DOI: 10.37863/umzh.v75i3.6788

UDC 517.5

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QUANTITATIVE DEPENDENCE OF SOME DISCRETE LIMITING CLASSES ON THE MUCKENHOUPT $\mathcal{A}_1(u)$ CLASS

КІЛЬКІСНА ЗАЛЕЖНІСТЬ ДЕЯКИХ ДИСКРЕТНИХ ГРАНИЧНИХ КЛАСІВ ВІД КЛАСУ МАКЕНХАУПТА $\mathcal{A}_1(u)$

We prove some relations between the discrete Gehring classes $\mathcal{G}_q$ and the discrete Muckenhoupt classes $\mathcal{A}_p$. Specifically, by using some known Hardy-type and Carleman-type inequalities, we study the relationship between $\mathcal{G}_1$, $\mathcal{A}_\infty$ and $\mathcal{A}_1$ for nonincreasing and nondecreasing weights. Finally, we establish some general results by introducing the notions of $\mathcal{G}_\varphi$ classes defined for nonnegative convex function φ.

Доведено деякі співвідношення між дискретними класами Герінга $\mathcal{G}_q$ і дискретними класами Макенхаупта $\mathcal{A}_p$. Зокрема, застосовуючи деякі відомі нерівності типу Гарді та Карлемана, вивчено зв'язок між $\mathcal{G}_1$, $\mathcal{A}_\infty$ і $\mathcal{A}_1$ для незростаючих та неспадаючих ваг. Крім того, встановлено деякі загальні результати на підставі введеного поняття класів $\mathcal{G}_\varphi$, що визначені для невід'ємної опуклої функції φ.

1. Introduction. In harmonic analysis it is well-known that the boundedness of a series of classical operators (Hardy – Littlewood maximal operator, Hardy's operator, Hilbert's operator, Calderón – Zygmund's operator, etc.) in the weighted spaces $L^p_w(\mathbb{R}^+)$ depends on the A_p-Muckenhoupt condition on the weight w (see [8]). In recent years, the discrete analogues in harmonic analysis becomes an active field of research and some results related to the boundedness of the discrete Hardy – Littlewood maximal operator has been established in [1, 11]. In particular, it has been proved that discrete Hardy – Littlewood maximal operator is bounded in the weighted spaces $\ell^p_w(\mathbb{Z}^+)$ if and only if the discrete weight w satisfies the $\mathcal{A}_p$-Muckenhoupt condition (see [11]). This fact motivated us to study in depth the structure of the discrete Muckenhoupt class and related spaces.

In the following, for the sake of completeness, we present the background and the basic definitions that will be used in this paper. Throughout this paper, $\mathbb{Z}_+$ stands for the set of nonnegative integers, i.e., $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. By an interval J, we mean a finite subset of $\mathbb{Z}_+$ consisting of consecutive integers, i.e., $J = \{a, a+1, \ldots, a+n\}$, $a,\ n \in \mathbb{Z}_+$, and $|J|$ stands for its cardinality. A discrete nonnegative sequence u belongs to the discrete Muckenhoupt class $\mathcal{A}_1(A)$ on the interval $J \subset \mathbb{Z}_+$ for $p > 1$ and $A > 1$, if the inequality

$$\frac{1}{|J|}\sum_{n\in J} u(n) \leq Au(n)$$

holds for every subinterval J and $|J|$ is the cardinality of the set J. That is, $\mathcal{A}_1$-norm is defined by the following quantity:

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$$\mathcal{A}_1(u) = \sup_{J\subset\mathbb{Z}_+} \frac{1}{|J|}\left(\frac{1}{u}\sum_J u\right)$$

or, equivalently,

$$\mathcal{A}_1(u) = \sup_{J\subset\mathbb{Z}_+} \frac{1}{|J|}\left(\sum_J \frac{u}{\operatorname*{ess\,inf}_J u}\right).$$

The sequence u is said belong to the class $\mathcal{A}_1$, if $\mathcal{A}_1(u) < +\infty$. A discrete nonnegative sequence u belongs to the discrete Muckenhoupt class $\mathcal{A}_p(A)$ on the interval J for $p > 1$ and $A > 1$, if the inequality

$$\left(\frac{1}{|J|}\sum_J u\right)\left(\frac{1}{|J|}\sum_J u^{\frac{1}{1-p}}\right)^{p-1} \leq A$$

holds on every subinterval $J \subset \mathbb{Z}_+$. For a given exponent $p > 1$, we define the $\mathcal{A}_p$-norm of the discrete weight u by the quantity

$$\mathcal{A}_p(u) := \sup_{J\subset\mathbb{Z}_+}\left(\frac{1}{|J|}\sum_J u\right)\left(\frac{1}{|J|}\sum_J u^{\frac{1}{1-p}}\right)^{p-1},$$

where the supremum is taken over all intervals $J \subset \mathbb{Z}_+$. When we fix a constant $A > 1$ the couple of real numbers (p, A) defines the discrete Muckenhoupt class $\mathcal{A}_p(A)$:

$$u \in \mathcal{A}_p(A) \iff \mathcal{A}_p(u) \leq A,$$

and we will refer to A as the $\mathcal{A}_p$-constant of the class. The discrete weight u is said belong to the discrete Muckenhoupt $\mathcal{A}_\infty(A)$, if

$$\left(\frac{1}{|J|}\sum_J u\right)\exp\left(\frac{1}{|J|}\sum_J \log\frac{1}{u}\right) \leq A \quad \text{for} \quad A > 1$$

for every interval $J \subset \mathbb{Z}_+$ or, equivalently,

$$\left(\frac{1}{|J|}\sum_J u\right) \leq A\exp\left(\frac{1}{|J|}\sum_J \log u\right),$$

that is, we define $\mathcal{A}_\infty$-norm by the quantity

$$\mathcal{A}_\infty(u) = \sup_{J\subset\mathbb{Z}_+}\left(\frac{1}{|J|}\sum_J u\right)\exp\left(\frac{1}{|J|}\sum_J \log\frac{1}{u}\right)$$

for every subinterval $J \subset \mathbb{Z}_+$. The discrete weight u belongs to the discrete Gehring class $\mathcal{G}_1(\mathcal{K})$, if

$$\exp\left(\frac{1}{|J|}\sum_J \frac{u}{\frac{1}{|J|}\sum_J u}\log\frac{u}{\frac{1}{|J|}\sum_J u}\right) \leq \mathcal{K} \quad \text{for} \quad \mathcal{K} > 1$$

for every subinterval $J \subset \mathbb{Z}_+$. We define $\mathcal{G}_1$-norm by the quantity

$$\mathcal{G}_1(u) := \sup_{J\subset\mathbb{Z}_+}\left(\exp\left(\frac{1}{|J|}\sum_J \frac{u}{\frac{1}{|J|}\sum_J u}\log\frac{u}{\frac{1}{|J|}\sum_J u}\right)\right).$$

