. Math. Soc., **38**, 261 – 271 (1973).
22. H. M. Srivastava, D. Bansal, *Coefficient estimates for a subclass of analytic and bi-univalent functions*, J. Egyptian Math. Soc., **23**, № 2, 242 – 246 (2015).
23. H. M. Srivastava, A. K. Mishra, P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett., **23**, 1188 – 1192 (2010).
24. D. Styer, D. J. Wright, *Results on bi-univalent functions*, Proc. Amer. Math. Soc., **82**, № 2, 243 – 248 (1981).
25. D. L. Tan, *Coefficient estimates for bi-univalent functions*, Chin. Ann. Math. Ser. A, **5**, 559 – 568 (1984).
26. A. Vasudevarao, H. Yanagihara, *On the growth of analytic functions in the class* $U(\lambda)$, Comput. Methods Funct. Theory, **13**, 613 – 634 (2013).
27. P. Zaprawa, *On the Fekete – Szegö problem for classes of bi-univalent functions*, Bull. Belg. Math. Soc. Simon Stevin, **21**, № 1, 169 – 178 (2014).

Received 02.03.21