DOI: 10.37863/umzh.v75i3.6903

UDC 517.9

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RESULTS FOR RETARDED NONLINEAR INTEGRAL INEQUALITIES WITH MIXED POWERS AND THEIR APPLICATIONS TO DELAY INTEGRO-DIFFERENTIAL EQUATIONS

РЕЗУЛЬТАТИ ДЛЯ СПОВІЛЬНЕНИХ НЕЛІНІЙНИХ ІНТЕГРАЛЬНИХ НЕРІВНОСТЕЙ З МІШАНИМИ СТЕПЕНЯМИ ТА ЇХ ЗАСТОСУВАННЯ ДО ІНТЕГРО-ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ІЗ ЗАПІЗНЕННЯМ

We present new retarded nonlinear integral inequalities with mixed powers. The obtained inequalities can be used to study the boundedness and global existence of the solutions of integro-differential equation with delay and Volterra-type integral equation with delay. These inequalities extend some results available in the literature. Finally, we present two examples to demonstrate the usefulness of our main results.

Запропоновано нові сповільнені нелінійні інтегральні нерівності з мішаними степенями. Отримані нерівності можуть бути використані для дослідження обмеженості та глобального існування розв'язків інтегро-диференціального рівняння із запізненням та інтегрального рівняння типу Вольтерра із запізненням. Ці нерівності розширюють деякі результати, наведені в літературі. Крім того, розглянуто два приклади, що демонструють корисність отриманих основних результатів.

1. Introduction. It is well-known that there exists a class of mathematical models described by differential equations and a lot of differential equations do not posses the exact solutions or existence of solution or boundedness of solution. On the other hand, integral inequalities have significant applications and tools in the study of existence [1 – 3], stability, boundness, uniqueness, asymptotic behavior, quantitative as well as qualitative properties for the solution of nonlinear differential equations and integro-differential equations [4 – 9]. Gronwall [10] established the following inequality that estimate the solution of linear differential equation:

Gronwall inequality [10]. Let ω be a nonnegative continuous function defined on $\mathbb{J}_1 = [a, b]$ and $\omega_0, k \geq 0$ such that

$$\omega(t) \leq \omega_0 + k \int_a^t \omega(s)ds \quad \forall t \in \mathbb{J}_1.$$

Then

$$\omega(t) \leq \omega_0 \exp(k(t-a)) \quad \forall t \in \mathbb{J}_1.$$

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A lot of number of mathematicians and scientists have shown their considerable interest after the discovery of above inequality to generalized the original form of the Gronwall inequality. An important generalization of above inequality is established by Bellman which is stated below:

Gronwall – Bellman inequality [11]. Let ω and h be nonnegative continuous functions defined on $J_1 = [a, b]$, and suppose that ω_0 be nonnegative constant, for which the inequality

$$\omega(t) \leq \omega_0 + \int\limits_a^t h(s)\omega(s)ds \quad \forall t \in J_1,$$

holds. Then

$$\omega(t) \leq \omega_0 \exp\left(\int\limits_a^t h(s)ds\right) \quad \forall t \in J_1.$$

A large number of articles, monographs and books have been appeared during the last century in the literature that covered the generalization of above inequalities and their applications (see [6, 12 – 20]). So another significant generalization of above inequality is given by Pachpatte which is stated below:

Pachpatte inequality [6]. Let ω, h_1 and h_2 be nonnegative continuous functions defined on $J_1 = [a, b]$, and ω_0 be positive constant for which the inequality

$$\omega(t) \leq \omega_0 + \int\limits_a^t h_1(s)\left(\omega(s) + \int\limits_a^s h_2(r)\omega(r)dr\right)ds \quad \forall t \in \mathbb{J}_1$$

holds, then

$$\omega(t) \leq \omega_0\left(1 + \int\limits_a^t h_1(s)\exp\left(\int\limits_a^s \big(h_1(r) + h_2(r)\big)dr\right)ds\right) \quad \forall t \in \mathbb{J}_1.$$

Retarded integral inequalities (where non-retarded argument t is shifted into retarded argument $\alpha(t)$) have been introduced in differential and integral equations to solve real-life problems such as involvement of remarkable memory effect in a refined model. So, retarded nonlinear integral inequalities have been established by many mathematicians and scientists to handle such type of problems, one of them is given below:

Lipovan inequality [13]. Let ω and h be nonnegative continuous functions defined on $J_1 = [a, b]$ with $\alpha(t) \leq t$ on $[t_0, T_0]$ and k be any constant. Then the inequality

$$\omega(t) \leq k + \int\limits_{\alpha(t_0)}^{\alpha(t)} h(s)\omega(s)ds, \quad t_0 < t < T_0,$$

implies that

$$\omega(t) \leq k\left(\int\limits_{\alpha(t_0)}^{\alpha(t)} h(s)ds\right), \quad t_0 < t < T_0.$$

However, from the above inequalities we can see that there is a need to study some more general results that will cover the above inequalities as well other inequalities in the literature. Up till now, retarded nonlinear delay integral inequalities with mix powers have considered less attention.

In this paper, we present few new retarded nonlinear delay integral inequalities with mix powers which will generalize and cover the inequalities presented in [6, 11, 15, 16, 20]. These inequalities can be used to estimate the existence, uniqueness, boundedness, stability, asymptotic behavior, quantitative and qualitative properties of solution of differential, integral and integro-differential equations.