For a given exponent $q > 1$ and a constant $\mathcal{K} > 1$, a discrete nonnegative weight u is said belong to the discrete Gehring class $\mathcal{G}_q(\mathcal{K})$ (or satisfies the reverse Hölder inequality), if

$$\left(\frac{1}{|J|}\sum_J u^q\right)^{\frac{1}{q}} \leq \mathcal{K}\frac{1}{|J|}\sum_J u$$

for every subinterval $J \subset \mathbb{Z}_+$. For a given exponent $q > 1$, we define the $\mathcal{G}_q$-norm of u as

$$\mathcal{G}_q(u) := \sup_{J\subset\mathbb{Z}_+}\left(\left(\frac{1}{|J|}\sum_J u\right)^{-1}\left(\frac{1}{|J|}\sum_J u^q\right)^{\frac{1}{q}}\right)^{\frac{q}{q-1}},$$

where the supremum is taken over all intervals $J \subset \mathbb{Z}_+$ and represents the best constant for which the $\mathcal{G}_q$-condition holds true independently on the interval $J \subset \mathbb{Z}_+$. In [2], Böttcher and Seybold proved that if $u \in \mathcal{A}_p(\mathcal{C})$, then there exist a constants $\delta > 0$ and $\mathcal{C} < \infty$, depending only on p and u, such that

$$\frac{1}{|J|}\sum_J u^{p(1+\varepsilon)} \leq \mathcal{C}\left(\frac{1}{|J|}\sum_J u^p\right)^{1+\varepsilon} \tag{1.1}$$

for all $\varepsilon \in [0, \delta]$ and all J of the form $|J| = 2^r$ with $r \in \mathbb{N}$, the set of natural numbers. This means that if $u \in \mathcal{A}_p(\mathcal{K})$, then there exists $\varepsilon > 0$ such that $u \in \mathcal{G}_{p(1+\varepsilon)}(\mathcal{K})$ and

$$\mathcal{A}_p(\mathcal{K}) \subset \mathcal{G}_{p(1+\varepsilon)}(\mathcal{K}).$$

This shows that any Muckenhoupt weight belongs to some Gehring class (*a transition property*). Note that the inequality (1.1) is the reverse of the inequality

$$\left(\frac{1}{|J|}\sum_J u^p\right)^{1+\varepsilon} \leq \frac{1}{|J|}\sum_J u^{p(1+\varepsilon)},$$

which can be obtained directly by a simple application of Hölder's inequality. We say that u is a discrete *Gehring* $\mathcal{G}_q$-weight if its $\mathcal{G}_q$-norm is finite, i.e.,

$$u \in \mathcal{G}_q \iff \mathcal{G}_q(u) < \infty.$$

More generally, for any convex function $\phi\colon (0, +\infty) \to [0, +\infty)$ we set, for $u\colon (0, +\infty) \to [0, +\infty)$,

$$\mathcal{G}_\phi(u) := \sup_{J\subset\mathbb{Z}_+}\frac{\frac{1}{|J|}\sum_J \phi(u)}{\phi\left(\frac{1}{|J|}\sum_J u\right)},$$

and we say that u belongs to the class $\mathcal{G}_\phi$ if and only if $\mathcal{G}_\phi(u) < +\infty$. For more details related to these classes see [13]. To provide more clarity on the previous concepts, the authors in [12] investigated following estimates for a power low sequence weight by making use of [7, Lemma 2.2] and [6, Lemma 2.2], respectively. These estimates will play a crucial role for the sharpness of our constants.

Lemma 1.1. *If $p > 1$ and $-1 < \lambda < p-1$, then we have*

$$\mathcal{A}_p(n^\lambda) \simeq \begin{cases} \dfrac{1}{1+\lambda}\left(\dfrac{p-1}{p-\lambda-1}\right)^{p-1}, & \lambda < 0, \\ \left(\dfrac{p-1}{p-\lambda-1}\right)^{p-1}, & \lambda > 0. \end{cases}$$

Lemma 1.2. *If $p > 1$ and $\lambda > -1/p$, then we have*

$$\left(\mathcal{G}_p(n^\lambda)\right)^{\frac{p-1}{p}} \simeq \begin{cases} \dfrac{1+\lambda}{(1+p\lambda)^{1/p}}, & \lambda < 0, \\ \dfrac{(1+\lambda)}{(1+p\lambda)^{1/p}}, & \lambda > 0. \end{cases}$$

Our aim in this paper is to establish some embedding relations between the two limiting cases $\mathcal{A}_\infty(u)$ and $\mathcal{G}_1(u)$ in terms of $\mathcal{A}_1$-weights. Also, the relations between the norm

$$\mathcal{A}_1(\mathcal{H}u(n)), \text{ where } \mathcal{H}u(n) = \frac{1}{n}\sum_{k=1}^{n} u(k),$$

and the norms of $\mathcal{A}_p(u)$ and $\mathcal{G}_q(u)$ also will be investigated. Since the dependence on the constant $\mathcal{A}_1(\mathcal{H}u(n))$ is precisely preserved with optimal two-sided bounds, we will give a full characterization of the weight u which imply that the discrete classical Hardy operator $\mathcal{H}u(n)$ belongs to $\mathcal{A}_1$-class.

The paper is organized as follows. In Section 2, we will provide the basic notions and lemmas that will be used later. Immediately after that, we will state some basic inequalities, some of them are considered as the limit cases of Hardy-type inequalities as $p \to 1$. In Section 3, we prove the main results of this paper, which give the relationships between $\mathcal{A}_p$, $\mathcal{G}_q$, $\mathcal{G}_1$, $\mathcal{A}_\infty$ and $\mathcal{A}_1$ for both nonincreasing and nondecreasing sequences $\{u_n\}_{n=1}^\infty$. Finally, in Section 4, by presenting $\mathcal{G}_\varphi$-classes defined for nonnegative convex function φ, we prove some generalized results.

2. Basic lemmas. In this section, we will introduce some notations and classical inequalities that will be needed later. Throughout the rest of the paper, we will assume that the weights are nonnegative sequences defined on $\mathbb{Z}_+$. The classical Hölder inequality for every measurable sequences f_n and g_n, defined on $J \subset \mathbb{Z}_+$, is given by

$$\left(\sum_J |f(n)g(n)|\right) \le \left(\sum_J |f(n)|^p\right)^{\frac{1}{p}} \left(\sum_J |g(n)|^q\right)^{\frac{1}{q}}, \tag{2.1}$$

where q is the Hölder conjugate of $p > 1$. The classical discrete Hardy inequality (see [3, Theorem 327]) is given by

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} f(k)\right)^p \le \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} f^p(n), \quad p > 1, \tag{2.2}$$

where $\{f(n)\}_{n=1}^\infty$ is a sequence of nonnegative real numbers. When $p \to +\infty$, the inequality (2.2), written for $f^{\frac{1}{p}}$, tends to Carleman inequality (cf. [5, Theorem 2.4])

$$\sum_{n=1}^{\infty} \exp\left(\frac{1}{n}\sum_{k=1}^{n} \log f(k)\right) \le e \sum_{n=1}^{\infty} f(n). \tag{2.3}$$

Moreover, Hardy inequalities with negative exponents (cf. [9, Corollary 2.1]) is given by

$$\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{-r} \leq \left(\frac{r+1}{r}\right)^{r}\sum_{n=1}^{N}u^{-r}(n), \quad r>0. \tag{2.4}$$

The next lemma gives the discrete version of Fubini's theorem for two different sequences and is adopted from [14].

Lemma 2.1. *Assume that φ, ψ are nonnegative sequences. Then*

$$\sum_{n=1}^{\infty}\varphi(n)\left(\sum_{k=n}^{\infty}\psi(k)\right) = \sum_{n=1}^{\infty}\psi(n)\left(\sum_{k=1}^{n}\varphi(k)\right).$$

In the next lemma, we will prove a characterization of both $\mathcal{G}_1(u)$- and $\mathcal{A}_\infty(u)$- norms for nonincreasing sequence u.

Lemma 2.2. *If u is a nonincreasing sequence, we have that*

i) $\mathcal{G}_1(u) = \sup\limits_J \dfrac{1}{|J|}\sum_J \dfrac{u(s)}{\frac{1}{|I|}\sum_I u(n)}\left(1+\log\dfrac{u(s)}{\frac{1}{|I|}\sum_I u(n)}\right).$

ii) $\mathcal{A}_p \subset \mathcal{A}_\infty$ *for all* $p>1$. *Moreover,* $\mathcal{A}_\infty = \bigcup_{p>1}\mathcal{A}_p$ *and* $\mathcal{A}_\infty = \lim_{p\to\infty}\mathcal{A}_p$.

Proof. i) We can write

$$\begin{aligned}\mathcal{G}_1(u) &= 1+\sup_J \frac{1}{|J|}\sum_J \frac{u(s)}{\frac{1}{|I|}\sum_I u(n)}\log\frac{u(s)}{\frac{1}{|I|}\sum_I u(n)} \\ &= 1+\log\sup_J \exp\frac{1}{|J|}\sum_J \frac{u(s)}{\frac{1}{|I|}\sum_I u(n)}\log\frac{u(s)}{\frac{1}{|I|}\sum_I u(n)}.\end{aligned}$$

By recalling the identity mentioned in [5, Remark 1] that

$$\exp\left(\frac{1}{|J|}\sum_J \log u\right) = \lim_{q\to\infty}\left(\frac{1}{|J|}\sum_J u^{\frac{1}{q}}\right)^{q}, \tag{2.5}$$

we conclude that

$$\lim_{q\to 1}\left(\frac{\left(\frac{1}{|J|}\sum_J u^q(n)\right)^{\frac{1}{q}}}{\frac{1}{|J|}\sum_J u(n)}\right)^{q'} = \exp\frac{1}{|J|}\sum_J \frac{u(s)}{\frac{1}{|I|}\sum_I u(s)}\log\frac{u(s)}{\frac{1}{|I|}\sum_I u(n)},$$

which leads directly to

$$\sup_J \exp\frac{1}{|J|}\sum_J \frac{u(s)}{\frac{1}{|I|}\sum_I u(s)}\log\frac{u(s)}{\frac{1}{|I|}\sum_I u(n)} = \lim_{q\to 1}\sup_J\left(\frac{\left(\frac{1}{|J|}\sum_J u^q(n)\right)^{\frac{1}{q}}}{\frac{1}{|J|}\sum_J u(n)}\right)^{q'}.$$

Similarly, we obtain

$$\sup_J \exp \frac{1}{|J|} \sum_J \frac{u(s)}{\frac{1}{|I|} \sum_I u(s)} \log \frac{u(s)}{\frac{1}{|I|} \sum_I u(n)} = \lim_{q \to 1} \mathcal{G}_q(u).$$

ii) Suppose that $u \in \mathcal{A}_p$ for $p > 1$, that is, there exists $A > 1$ such that

$$\left(\frac{1}{|J|} \sum_J u \right) \left(\frac{1}{|J|} \sum_J u^{\frac{1}{1-p}} \right)^{p-1} \leq A \tag{2.6}$$

for all $J \subset \mathbb{Z}_+$. Taking limit for both sides as $p \to \infty$ in (2.6), we get (cf. [10])

$$\begin{aligned} \left(\frac{1}{|J|} \sum_J u \right) \exp \left(\frac{1}{|J|} \sum_J \log \frac{1}{u} \right) &= \left(\frac{1}{|J|} \sum_J u \right) \exp \left(\frac{1}{|J|} \sum_J \log u \right)^{-1} \\ &= \left(\frac{1}{|J|} \sum_J u \right) \lim_{p \to \infty} \left(\frac{1}{|J|} \sum_J u^{\frac{1}{1-p}} \right)^{p-1} \\ &\leq \lim_{p \to \infty} A = A. \end{aligned}$$

That is, $u \in \mathcal{A}_\infty$, which implies for any $1 < p < \infty$ that $\mathcal{A}_p \subset \mathcal{A}_\infty$ and

$$\bigcup_{1 \leq p < \infty} \mathcal{A}_p \subseteq \mathcal{A}_\infty. \tag{2.7}$$

Conversely, assume that $u \in \mathcal{A}_\infty$ and assume, on the contrary, that, for all $1 \leq p < \infty$, $u \notin \mathcal{A}_p$. Then, for all $1 \leq p < \infty$, we have

$$\sup_{J \subset \mathbb{Z}_+} \left(\frac{1}{|J|} \sum_J u \right) \left(\frac{1}{|J|} \sum_J u^{\frac{1}{1-p}} \right)^{p-1} = \infty,$$

which, by taking the limit as $p \to \infty$, implies that (see (2.5))

$$\sup_{J \subset \mathbb{Z}_+} \left(\frac{1}{|J|} \sum_J u \right) \left(\exp \frac{1}{|J|} \sum_J \log \frac{1}{u} \right) = \infty.$$

This contradicts the assumption that $u \in \mathcal{A}_\infty$. Then $u \in \mathcal{A}_\infty$ implies that $u \in \mathcal{A}_p$ for some $1 \leq p < \infty$, and hence

$$u \in \bigcup_{1 \leq p < \infty} \mathcal{A}_p.$$

Thus,

$$\mathcal{A}_\infty \subseteq \bigcup_{1 \leq p < \infty} \mathcal{A}_p. \tag{2.8}$$