Let $\omega, h_1, h_2, h_3 \in \mathcal{C}(\mathbb{J}, \mathbb{R}_+)$ and $p, \alpha \in \mathcal{C}'(\mathbb{J}, \mathbb{R}_+)$ be nondecreasing with $p(t) \geq 1$ and $\alpha(t) \leq t$. We take the following inequality:

$$\omega(t) \leq p(t) + \int_a^{\alpha(t)} h_1(s)\omega(s)ds + \int_a^{\alpha(t)} h_2(s)\left(\omega(s) + \int_a^s h_3(r)\omega(r)dr\right)^n ds \quad \forall t \in \mathbb{J}. \tag{1.1}$$

To the best of our knowledge, inequalities given in the literature hold only for $n = 1$, so here we discus the case when $n > 1$ and $0 < n \leq 1$ in the above inequality.

Consider

$$\omega(t) \leq p(t) + \int_a^{\alpha(t)} h_1(s)\omega^{n_1}(s)ds + \int_a^{\alpha(t)} h_2(s)\left(\omega^{n_1}(s) + \int_a^s h_3(r)\omega^{n_2}(r)dr\right)^n ds \quad \forall t \in \mathbb{J}, \tag{1.2}$$

where n_1, n_2 and n are nonnegative constants satisfying $0 < n_1,\ n_2 \leq 1$, and $n > 1$.

Now we consider another inequality

$$\omega^m(t) \leq p(t) + \int_a^{\alpha(t)} h_1(s)\omega^{n_1}(s)ds + \int_a^{\alpha(t)} h_2(s)\Big(\omega^{n_1}(s) + \int_a^s h_3(r)\omega^{n_2}(r)dr\Big)^n \quad \forall t \in \mathbb{J}, \tag{1.3}$$

where $n,\ n_1,\ n_2,\ m$ are nonnegative constants with $m \geq n_1 > 0,\ m \geq n_2 > 0$, and $n > 1$.

At the end of this paper, as an application of inequality (1.3), can be applied on the following system of delay integro-differential equation to estimate the bounds and existence of solution:

$$(x^m(t))' = H_1\big(t, x(\tau(t))\big)F\left(t, x(\tau(t)), \int_0^t H_2\big(r, x(\tau(r))\big)dr\right) \quad \forall t \in \mathbb{J},$$
$$x(t) = \phi(t), \quad t \in [d, 0], \quad -\infty < d = \inf(\tau(t), t \in \mathbb{J}) \leq 0,$$
$$\tau(t) \leq t, \quad x(0) = c,$$

where $x(t)$ and $x(\tau(t))$ are the state and state delay, respectively.

2. Preliminaries. Throughout this paper, the set of real numbers is denoted by $\mathbb{R}$, while $\mathbb{R}_+ = [0, \infty)$ is the subset of $\mathbb{R}$ and $'$ will stand for first derivative of any function. Moreover, the set of all nonnegative continuous functions and nonnegative continuously differentiable functions from $\mathbb{J}$ into $\mathbb{R}_+$ are presented by $\mathcal{C}(\mathbb{J}, \mathbb{R}_+)$ and $\mathcal{C}'(\mathbb{J}, \mathbb{R}_+)$, respectively, where $\mathbb{J}$ is the subset of $\mathbb{R}_+$. Now, we introduce the following basic lemmas which will be very helpful in the proof of main results.

Lemma 2.1 [15]. *Let $a \geq 0$ and $n_1 \geq n_2 > 0$. Then*

$$a^{\frac{n_2}{n_1}} \leq \frac{n_2}{n_1}a + \frac{n_1 - n_2}{n_1}.$$

Lemma 2.2 [9]. *Assume that* $u, v \geq 0$ *and* $n > 1$. *Then*

$$(u+v)^n \leq 2^{n-1}(u^n + v^n).$$

3. Main results. In this section, we state and prove new retarded nonlinear delay integral inequalities for Volterra type with mix powers which can be used to examine the existence, uniqueness, boundedness, stability, asymptotic behavior, quantitative and qualitative properties of solution of differential, integral and integro-differential equations. These inequalities will generalize some existing important results in [6, 11, 15, 16, 20]. We start with following theorem.

Theorem 3.1. *If the inequality* (1.1) *holds*, *then*

(a) *for* $n > 1$,

$$\omega(t) \leq p(a)\exp\left(\int\limits_a^{\alpha(t)} h_1(s)ds\right) + \left(p(t) + 2^{n-1}\int\limits_a^{\alpha(t)} h_2(s)\beta_1^{-1}(s)ds\right)\exp\left(\int\limits_s^{\alpha(t)} h_1(r)dr\right) \tag{3.1}$$

for all $t \in \mathbb{J}$, *where*

$$\beta_1(t) = \left(p^{n(1-n)}(a) + (1-n)\int\limits_a^{\alpha(t)} \big(np'(\alpha^{-1}(s)) + nh_1(s) + 2^{n-1}nh_2(s) + h_3^n(s)\big)ds\right)^{\frac{1}{1-n}}; \tag{3.2}$$

(b) *for* $0 < n \leq 1$,

$$\omega(t) \leq p(a)\exp\left(\int\limits_a^{\alpha(t)} h_1(s)ds\right) + \int\limits_a^{\alpha(t)} \big(p'(\alpha^{-1}(s))$$

$$+ nh_2(s)\beta_2(\alpha^{-1}(s)) + (1-n)h_2(s)\big)\exp\left(\int\limits_s^{\alpha(t)} h_1(r)dr\right)ds \quad \forall t \in \mathbb{J}, \tag{3.3}$$

where

$$\beta_2(t) = p(a)\exp\left(\int\limits_a^{\alpha(t)} \big(h_1(s) + nh_2(s) + h_3(s)\big)ds\right)$$

$$+ \int\limits_a^{\alpha(t)} \big(p'(\alpha^{-1}(s)) + (1-n)h_2(s)\big)\exp\left(\int\limits_s^{\alpha(t)} \big(h_1(r) + nh_2(r) + h_3(r)\big)dr\right)ds \quad \forall t \in \mathbb{J}. \tag{3.4}$$