Combining (2.7) and (2.8), we obtain

$$\mathcal{A}_\infty = \bigcup_{1 \leq p < \infty} \mathcal{A}_p.$$

Moreover, again making use of [5, Remark 1], we get

$$\lim_{p\to\infty}\mathcal{A}_p(u)=\lim_{p\to\infty}\sup_{J\subset\mathbb{Z}_+}\left(\frac{1}{|J|}\sum_J u\right)\left(\frac{1}{|J|}\sum_J u^{\frac{1}{1-p}}\right)^{p-1}$$
$$=\sup_{J\subset\mathbb{Z}_+}\lim_{p\to\infty}\left(\frac{1}{|J|}\sum_J u\right)\left(\frac{1}{|J|}\sum_J u^{\frac{1}{1-p}}\right)^{p-1}$$
$$=\sup_{J\subset\mathbb{Z}_+}\left(\frac{1}{|J|}\sum_J u\right)\lim_{p\to\infty}\left(\frac{1}{|J|}\sum_J u^{\frac{1}{1-p}}\right)^{p-1}$$
$$=\sup_{J\subset\mathbb{Z}_+}\left(\frac{1}{|J|}\sum_J u\right)\left(\exp\frac{1}{|J|}\sum_J\log\frac{1}{u}\right)=\mathcal{A}_\infty(u),$$

which claims the assertion.

Lemma 2.1 is proved.

3. Main results. In the sequel, we will start with the main results for this paper and without loss of generality, we will assume that $J=\{1,2,\ldots,n\}$, $n\in\mathbb{Z}_+$, and $|J|=n$ stands for its cardinality. We shall study the two equivalences

$$u\in\mathcal{G}_q\Leftrightarrow\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^q\in\mathcal{A}_1,\quad q>1,$$

and

$$u\in\mathcal{A}_p\Leftrightarrow\frac{1}{n}\sum_{k=1}^{n}u(k)\in\mathcal{A}_1$$

for a nonnegative monotonic sequence u. Some additional limiting cases for these cases are also indicated.

Theorem 3.1. *For $q>1$, $\Big((1/n)\sum_{k=1}^{n}u(k)\Big)^q\in\mathcal{A}_1$ if and only if it belongs to $\mathcal{G}_q$ and*

$$(\mathcal{G}_q(u))^{\frac{1}{q'}}\le\left(\mathcal{A}_1\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^q\right)^{\frac{1}{q}}\le\frac{q}{q-1}(\mathcal{G}_q(u))^{\frac{1}{q'}}\qquad(3.1)$$

with $q'=q/(q-1)$. The inequalities in (3.1) *are sharp in the sense that the constants* 1 *and* $q/(q-1)$ *cannot be improved.*

Proof. Assume that $c>1$ and u belongs to $\mathcal{G}_q$ with constant $\mathcal{G}_q(u)=c^{\frac{q'}{q}}$, that is,

$$\frac{1}{N}\sum_{n=1}^{N}(u(n))^q\le c\left(\frac{1}{N}\sum_{k=1}^{N}u(k)\right)^q$$

for $N>1$. By Hardy's inequality (2.2) we deduce that

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^q\le\left(\frac{q}{q-1}\right)^q\left(\frac{1}{N}\sum_{n=1}^{N}u(n)^q\right)$$

$$\leq c\left(\frac{q}{q-1}\right)^q\left(\frac{1}{N}\sum_{k=1}^{N}u(k)\right)^q.$$

Thus, we get

$$\mathcal{A}_1\left(\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^q\right) \leq c\left(\frac{q}{q-1}\right)^q,$$

which implies the second inequality in (3.1). Suppose now that $\left((1/n)\sum_{k=1}^{n}u(k)\right)^q$ belongs to $\mathcal{A}_1$ with constant

$$\mathcal{A}_1\left(\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^q\right) = \lambda.$$

Then, for $N > 1,$ we have

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^q \leq \lambda\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right)^q. \tag{3.2}$$

Since u is nonincreasing, the inequality

$$u(k) \leq \frac{1}{k}\sum_{s=1}^{k}u(s)$$

holds and by (3.2)

$$\frac{1}{N}\sum_{n=1}^{N}(u(n))^q \leq c\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right)^q,$$

which implies the first inequality in (3.1). Let us now check that the inequalities in (3.1) are optimal. Indeed, for ε in $(0,1),$ the nonincreasing sequence $u_\varepsilon(n) = n^{\frac{\varepsilon-1}{q}}$ belongs to $\mathcal{G}_q$ and

$$(\mathcal{G}_q(u_\varepsilon))^{\frac{1}{q'}} \simeq \left(\frac{1}{\varepsilon}\right)^{\frac{1}{q}}\frac{q-1+\varepsilon}{q}$$

with $\alpha = \dfrac{\varepsilon-1}{q}$ and $p = q$ in Lemma 1.2, and

$$\left(\mathcal{A}_1\left(\left(\frac{1}{n}\sum_{k=1}^{n}u_\varepsilon\right)^q\right)\right)^{\frac{1}{q}} \simeq \left(\frac{1}{\varepsilon}\right)^{\frac{1}{q}}.$$

So, we have

$$\frac{\left(\mathcal{A}_1\left(\left(\frac{1}{n}\sum_{k=1}^{n}u_\varepsilon\right)^q\right)\right)^{\frac{1}{q}}}{(\mathcal{G}_q(u_\varepsilon))^{\frac{1}{q'}}} \simeq \frac{q}{q-1+\varepsilon},$$

that tends to $q' = q/(q-1)$ as $\varepsilon \to 0$ and tends to 1 as $\varepsilon \to 1.$

Theorem 3.1 is proved.

Theorem 3.2. *If $u \in \mathcal{G}_1$, then the average $(1/n)\sum_{k=1}^{n} u(k) \in \mathcal{A}_1$ and*

$$\mathcal{A}_1\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right) \leq \alpha_e \mathcal{G}_1(u), \tag{3.3}$$

where $\alpha_e = e/(e-1)$.The inequality (3.3) *is sharp in the sense that the constants α_e cannot be improved.*

Proof. Assume that $\mathcal{G}_1(u) = c$, that is,

$$\frac{1}{N}\sum_{n=1}^{N} \frac{u(n)}{\frac{1}{n}\sum_{k=1}^{n} u(k)}\left(1 + \log \frac{u(n)}{\frac{1}{n}\sum_{k=1}^{n} u(k)}\right) \leq c \quad \text{for all} \quad N > 1. \tag{3.4}$$

By the inequality

$$\frac{1}{\frac{1}{n}\sum_{k=1}^{n} u(k)}\left(\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right)\right) \leq \frac{e}{e-1}\left(\frac{1}{N}\sum_{n=1}^{N} \frac{u(n)}{\frac{1}{n}\sum_{k=1}^{n} u(k)}\right.$$
$$\left.\times\left(1 + \log \frac{u(n)}{\frac{1}{n}\sum_{k=1}^{n} u(k)}\right)\right)$$

and by (3.4) it follows that

$$\frac{1}{\frac{1}{n}\sum_{k=1}^{n} u(k)}\left(\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right)\right) \leq c\frac{e}{e-1}.$$

Then $(1/n)\sum_{k=1}^{n} u(k) \in \mathcal{A}_1$ and the inequality (3.3) holds. The inequality (3.3) is sharp. Indeed the sequence $u(n) = n^{\frac{1-e}{e}}$ is in $\mathcal{A}_\infty$ and it results

$$\mathcal{G}_1(u) \simeq e - 1$$

and

$$\mathcal{A}_1\left(\frac{1}{n}\sum_{k=1}^{n} u\right) \simeq e.$$

Theorem 3.2 is proved.

The equivalence

$$u \in \mathcal{A}_\infty \Leftrightarrow \frac{1}{n}\sum_{k=1}^{n} u(k) \in \mathcal{A}_1$$

can be also shown by using $\mathcal{A}_\infty$ constants.

Theorem 3.3. *If* $(1/n)\sum_{k=1}^{n} u(k) \in \mathcal{A}_1$, *then*

$$\mathcal{A}_\infty(u) \le \mathcal{A}_1\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right) \le e\mathcal{A}_\infty(u). \tag{3.5}$$

The inequalities in (3.5) *are sharp in the sense that the constants* 1 *and* e *cannot be improved.*

Proof. Assume that $\mathcal{A}_\infty(u) = c$, that is, for $n > 0$,

$$\frac{1}{n}\sum_{k=1}^{n} u(k) \le c e^{\frac{1}{n}\sum_{k=1}^{n}\log u(k)}.$$

Then, by Carleman inequality (2.3), we have

$$\frac{1}{N}\sum_{n=1}^{N} e^{\frac{1}{n}\sum_{k=1}^{n}\log u(k)} \le e\left(\frac{1}{N}\sum_{n=1}^{N} u(n)\right)$$

and

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right) \le ce\left(\frac{1}{N}\sum_{n=1}^{N} u(n)\right),$$

which implies the second inequality in (3.5). Conversely, if $\mathcal{A}_1\left(\frac{1}{n}\sum_{k=1}^{n} u\right) = C > 0$, we obtain

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right) \le C\left(\frac{1}{N}\sum_{n=1}^{N} u(n)\right),$$

and, by the monotonicity of $\log u$, it follows that

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right) \le C\left(\frac{1}{N}\sum_{n=1}^{N} e^{\frac{1}{n}\sum_{k=1}^{n}\log u(k)}\right),$$

that is,

$$\frac{1}{N}\sum_{n=1}^{N} u(n) \le C e^{\frac{1}{N}\sum_{n=1}^{N}\log u}$$

and

$$\mathcal{A}_\infty(u) \le \mathcal{A}_1\left(\frac{1}{n}\sum_{k=1}^{n} u\right).$$

In the inequalities (3.5) the constants 1 and e cannot be improved. Indeed the nonincreasing sequence $u_\varepsilon(n) = n^{-\varepsilon}$ belongs to $\mathcal{A}_\infty$ and it results

$$\mathcal{A}_\infty(u_\varepsilon) \simeq \frac{1}{e^\varepsilon(1-\varepsilon)}$$

and

$$\mathcal{A}_1\left(\frac{1}{n}\sum_{k=1}^{n}u_\varepsilon\right)\simeq\frac{1}{1-\varepsilon}.$$

So,

$$\frac{\mathcal{A}_1\left(\frac{1}{n}\sum_{k=1}^{n}u_\varepsilon\right)}{\mathcal{A}_\infty(u_\varepsilon)}\simeq e^{\varepsilon}$$

that tends to 1 as $\varepsilon\to 0$ and tends to e as $\varepsilon\to 1$.

Theorem 3.3 is proved.

Theorem 3.4. *Assume that u is nonnegative and nonincreasing, then*

$$\mathcal{A}_1\left(\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^p\right)\leq\frac{1}{1-p}$$

for $0<p<1$. The above inequality is sharp in the sense that the constants $\dfrac{1}{1-p}$ cannot be improved.

Proof. Applying Hölder's inequality (2.1) with exponents $1/(1-p)$ and $1/p$, we get

$$\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^p=\sum_{n=1}^{N}\frac{1}{n^{p(1-p)}}\left(\frac{1}{n^p}\sum_{k=1}^{n}u(k)\right)^p$$
$$\leq\left(\sum_{n=1}^{N}\frac{1}{n^p}\right)^{1-p}\left(\sum_{n=1}^{N}\frac{1}{n^p}\sum_{k=1}^{n}u(k)\right)^p.$$

Applying Lemma 2.1, we get

$$\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^p\leq\left(\sum_{n=1}^{N}\frac{1}{n^p}\right)^{1-p}\left(\sum_{n=1}^{N}u(k)\left(\sum_{k=n}^{N}\frac{1}{n^p}\right)\right)^p. \tag{3.6}$$

To estimate the summation $\left(\sum_{n=1}^{N}\dfrac{1}{n^p}\right)$, we employ the inequality [3, p. 39] (for $0<p<1$)

$$\gamma x^{\gamma-1}(x-y)\leq x^\gamma-y^\gamma\leq\gamma y^{\gamma-1}(x-y)\quad\text{for}\quad x\geq y>0,$$

which yields to (with $\gamma=p$)

$$\Delta(k-1)^{1-\gamma}=(k)^{1-\gamma}-(k-1)^{1-\gamma}\geq(1-\gamma)k^{-\gamma},$$

and then we have

$$k^{-\gamma}\leq\frac{1}{(1-\gamma)}\Delta(k-1)^{1-\gamma}.$$

Substituting this in (3.6), we obtain

$$\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{p}\leq\frac{N^{(1-p)^{2}}}{(1-p)^{1-p}}\left(\sum_{n=1}^{N}u(k)\left(\sum_{k=n}^{N}\frac{1}{n^{p}}\right)\right)^{p}$$
$$\leq\frac{N^{1-p}}{1-p}\left(\sum_{n=1}^{N}u(n)\right)^{p},$$

which yield to

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{p}\leq\frac{1}{1-p}\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right)^{p},$$

that is, $\left((1/n)\sum_{k=1}^{n}u(k)\right)^{p}\in\mathcal{A}_{1}$ and

$$\mathcal{A}_{1}\left(\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{p}\right)\leq\frac{1}{1-p}.$$

The reverse implication is straightforward. In order to complete the proof we remark that

$$\mathcal{A}_{1}\left(\frac{1}{n}\sum_{k=1}^{n}u_{\varepsilon}(k)\right)^{p}\simeq\frac{1}{1-p\varepsilon}$$

for $u_{\varepsilon}(n)=n^{-\varepsilon}$ and $\varepsilon\in(0,1)$. Thus,

$$\mathcal{A}_{1}\left(\left(\frac{1}{n}\sum_{k=1}^{n}u_{\varepsilon}(k)\right)^{p}\right)\to\frac{1}{1-p}$$

as $\varepsilon\to 1$.