Proof. (a) Assume that $Z_1(t)$ be the right-hand side of (1.1), then $Z_1(a) = p(a)$. By using the monotonicity of $Z_1(t)$, we get

$$\omega(t) \leq Z_1(t), \qquad \omega(\alpha(t)) \leq Z_1(\alpha(t)) \leq Z_1(t) \quad \forall t \in \mathbb{J}. \tag{3.5}$$

Applying differentiation on $Z_1(t)$ and using (3.5), we obtain

$$Z_1'(t) = p'(t) + \alpha'(t)h_1(\alpha(t))\omega(\alpha(t)) + \alpha'(t)h_2(\alpha(t))\left(\omega(\alpha(t)) + \int\limits_a^{\alpha(t)} h_3(s)\omega(s)ds\right)^n$$

$$\le p'(t) + \alpha'(t)h_1(\alpha(t))Z_1(t) + \alpha'(t)h_2(\alpha(t))\left(Z_1(t) + \int\limits_a^{\alpha(t)} h_3(s)Z_1(s)ds\right)^n \qquad (3.6)$$

for all $t \in \mathbb{J}$. Applying Lemma 2.2 on the inequality (3.6), we have

$$Z_1'(t) \le p'(t) + \alpha'(t)h_1(\alpha(t))Z_1(t) + 2^{n-1}\alpha'(t)h_2(\alpha(t))\left(Z_1^n(t) + \int\limits_a^{\alpha(t)} h_3^n(s)Z_1^n(s)ds\right)$$

$$\le p'(t) + \alpha'(t)h_1(\alpha(t))Z_1(t) + 2^{n-1}\alpha'(t)h_2(\alpha(t))M_1(t) \quad \forall t \in \mathbb{J}, \qquad (3.7)$$

where

$$M_1(t) = \left(Z_1^n(t) + \int\limits_a^{\alpha(t)} h_3^n(s)Z_1^n(s)ds\right) \quad \forall t \in \mathbb{J}. \qquad (3.8)$$

Thus, we have $M_1(a) = Z_1^n(a) = p^n(a)$, $Z_1(t) \le M_1(t)$ and $Z_1(\alpha(t)) \le M_1(\alpha(t)) \le M_1(t)$. Now, differentiating (3.8) with respect to t and using (3.7), we get

$$\begin{aligned} M_1'(t) &= nZ_1^{n-1}(t)Z_1'(t) + \alpha'(t)h_3^n(\alpha(t))Z_1^n(t) \\ &\le nM_1^{n-1}(t)\Big(p'(t) + \alpha'(t)h_1(\alpha(t))M_1(t) + 2^{n-1}\alpha'(t)h_2(\alpha(t))M_1(t)\Big) \\ &\qquad + \alpha'(t)h_3^n(\alpha(t))M_1^n(t) \\ &\le M_1^n(t)\Big(np'(t) + n\alpha'(t)h_1(\alpha(t)) + 2^{n-1}n\alpha'(t)h_2(\alpha(t)) + \alpha'(t)h_3^n(\alpha(t))\Big) \quad \forall t \in \mathbb{J} \end{aligned}$$

or, equivalently,

$$M_1^{-n}(t)M_1'(t) \le np'(t) + n\alpha'(t)h_1(\alpha(t)) + 2^{n-1}n\alpha'(t)h_2(\alpha(t)) + \alpha'(t)h_3^n(\alpha(t)) \quad \forall t \in \mathbb{J}. \quad (3.9)$$

Multiply $(1-n)$ on both sides of (3.9) and we obtain the following estimation for $M_1(t)$ after applying integration from a to t on the inequality (3.9):

$$M_1(t) \le \left(p^{n(1-n)}(a) + (1-n)\int\limits_a^{\alpha(t)} \big(np'(\alpha^{-1}(s)) + nh_1(s) + 2^{n-1}nh_2(s) + h_3^n(s)\big)ds\right)^{\frac{1}{1-n}} \quad \forall t \in \mathbb{J}.$$

Substituting above inequality in (3.7), we have

$$Z_1'(t) \le p'(t) + \alpha'(t)h_1(\alpha(t))Z_1(t) + 2^{n-1}\alpha'(t)h_2(\alpha(t))\beta_1(t) \quad \forall t \in \mathbb{J}, \qquad (3.10)$$

where $\beta_1(t)$ is defined in (3.2), and applying integration from a to t on the inequality (3.10), we get

$$Z_1(t) \le p(a)\exp\left(\int_a^{\alpha(t)} h_1(s)ds\right) + \left(p(t) + 2^{n-1}\int_a^{\alpha(t)} h_2(s)\beta_1(\alpha^{-1}(s))ds\right)\exp\left(\int_s^{\alpha(t)} h_1(r)dr\right)$$

for all $t \in \mathbb{J}$. Substituting above inequality into (3.5), we obtain the required inequality (3.1).

(b) Applying Lemma 2.1 on the inequality (3.6), we get

$$\begin{aligned} Z_1'(t) &\le p'(t) + \alpha'(t)h_1(\alpha(t))Z_1(t) + \alpha'(t)h_2(\alpha(t))\left(n\left(Z_1(t) + \int_a^{\alpha(t)} h_3(s)Z_1(s)ds\right) + 1 - n\right) \\ &\le p'(t) + \alpha'(t)h_1(\alpha(t))Z_1(t) + n\alpha'(t)h_2(\alpha(t))\left(Z_1(t) + \int_a^{\alpha(t)} h_3(s)Z_1(s)ds\right) \\ &\quad + (1-n)\alpha'(t)h_2(\alpha(t)) \\ &\le p'(t) + \alpha'(t)h_1(\alpha(t))Z_1(t) + n\alpha'(t)h_2(\alpha(t))M_2(t) + (1-n)\alpha'(t)h_2(\alpha(t)) \quad \forall t \in \mathbb{J}, \end{aligned} \tag{3.11}$$

where

$$M_2(t) = Z_1(t) + \int_a^{\alpha(t)} h_3(s)Z_1(s)ds. \tag{3.12}$$

Thus, we have $M_2(a) = Z_1(a) = p(a),\ Z_1(t) \le M_2(t)$ and $Z_1(\alpha(t)) \le M_2(\alpha(t)) \le M_2(t)$. Now, differentiating (3.12) with respect to t and using (3.11), we get

$$\begin{aligned} M_2'(t) &= Z_1'(t) + \alpha'(t)h_3(\alpha(t))Z_1(t) \\ &\le p'(t) + (1-n)\alpha'(t)h_2(\alpha(t)) \\ &\quad + \alpha'(t)\big(h_1(\alpha(t)) + nh_2(\alpha(t)) + h_3(\alpha(t))\big)M_2(t) \quad \forall t \in \mathbb{J}. \end{aligned} \tag{3.13}$$

We obtain the following estimation for $M_2(t)$ after applying integration from a to t on the inequality (3.13):

$$\begin{aligned} M_2(t) \le p(a)\exp\left(\int_a^{\alpha(t)} \big(h_1(s) + nh_2(s) + h_3(s)\big)ds\right) + \int_a^{\alpha(t)} \big(p'(\alpha^{-1}(s)) + (1-n)h_2(s)\big) \\ \times \exp\left(\int_s^{\alpha(t)} \big(h_1(r) + nh_2(r) + h_3(r)\big)dr\right)ds \quad \forall t \in \mathbb{J}. \end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.11), we have

$$Z_1'(t) \leq p'(t) + \alpha'(t)h_1(\alpha(t))z_2(t) + n\alpha'(t)h_2(\alpha(t))\beta_2(t) + (1-n)\alpha'(t)h_2(\alpha(t)) \quad \forall t \in \mathbb{J}, \tag{3.15}$$

where $\beta_2(t)$ is defined in (3.4). We obtain the following estimation for $M_2(t)$ after applying integration from a to t on the inequality (3.15):

$$Z_1(t) \leq p(a)\exp\left(\int\limits_a^{\alpha(t)} h_1(s)ds\right) + \int\limits_a^{\alpha(t)} \big(p'(\alpha^{-1}(s)) + nh_2(s)\beta_2(\alpha^{-1}(s)) + (1-n)h_2(s)\big) \times \exp\left(\int\limits_s^{\alpha(t)} h_1(r)dr\right)ds \quad \forall t \in \mathbb{J}. \tag{3.16}$$

Substituting (3.16) in (3.5), we obtain the required inequality (3.3).

Theorem 3.1 is proved.

***Remark* 3.1.** It is very interesting to note that when we change the assumptions of Theorem 3.1, we get the following results:

1. If we put $h_1(t) = 0$ and $\alpha(t) = t$, then the part (b) of Theorem 3.1 is converted into the Theorem 2.9 [20].

2. When we take $p(t) = \omega_0$, $\alpha(t) = t$ and $h_2(t) = 0$, then Theorem 3.1 becomes Gronwall–Bellman inequality [11].

3. If $p(t) = \omega_0$, $\alpha(t) = t$, $n = 1$ and $h_1(t) = 0$, then part (b) of Theorem 3.1 reduced to Pachpatte inequality [6].

Here, we present another more general result that will cover the existing results in [6, 11, 15, 16].

Theorem 3.2. *If the inequality* (1.2) *holds*, *then*

$$\omega(t) \leq p(t) + 2^{2(n-1)} \int\limits_a^{\alpha(t)} \big(\beta_3(\alpha^{-1}(s)) + h_2(s)\beta_4(\alpha^{-1}(s))\big)ds \exp\left(n_1 \int\limits_s^{\alpha(t)} h_1(r)dr\right) \tag{3.17}$$

for all $t \in \mathbb{J}$, *where*

$$\beta_3(t) = n_1\alpha'(t)h_1(\alpha(t))p(t) + \alpha'(t)h_1(\alpha(t))(1-n_1) + 2^{n-1}\alpha'(t)h_2(\alpha(t)) \times \left(n_1p(t) + (1-n_1) + \int\limits_a^{\alpha(t)} n_2p(s)h_3(s)ds + \int\limits_a^{\alpha(t)} h_3(s)(1-n_2)ds\right)^n \tag{3.18}$$

and

$$\beta_4(t) = \left((1-n)\int\limits_a^{\alpha(t)} \big(\beta_3(\alpha^{-1}(s)) + nn_1^2h_1(s) + 2^{2(n-1)}nn_1h_2(s) + n_2h_3^n(\alpha(s))\big)ds\right)^{\frac{1}{1-n}}. \tag{3.19}$$