Theorem 3.4 is proved.

Theorem 3.5. *Let* $\alpha=-\dfrac{1}{p-1}$ *for* $p>1$ *and* u *is nondecreasing. Then* $\left(\dfrac{1}{n}\sum_{k=1}^{n}u(k)\right)^{\alpha}\in\mathcal{A}_{1}$ *if and only if* $u\in\mathcal{A}_{p}$ *and*

$$(\mathcal{A}_{p}(u))^{-\alpha}\leq\mathcal{A}_{1}\left(\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{\alpha}\right)\leq p^{-\alpha}(\mathcal{A}_{p}(u))^{-\alpha}. \tag{3.7}$$

The inequalities in (3.7) *are sharp in the sense that the constants* 1 *and* $p^{\frac{1}{p-1}}$ *cannot be improved.*

Proof. Let u belongs to $\mathcal{A}_{p}$ and $\mathcal{A}_{p}(u)=c$. Then we have

$$\left(\frac{1}{N}\sum_{n=1}^{N}(u(n))^{-\frac{1}{p-1}}\right)^{p-1}\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right)\leq c.$$

Applying Hardy's inequality (2.4) with $r=1/(p-1)=-\alpha$, gives

$$\left(\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{-\frac{1}{p-1}}\right)^{p-1}\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right)$$

$$\leq p\left(\frac{1}{N}\sum_{n=1}^{N}(u(n))^{-\frac{1}{p-1}}\right)^{p-1}\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right)\leq cp,$$

and then

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{-\frac{1}{p-1}}\leq (cp)^{\frac{1}{p-1}}\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right)^{-\frac{1}{p-1}}.$$

Thus,

$$\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{\alpha}\in\mathcal{A}_1,$$

and the second inequality in (3.7) holds. Conversely, let $\left((1/n)\sum_{k=1}^{n}u(k)\right)^{\alpha}\in\mathcal{A}_1$ and

$$\mathcal{A}_1\left(\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{\alpha}\right)=\lambda.$$

In virtue of the inequality

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{-\frac{1}{p-1}}\leq\lambda\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right)^{-\frac{1}{p-1}}$$

we have

$$\left(\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{-\frac{1}{p-1}}\right)^{p-1}\leq\lambda^{p-1}\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right)^{-1}. \tag{3.8}$$

Since u is nondecreasing, it follows that

$$\frac{1}{n}\sum_{k=1}^{n}u(k)\leq u(n).$$

This gives

$$(u(n))^{-\frac{1}{p-1}}\leq\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{-\frac{1}{p-1}}.$$

So, from (3.8) it follows that

$$\left(\frac{1}{N}\sum_{n=1}^{N}(u(n))^{-\frac{1}{p-1}}\right)^{p-1}\leq\left(\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)^{-\frac{1}{n-1}}\right)^{p-1}$$

$$\leq\lambda^{p-1}\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right)^{-1},$$

that is, $u \in \mathcal{A}_p$, and the first inequality in (3.7) holds. The inequalities (3.7) are optimal. Indeed, for $\varepsilon \in (0,1)$, the sequence $u_\varepsilon(n) = n^{\varepsilon(p-1)}$ lies in $\mathcal{A}_p$ and it results

$$\mathcal{A}_p(u_\varepsilon) \simeq \frac{1}{(1-\varepsilon)^{p-1}(1+\varepsilon(p-1))}$$

with $\alpha = \varepsilon(p-1)$ in Lemma 1.1, and

$$\mathcal{A}_1\left(\left(\frac{1}{n}\sum_{k=1}^{n} u_\varepsilon(k)\right)^{-\frac{1}{p-1}}\right) \simeq \frac{1}{1-\varepsilon}.$$

So,

$$\frac{\mathcal{A}_1\left(\left(\frac{1}{n}\sum_{k=1}^{n} u_\varepsilon(k)\right)^{-\frac{1}{p-1}}\right)}{(\mathcal{A}_p(u_\varepsilon))^{\frac{1}{p-1}}} \simeq (1+\varepsilon(p-1))^{\frac{1}{p-1}},$$

which tends to 1 as $\varepsilon \to 0$, and tends to $p^{\frac{1}{p-1}}$ as $\varepsilon \to 1$.

Theorem 3.5 is proved.

***Remark* 3.1.** Theorem 3.3 gives us explicit examples of monotonic $\mathcal{A}_p$-sequences in a similar way as Theorem 3.4 gives us monotonic $\mathcal{A}_1$-sequences.

Proposition 3.1. *For* $p > 1$, *if* v *is nonincreasing sequence and* $0 < \beta < 1$, *then*

$$\mathcal{A}_p\left(\left(\frac{1}{n}\sum_{k=1}^{n} v(k)\right)^{-\beta}\right) \le \left(\frac{p-1}{p-1-\beta}\right)^{p-1}.$$

Proof. Let us first note that if u is a nondecreasing, then, for any $n \ge 1$, we have

$$\frac{1}{n}\sum_{k=1}^{n} u(k) \le u(n),$$

and also

$$\frac{1}{n}\sum_{k=1}^{n} u(k)\left(\frac{1}{n}\sum_{k=1}^{n}(u(k))^{-\frac{1}{p-1}}\right)^{p-1} \le \left(\frac{\frac{1}{n}\sum_{k=1}^{n} u(k)^{-\frac{1}{p-1}}}{u(n)^{-\frac{1}{p-1}}}\right)^{p-1}.$$

This inequality implies that $\left(\text{with } u(n) = (1/n)\sum_{k=1}^{n} v(k)^{-\beta}\right)$

$$\mathcal{A}_p\left(\left(\frac{1}{n}\sum_{k=1}^{n} v(k)\right)^{-\beta}\right) = \mathcal{A}_p(u) \le \left[\mathcal{A}_1\left(u^{-\frac{1}{p-1}}\right)\right]^{p-1}$$

$$= \left(\mathcal{A}_1\left(\left[\frac{1}{n}\sum_{k=1}^{n} v(k)\right]^{\frac{\beta}{p-1}}\right)\right)^{p-1}.$$

Applying Theorem 3.4 to the nondecreasing sequence $\Big((1/n)\sum_{k=1}^{n} v(k)\Big)^{-\beta}$, we deduce that

$$\mathcal{A}_1\left(\left(\frac{1}{n}\sum_{k=1}^{n} v(k)\right)^{\frac{\beta}{p-1}}\right) \leq \frac{p-1}{p-1-\beta}.$$

Combining the above inequalities, we get our result.

Proposition 3.1 is proved.