Proof. Define a function

$$Z_2(t) = \int\limits_a^{\alpha(t)} h_1(s)\omega^{n_1}(s)ds + \int\limits_a^{\alpha(t)} h_2(s)\left(\omega^{n_1}(s) + \int\limits_a^{s} h_3(r)\omega^{n_2}(r)dr\right)^n ds \quad \forall t \in \mathbb{J}, \tag{3.20}$$

then we have $Z_2(a) = 0$, and using the monotonicity of $Z_2(t)$, we get

$$\omega(t) \le p(t) + Z_2(t) \quad \forall t \in \mathbb{J}. \tag{3.21}$$

Applying differentiation on $Z_2(t)$ and using (3.21), we obtain

$$\begin{aligned} Z_2'(t) &= \alpha'(t)h_1(\alpha(t))\omega^{n_1}(\alpha(t)) + \alpha'(t)h_2(\alpha(t))\left(\omega^{n_1}(\alpha(t)) + \int\limits_a^{\alpha(t)} h_3(s)\omega^{n_2}(s)\right) \\ &\le \alpha'(t)h_1(\alpha(t))\big(p(t) + Z_2(t)\big)^{n_1} \\ &\quad + \alpha'(t)h_2(\alpha(t))\left(\big(p(t) + Z_2(t)\big)^{n_1} + \int\limits_a^{\alpha(t)} h_3(s)\big(p(t) + Z_2(t)\big)^{n_2}ds\right)^n \quad \forall t \in \mathbb{J}. \end{aligned} \tag{3.22}$$

Applying Lemmas 2.1 and 2.2 in (3.22), and after some computations, we have

$$Z_2'(t) \le n_1\alpha'(t)h_1(\alpha(t))Z_2(t) + 2^{2(n-1)}\alpha'(t)h_2(\alpha(t))M_3(t) + \beta_3(t) \quad \forall t \in \mathbb{J}, \tag{3.23}$$

where $\beta_3(t)$ is defined in (3.18) and

$$M_3(t) = n_1Z_2^n(t) + \int\limits_a^{\alpha(t)} n_2h_3^n(s)Z_2^n(s)ds \quad \forall t \in \mathbb{J}. \tag{3.24}$$

Thus, we obtain $M_3(a) = 0,\ Z_2(t) \le M_3(t)$ and $Z_2(\alpha(t)) \le M_3(\alpha(t)) \le M_3(t)$. Now, differentiating (3.24) with respect to t and using (3.23), we get

$$\begin{aligned} M_3'(t) &= nn_1Z_2^{n-1}(t)Z_2'(t) + \alpha'(t)n_2h_3^n(\alpha(t))Z_2^n(\alpha(t)) \\ &\le nn_1M_3^{n-1}\big(n_1\alpha'(t)h_1(\alpha(t))M_3(t) + 2^{2(n-1)}\alpha'(t)h_2(\alpha(t))M_3(t) \\ &\qquad + \beta_3(t)\big) + n_2\alpha'(t)h_3^n(\alpha(t))M_3^n(\alpha(t)) \\ &\le \alpha'(t)\big(nn_1^2h_1(\alpha(t)) + 2^{2(n-1)}nn_1h_2(\alpha(t)) + n_2h_3^n(\alpha(t))\big)M_3^n(t) + \beta_3(t) \end{aligned}$$

or, equivalently,

$$M_3^{-n}(t)M_3'(t) \le \alpha'(t)\big(a^2nh_1(\alpha(t)) + 2^{2(n-1)}nn_1h_2(\alpha(t)) + bh_3^n(\alpha(t))\big) + \beta_3(t) \quad \forall t \in \mathbb{J}. \tag{3.25}$$

Multiply $(1-n)$ on both sides of (3.25) and we obtain the following estimation for $M_3(t)$ after applying integration from a to t on the inequality (3.25):

$$M_3(t) \le \left((1-n)\int\limits_a^{\alpha(t)} \big(\beta_3(\alpha^{-1}(s)) + nn_1^2h_1(s) + 2^{2(n-1)}nn_1h_2(s) + n_2h_3^n(\alpha(s))\big)ds\right)^{\frac{1}{1-n}} \quad \forall t \in \mathbb{J}.$$

Substituting above inequality in (3.23), we have

$$Z_2'(t) \le n_1\alpha'(t)h_1(\alpha(t))Z_2(t) + 2^{2(n-1)}\alpha'(t)h_2(\alpha(t))\beta_4(t) + \beta_3(t) \quad \forall t \in \mathbb{J}, \tag{3.26}$$

where $\beta_4(t)$ is defined in (3.19). We obtain the following estimation for $Z_2(t)$ after applying integration from a to t on the inequality (3.26):

$$Z_2(t) \le 2^{2(n-1)} \int\limits_a^{\alpha(t)} \big(h_2(s)\beta_4(\alpha^{-1}(s)) + \beta_3(\alpha^{-1}(s))\big)ds \exp\left(n_1 \int\limits_s^{\alpha(t)} h_1(r)dr\right) \quad \forall t \in \mathbb{J}. \tag{3.27}$$

Substituting (3.27) in (3.21), we get required inequality (3.17).

Theorem 3.2 is proved.

***Remark* 3.2.** It is very interesting to note that when we change the assumptions of Theorem 3.2, we obtain the following results:

1. If we put $h_1(t) = 0$, then Theorem 3.2 converted into the Theorem 2.1 [16].
2. When we take $h_1(t) = 0$ in (1.6), then the remaining inequality has been studied in Theorem 2.1 [15] for the case $0 < p \le 1$. So, in some extent Theorem 3.2 extends Theorem 2.1 [15].
3. If we take $h_2(t) = 0$, $\alpha(t) = t$, and $n_1 = 1$, then Theorem 3.2 reduced to Gronwall – Bellman inequality [11].
4. When we put $p(t) = \omega_0$, $\alpha(t) = t$, $h_1(t) = 0$, and $n_1 = n_2 = 1$ in (1.6), then remaining inequality has been studied in [6] for the case $n = 1$. Somehow Theorem 3.2 extends Pachpatte inequality [6].

Now, we state and prove the more general inequality which can be used to study the boundedness and existence of solution of delay integro-differential equation.

Theorem 3.3. *If the inequality* (1.3) *holds*, *then*

$$\omega(t) \le \left(p(t) + \int\limits_a^{\alpha(t)} \big(\beta_5(\alpha^{-1}(s)) + 2^{2(n-1)}\beta_6(\alpha^{-1}(s))h_2(s)\big) \exp\left(\frac{n_1}{m}\int\limits_s^{\alpha(t)} h_1(r)dr\right)ds\right)^{\frac{1}{m}} \tag{3.28}$$

for all $t \in \mathbb{J}$, *where*

$$\beta_5(t) = \frac{n_1}{m}\alpha'(t)h_1(\alpha(t))p(t) + \frac{m-n_1}{m}\alpha'(t)h_1(\alpha(t)) + 2^{n-1}\alpha'(t)h_2(\alpha(t))$$
$$\times \left(\frac{n_1}{m}p(t) + \frac{m-n_1}{m} + \int\limits_a^{\alpha(t)} \frac{n_2}{m}h_3(s)p(s)ds + \int\limits_a^{\alpha(t)} \frac{m-n_2}{m}h_3(s)ds\right)^n, \tag{3.29}$$

$$\beta_6(t) = \left((1-n)\int\limits_a^{\alpha(t)} \left(\beta_5(\alpha^{-1}(s)) + \frac{nn_1^2}{m^2}h_1(s) + 2^{2(n-1)}\frac{nn_1}{m}h_2(s) + \frac{n_2}{m}h_3^n(s)\right)ds\right)^{\frac{1}{1-n}}. \tag{3.30}$$

Proof. Define a function

$$Z_3(t) = \int_a^{\alpha(t)} h_1(s)\omega^{n_1}(s)ds + \int_a^{\alpha(t)} h_2(s)\left(\omega^{n_1}(s) + \int_a^s h_3(r)\omega^{n_2}(r)dr\right)^n \quad \forall t \in \mathbb{J},$$

then we have $Z_3(a) = 0$, and using the monotonicity of $Z_3(t)$, we get

$$\omega^m(t) \le p(t) + Z_3(t) \quad \forall t \in \mathbb{J}. \tag{3.31}$$

Applying differentiation on $Z_3(t)$ and using (3.31), we obtain

$$\begin{aligned} Z_3'(t) &= \alpha'(t)h_1(\alpha(t))\omega^{n_1}(\alpha(t)) + \alpha'(t)h_2(\alpha(t))\left(\omega^{n_1}(\alpha(t)) + \int_a^{\alpha(t)} h_3(s)\omega^{n_2}(s)ds\right)^n \\ &\le \alpha'(t)h_1(\alpha(t))\Big(p(t) + Z_3(t)\Big)^{\frac{n_1}{m}} + \alpha'(t)h_2(\alpha(t))\left(\big(p(t) + Z_3(t)\big)^{\frac{n_1}{m}}\right. \\ &\quad \left. + \int_a^{\alpha(t)} h_3(s)\big(p(s) + Z_3(s)\big)^{\frac{n_2}{m}} ds\right)^n \quad \forall t \in \mathbb{J}. \end{aligned} \tag{3.32}$$

Applying Lemmas 2.1 and 2.2 on the inequality (3.32), after few computations, we have

$$Z_3'(t) \le \beta_5(t) + \frac{n_1}{m}\alpha'(t)h_1(\alpha(t))Z_3(t) + 2^{2(n-1)}\alpha'(t)h_2(\alpha(t))M_4(t) \quad \forall t \in \mathbb{J}, \tag{3.33}$$

where $\beta_5(t)$ is defined in (3.29) and

$$M_4(t) = \frac{n_1}{m}Z_3^n(t) + \int_a^{\alpha(t)} \frac{n_2}{m}h_3^n(s)Z_3^n(s)ds \quad \forall t \in \mathbb{J}.$$

Thus, we get $M_4(a) = 0$ and $Z_3(t) \le M_5(t)$. Now, differentiating $M_5(t)$ with respect to t and using (3.33), we obtain

$$\begin{aligned} M_4'(t) &= \frac{nn_1}{m}Z_3^{n-1}(t)Z_3'(t) + \frac{n_2}{m}\alpha'(t)Z_3^n(t) \\ &\le \frac{nn_1}{m}M_4^{n-1}(t)\Big(\beta_5(t) + \frac{n_1}{m}\alpha'(t)h_1(\alpha(t))M_4(t) + 2^{2(n-1)}\alpha'(t)h_2(\alpha(t))M_4(t)\Big) \\ &\quad + \frac{n_2}{m}\alpha'(t)M_4^n(t) \end{aligned}$$

or, equivalently,

$$\begin{aligned} M_4^{-n}(t)M_4'(t) &\le \beta_3(t) + \frac{nn_1^2}{m^2}\alpha'(t)h_1(\alpha(t)) \\ &\quad + 2^{2(n-1)}\frac{nn_1}{m}\alpha'(t)h_2(\alpha(t)) + \frac{n_2}{m}\alpha'(t)h_3^n(\alpha(t)). \end{aligned} \tag{3.34}$$