4. Gehring class $\mathcal{G}_\phi$ for convex function. We begin this section by a generalization of Hardy's (2.2), which is Hardy – Levinson inequality [4, Theorem 1]. Let Φ be the set of nonnegative functions ϕ defined over a real interval J, with second derivative ϕ'' positive in the inner of J and verifying the Levinson condition

$$\exists\, p > 1 : \phi \cdot \phi'' \geq \left(1 - \frac{1}{p}\right)\left(\phi'\right)^2. \tag{4.1}$$

Theorem 4.1. *If $\phi \in \Phi$ and u is a nonnegative sequence and the condition* (4.1) *holds, then*

$$\frac{1}{N}\sum_{n=1}^{N} \phi\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right) \leq \left(\frac{p}{p-1}\right)^p \left(\frac{1}{N}\sum_{n=1}^{N} \phi(u(n))\right) \tag{4.2}$$

holds for every $p > 1$.

Proof. Let

$$\psi(u) = (\phi(u))^{1/p} \geq 0.$$

Then by (4.1) $\psi'' \geq 0$ where $\psi > 0$. Hence ψ is convex. Thus, by Jensen's inequality,

$$\psi\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right) \leq \frac{1}{n}\sum_{k=1}^{n} \psi(u(k)). \tag{4.3}$$

By Hardy's inequality (2.2) applied to $\psi(u(n))$

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} \psi(u(k))\right)^p \leq \left(\frac{p}{p-1}\right)^p \left(\frac{1}{N}\sum_{n=1}^{N} \psi^p(u(n))\right).$$

Using (4.3) and $\psi^p = \phi$, this completes the proof of the theorem.

An application of Theorem 4.1, gives us the following result.

Theorem 4.2. *Let $\phi \in \Phi$ be increasing* (*respectively decreasing*) *function and u is nonincreasing* (*respectively nondecreasing*) *sequence such that the condition* (4.1) *holds. Then $\phi\Big((1/n)\sum_{k=1}^{n} u(k)\Big) \in \mathcal{A}_1$ if and only if $u \in \mathcal{G}_\phi$ and*

$$\mathcal{G}_\phi(u) \leq \mathcal{A}_1\left(\phi\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right)\right) \leq \left(\frac{p}{p-1}\right)^p \mathcal{G}_\phi(u)$$

for every $p > 1$.

Proof. Let ϕ be increasing and u is nonincreasing. If u belongs to $\mathcal{G}_\phi$ and $\mathcal{G}_\phi(u) = c$, then

$$\frac{1}{N}\sum_{n=1}^{N}\phi(u(n)) \le c\phi\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right),$$

and from (4.2) it follows

$$\frac{1}{N}\sum_{n=1}^{N}\phi\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right) \le c\left(\frac{p}{p-1}\right)^{p}\phi\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right),$$

that ensures $\phi\Big((1/n)\sum_{k=1}^{n}u(k)\Big) \in \mathcal{A}_1$, and

$$\mathcal{A}_1\left(\phi\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)\right) \le c\left(\frac{p}{p-1}\right)^{p}.$$

Let us now suppose $\phi\Big((1/n)\sum_{k=1}^{n}u(k)\Big) \in \mathcal{A}_1$ and

$$\mathcal{A}_1\left(\phi\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)\right) = \lambda.$$

Then we have

$$\frac{1}{N}\sum_{n=1}^{N}\phi\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right) \le \lambda\phi\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right).$$

By the monotonicity of u and ϕ, we obtain

$$\phi(u(n)) \le \phi\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)$$

and

$$\frac{1}{N}\sum_{n=1}^{N}\phi(u(n)) \le \frac{1}{N}\sum_{n=1}^{N}\phi\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right) \le \lambda\phi\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right),$$

that is, u belongs to $\mathcal{G}_\phi$, and $\mathcal{G}_\phi(u) \le \lambda$. If ϕ is decreasing, we can get the result, related to a nondecreasing u in a similar way.

Theorem 4.2 is proved.

Remark **4.1.** It is obvious that Theorem 4.2 is a partial generalization of the Theorems 3.1 and 3.5.

Indeed for a general sequence ϕ optimal inequalities are not available. The following theorem introduces a characterization of of $\mathcal{A}_\infty$-condition for nonincreasing sequences.

Theorem 4.3. *Let $\phi \in \Phi$ and ϕ is an increasing function. Then the nonincreasing sequence $u \in \mathcal{A}_\infty$ if and only if $\phi^{-1}(u) \in \mathcal{G}_\phi$.*

Proof. Let $\phi^{-1}(u) \in \mathcal{G}_\phi$ and

$$\mathcal{G}_\phi(u) = \sup_{N\geq 1} \frac{\dfrac{1}{N}\sum_{n=1}^{N} \phi(u(n))}{\phi\left(\dfrac{1}{N}\sum_{n=1}^{N} u(n)\right)},$$

that is,

$$\big(\mathcal{G}_\phi(\phi^{-1}(u))\big) = \sup_{N\geq 1} \frac{\dfrac{1}{N}\sum_{n=1}^{N} \phi(\phi^{-1}(u(n)))}{\varphi\left(\dfrac{1}{N}\sum_{n=1}^{N} \varphi^{-1}(u(n))\right)} = \lambda > 0.$$

So, we have

$$\sup_{N\geq 1} \frac{1}{N}\sum_{n=1}^{N} u(n) \leq \lambda\varphi\left(\frac{1}{N}\sum_{n=1}^{N} \varphi^{-1}(u(n))\right).$$

Choosing $\phi = \exp$, then $\phi^{-1} = \log$, then we get

$$\sup_{N\geq 1} \frac{1}{N}\sum_{n=1}^{N} u(n) \leq \lambda \exp\left(\frac{1}{N}\sum_{n=1}^{N} \log(u(n))\right).$$

So, we obtain

$$\mathcal{A}_\infty(u) = \sup_{N\geq 1} \frac{\dfrac{1}{N}\sum_{n=1}^{N} u(n)}{\exp\left(\dfrac{1}{N}\sum_{n=1}^{N} \log(u(n))\right)} \leq \lambda.$$

Let $u \in \mathcal{A}_\infty$ and $[\mathcal{A}_\infty(u)] = \mathcal{C}$, then

$$\sup_{N\geq 1} \frac{1}{N}\sum_{n=1}^{N} u(n) \leq \mathcal{C} \exp\left(\frac{1}{N}\sum_{n=1}^{N} \log(u(n))\right).$$

Since that $\phi = \exp$, $\phi^{-1} = \log$, we obtain

$$\sup_{N\geq 1} \frac{1}{N}\sum_{n=1}^{N} u(n) \leq \mathcal{C}\phi\left(\frac{1}{N}\sum_{n=1}^{N} \phi^{-1}(u(n))\right).$$

Since $u(n) = \phi\big[\phi^{-1}(u(n))\big]$, we have

$$\big(\mathcal{G}_\varphi(\varphi^{-1}(u))\big) = \sup_{N\geq 1} \frac{\dfrac{1}{N}\sum_{n=1}^{N} \big[\phi(\phi^{-1}u(n))\big]}{\phi\left(\dfrac{1}{N}\sum_{n=1}^{N} \phi^{-1}(u(n))\right)}$$

$$= \sup_{N\geq 1} \frac{\dfrac{1}{N}\sum_{n=1}^{N} u(n)}{\phi\left(\dfrac{1}{N}\sum_{n=1}^{N}\phi^{-1}(u(n))\right)} \leq \mathcal{C}.$$

Theorem 4.3 is proved.