Multiply $(1-n)$ on both sides of (3.34) and we have the following estimation for $M_4(t)$ after applying integration from a to t on the inequality (3.34):

$$M_4(t) \leq \left((1-n) \int\limits_a^{\alpha(t)} \Big(\beta_5(\alpha^{-1}(s)) + \frac{nn_1^2}{m^2} h_1(s) + 2^{2(n-1)} \frac{nn_1}{m} h_2(s) + \frac{n_2}{m} h_3^n(s) \Big) ds \right)^{\frac{1}{1-n}} \quad \forall t \in \mathbb{J}.$$

Substituting above inequality in (3.33), we get

$$Z_3'(t) \leq \beta_5(t) + \frac{n_1}{m}\alpha'(t) h_1(\alpha(t)) Z_3(t) + 2^{2(n-1)}\alpha'(t) h_2(\alpha(t))\beta_6(t) \quad \forall t \in \mathbb{J}, \tag{3.35}$$

where $\beta_6(t)$ is defined in (3.30). We obtain the following estimation for $Z_3(t)$ after applying integration from a to t on the inequality (3.35):

$$Z_3(t) \leq \int\limits_a^{\alpha(t)} \big(\beta_5(\alpha^{-1}(s)) + 2^{2(n-1)}\beta_6(\alpha^{-1}(s)) h_2(s)\big) \exp\left(\frac{n_1}{m} \int\limits_s^{\alpha(t)} h_1(r) dr \right) ds \quad \forall t \in \mathbb{J}. \tag{3.36}$$

Substituting (3.36) into (3.31), we get the required inequality (3.28).

Theorem 3.3 is proved.

***Remark* 3.3.** It is very interesting to note that when we change the assumptions of Theorem 3.1, we get the following results:

1. If $h_1(t) = 0$, then Theorem 3.3 converted into the Theorem 2.2 [16].
2. When we put $h_1(t) = 0$, $\alpha(t) = t$, and $n_2 = 1$ in (1.3), then the remaining inequality has been studied in Theorem 2.3 [20] for the case $n = 1$. So, in some extent Theorem 3.3 extends Theorem 2.3 [20].
3. If we take $h_2(t) = 0$, $\alpha(t) = t$, and $m = n_1 = 1$, then Theorem 3.3 becomes Gronwall–Bellman inequality [11].
4. When we put $p(t) = \omega_0$, $\alpha(t) = t$, $h_1(t) = 0$, and $m = n_1 = n_2 = 1$ in (1.3), then the remaining inequality has been studied in [6] for the case $n = 1$. So, in somehow Theorem 3.3 extends Pachpatte inequality [6].

4. Applications. This section presents two applications to demonstrate the strength of our derived inequalities of Section 3 in estimating the boundedness and global existence of the solutions to delay integro-differential equation and Volterra-type integral equation with delay.

***Example* 4.1.** Consider the delay integro-differential equation

$$\begin{aligned} &(x^m(t))' = H_1\big(t, x(\tau(t))\big) F\left(t, x(\tau(t)), \int\limits_0^t H_2\big(r, x(\tau(r))\big) dr \right) \quad \forall t \in \mathbb{J}, \\ &x(t) = \phi(t), \quad t \in [d, 0], \qquad -\infty < d = \inf(\tau(t), t \in \mathbb{J}) \leq 0, \\ &\tau(t) \leq t, \quad x(0) = c, \end{aligned} \tag{4.1}$$

where $x(t)$ and $x(\tau(t))$ are the state and state delay, respectively, $F \in \mathcal{C}(\mathbb{J} \times \mathbb{R}_+^2, \mathbb{R}_+)$, $H_1, H_2 \in \mathcal{C}(\mathbb{J} \times \mathbb{R}_+, \mathbb{R}_+)$ satisfy the following conditions:

$$|H_1(t, x)| \leq h_1(t)|x|^{n_1} \quad \forall t \in \mathbb{J}, \tag{4.2}$$

$$|F(t,x,H_2)| \le h_2(t)\Big(|x|^{n_1} + |H_2|\Big)^n \quad \forall t \in \mathbb{J}, \tag{4.3}$$

$$|H_2(t,x)| \le h_3(t)|x|^{n_2} \quad \forall t \in \mathbb{J}, \tag{4.4}$$

where $h_1,\ h_2,\ h_3,\ m,\ n,\ n_1,\ n_2$ are defined in Theorem 3.3. Integrating (4.1) and using initial condition, we obtain

$$x^m(t) = c^m + \int\limits_0^t H_1\big(s, x(\tau(s))\big)ds + \int\limits_0^t F\left(s, x(\tau(s)), \int\limits_0^s H_2\big(r, x(\tau(r))\big)dr\right)ds \quad \forall t \in \mathbb{J}.$$

Letting $\omega(t) = |x(t)|$ and using (4.2) – (4.4), we have

$$\omega^m(t) \le c^m + \int\limits_0^t h_1(s)\omega^{n_1}(\tau(s))ds + \int\limits_0^t h_2(s)\left(\omega^{n_1}(s) + \int\limits_0^s h_3(r)\omega^{n_2}(\tau(r))dr\right)^n ds$$

$$\le c^m + \int\limits_0^{\tau(t)} \frac{h_1(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\omega^{n_1}(s)ds$$

$$+ \int\limits_0^{\tau(t)} \frac{h_2(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\left(\omega^{n_1}(s) + \int\limits_0^s \frac{h_3(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\omega^{n_2}(r)dr\right)^n ds \quad \forall t \in \mathbb{J}.$$