We conclude this section by establishing the following lemma that generalizes the equivalence

$$\frac{1}{n}\sum_{k=1}^{n} u(k) \in \mathcal{A}_1 \Leftrightarrow e^{\frac{1}{n}\sum_{k=1}^{n}\log u} \in \mathcal{A}_1.$$

Lemma 4.1. *Let ϕ be nonnegative, increasing and convex function. Then $\left((1/n)\sum_{k=1}^{n} u(k)\right) \in \mathcal{A}_1$ if and only if*

$$\phi\left(\frac{1}{n}\sum_{k=1}^{n}\phi^{-1}(u(k))\right) \in \mathcal{A}_1$$

for a nonincreasing sequence u.

Proof. Let $\left((1/n)\sum_{k=1}^{n} u(k)\right) \in \mathcal{A}_1$, that is,

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right) \leq \lambda\left(\frac{1}{N}\sum_{n=1}^{N} u(n)\right) \quad \text{for} \quad k > 1.$$

By the monotonicity of $\phi^{-1}(u)$, we get

$$\frac{1}{n}\sum_{k=1}^{n}\phi^{-1}(u(k)) \geq \phi^{-1}(u(n)),$$

and by

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right) \leq \lambda\left(\frac{1}{N}\sum_{n=1}^{N}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\phi^{-1}(u(k))\right)\right), \tag{4.4}$$

we obtain

$$\frac{1}{N}\sum_{n=1}^{N} u(n) \leq \lambda\phi\left(\frac{1}{N}\sum_{n=1}^{N}\phi^{-1}(u(n))\right). \tag{4.5}$$

By applying Jensen's inequality and the inequalities (4.4) and (4.5), we have

$$\frac{1}{N}\sum_{n=1}^{N}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\phi^{-1}(u(k))\right) \leq \frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n} u(k)\right)$$

$$\leq \lambda\left(\frac{1}{N}\sum_{n=1}^{N} u(n)\right) \leq \lambda^2\phi\left(\frac{1}{N}\sum_{n=1}^{N}\phi^{-1}(u(k))\right),$$

and so $\phi\Big((1/n)\sum_{k=1}^{n}\phi^{-1}(u(k))\Big)$ is in $\mathcal{A}_1$. Conversely, if

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}\phi^{-1}(u(k))\right)\leq c\phi\left(\frac{1}{N}\sum_{n=1}^{N}\phi^{-1}(u(n))\right),$$

by the monotonicity of $\phi^{-1}(u)$, we get

$$\frac{1}{N}\sum_{n=1}^{N}u(n)\leq\frac{1}{N}\sum_{n=1}^{N}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\phi^{-1}(u(k))\right)\leq c\phi\left(\frac{1}{N}\sum_{n=1}^{N}\phi^{-1}(u(n))\right)$$

and

$$\frac{1}{N}\sum_{n=1}^{N}\left(\frac{1}{n}\sum_{k=1}^{n}u(k)\right)\leq c\left(\frac{1}{N}\sum_{n=1}^{N}\phi\left(\frac{1}{n}\sum_{k=1}^{n}\phi^{-1}(u(k))\right)\right)$$

$$\leq c^2\phi\left(\frac{1}{N}\sum_{n=1}^{N}\phi^{-1}(u(n))\right)\leq c^2\left(\frac{1}{N}\sum_{n=1}^{N}u(n)\right).$$

Lemma 4.1 is proved.

References

1. G. Bennett, K.-G. Grosse-Erdmann, *Weighted Hardy inequalities for decreasing sequences and functions*, Math. Ann., **334**, 489 – 531 (2006).
2. A. Böttcher, M. Seybold, *Wackelsatz and Stechkin's inequality for discrete Muckenhoupt weights*, Preprint № 99-7, TU Chemnitz (1999).
3. G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, 2nd ed., Cambridge Univ. Press (1934).
4. N. Levinson, *Generalizations of an inequality of Hardy*, Duke Math. J., **31**, № 3, 389 – 394 (1964).
5. M. M. Iddrisu, C. A. Okpoti, *Applications of Taylor series for Carleman's inequality through Hardy inequality*, Korean J. Math., **23**, № 4, 655 – 664 (2015).
6. N. A. Malaksiano, *The exact inclusions of Gehring classes in Muckenhoupt classes* (in Russian), Mat. Zametki, **70**, № 5, 742 – 750 (2001); *English translation*: Math. Notes, **70**, № 5-6, 673 – 681 (2001).
7. N. A. Malaksiano, *The precise embeddings of one-dimensional Muckenhoupt classes in Gehring classes*, Acta Sci. Math., **68**, 237 – 248 (2002).
8. B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc., **165**, 207 – 226 (1972).
9. E. N. Nikolidakis, *A sharp integral Hardy type inequality and applications to Muckenhoupt weights on* $\mathbb{R}$, Ann. Acad. Sci. Fenn. Math., **39**, 887 – 896 (2014).
10. S. H. Saker, I. Kubiaczyk, *Higher summability and discrete weighted Muckenhoupt and Gehring type inequalities*, Proc. Edinb. Math. Soc., **62**, № 4, 949 – 973 ( ).
11. S. H. Saker, R. R. Mahmoud, *Boundedness of discrete Hardy – Littlewood maximal operator via Muchenhoupt weights*, Rocky Mountain J. Math., **51**, № 2, 733 – 746 (2021).
12. S. H. Saker, S. S. Rabie, Ghada AlNemer, M. Zakarya, *On structure of discrete Muckenhoupt and discrete Gehring classes*, J. Inequal. and Appl., **2020**, № 1, 1 – 18 (2020).
13. S. H. Saker, S. S. Rabie, J. Alzabut, D. O'Regan, R. P. Agarwal, *Some basic properties of discrete Muckenhoupt and Gehring classes and fundumental relations*, Adv. Difference Equat., **2021**, № 1, 1 – 22 (2021).
14. S. H. Saker, D. O'Regan, R. P. Agarwal, *Self-improving properties of discrete Muckenhoupt weights*, Analysis, **41**, № 3, 169 – 178 (2021).

Received 17.06.21