As an application of Theorem 3.3, we get

$$\omega(t) \le \left(|c|^m + \int\limits_0^{\alpha(t)}\left(\beta_7(\alpha^{-1}(s)) + 2^{2(n-1)}\beta_8(\alpha^{-1}(s))\frac{h_2(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\right)\right.$$

$$\left.\times \exp\left(\frac{n_1}{m}\int\limits_s^{\alpha(t)} \frac{h_1(\tau^{-1}(r))}{\tau'\tau^{-1}(r)}dr\right)ds\right)^{\frac{1}{m}} \quad \forall t \in \mathbb{J}, \tag{4.5}$$

where

$$\beta_7(t) = \frac{n_1}{m}\frac{\alpha'(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))}\frac{h_1(\alpha(\tau^{-1}(t)))}{\tau'(\tau^{-1}(t))}\frac{p(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))}$$

$$+ 2^{n-1}\frac{m-n_1}{m}\frac{\alpha'(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))}\frac{h_2(\alpha(\tau^{-1}(t)))}{\tau'(\tau^{-1}(t))}$$

$$\times\left(\frac{n_1}{m}\frac{p(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))} + \frac{m-n_1}{m} + \int\limits_0^{\tau(t)}\frac{n_2}{m}\frac{h_3(\alpha(\tau^{-1}(s)))}{\tau'(\tau^{-1}(s))}\frac{p(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}ds\right.$$

$$\left.+ \int\limits_0^{\tau(t)}\frac{m-n_2}{m}\frac{h_3(\alpha(\tau^{-1}(s)))}{\tau'(\tau^{-1}(s))}ds\right)^n \quad \forall t \in \mathbb{J}$$

and

$$\beta_8(t) = \left[(1-n)\int\limits_0^{\tau(t)} \left(\beta_5(\alpha^{-1}(s)) + \frac{nn_1^2}{m^2}\frac{h_1(\alpha(\tau^{-1}(s)))}{\tau'(\tau^{-1}(s))} + 2^{2(n-1)}\frac{nn_1}{m}\frac{h_2(\alpha(\tau^{-1}(s)))}{\tau'(\tau^{-1}(s))} + \frac{n_2}{m}\frac{h_3^n(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\right)ds\right]^{\frac{1}{1-n}} \quad \forall t \in \mathbb{J}.$$

Thus, the inequality (4.5) shows that the solution of the system (4.1) exists and is bounded.

Now, we give another example to demonstrate Theorem 3.2.

***Example* 4.2.** Consider the Volterra-type integral equation with delay

$$x(t) = p(t) + \int\limits_a^{\alpha(t)} h_1(s)x^{0.2}(s)ds + \int\limits_a^{\alpha(t)} h_2(s)\left(x^{0.2}(s) + \int\limits_a^{\alpha(s)} h_3(r)x^{0.5}(r)dr\right)^5 ds \quad \forall t \in \mathbb{J}, \tag{4.6}$$

where $p,\ h_1,\ h_2,\ h_3$ and α are defined in Theorem 3.2. Let $\omega(t) = |x(t)|$, then we have

$$\omega(t) \le |p(t)| + \int\limits_a^{\alpha(t)} |h_1(s)|\omega^{0.2}(s)ds + \int\limits_a^{\alpha(t)} |h_2(s)|\left(\omega^{0.2}(s) + \int\limits_a^{\alpha(s)} |h_3(r)|\omega^{0.5}(r)dr\right)^5 ds \quad \forall t \in \mathbb{J}. \tag{4.7}$$

The inequality (4.7) is the particular form of (1.2) and satisfies all the conditions of Theorem 3.2. Then, as an application of Theorem 3.2, we obtain

$$\omega(t) \le |p(t)| + 256\int\limits_a^{\alpha(t)} \big(\beta_9(\alpha^{-1}(s)) + |h_2(s)|\beta_{10}(\alpha^{-1}(s))\big)ds \exp\left(0.2\int\limits_s^{\alpha(t)} |h_1(r)|dr\right) \tag{4.8}$$

for all $t \in \mathbb{J}$, where

$$\beta_9(t) = 0.2\alpha'(t)h_1(\alpha(t))p(t) + 0.8\alpha'(t)h_1(\alpha(t)) + 16\alpha'(t)h_2(\alpha(t))\left(0.2p(t) + 0.8 + \int\limits_a^{\alpha(t)} 0.5p(s)h_3(s)ds + \int\limits_a^{\alpha(t)} 0.5h_3(s)ds\right)^5 \quad \forall t \in \mathbb{J}$$

and

$$\beta_{10}(t) = \left(4\int\limits_a^{\alpha(t)} \big(\beta_3(\alpha^{-1}(s)) + 0.2h_1(s) + 16h_2(s) + 0.5h_3^5(\alpha(s))\big)ds\right)^{\frac{1}{-4}} \quad \forall t \in \mathbb{J}.$$

Hence the estimation in (4.8) shows that the solution of retarded nonlinear Volterra-type integral equation with delay (4.6) exists and bounded.

5. Conclusion. It is well-known that there exists a class of mathematical models described by differential equations and a lot of differential equations do not posses the exact solutions or existence of solution or boundedness of solution. Also, the bounds and existence studied by integral inequalities given in the current literature (see references) are not directly fit, and not possible to examine the stability and asymptotic behavior of solutions of classes of more general retarded nonlinear differential, integral and integro-differential equations. However, we have presented more general form of retarded nonlinear delay integral inequalities with mix powers and we have used these inequalities to study the boundedness and global existence of the solutions to delay integro-differential equation and Volterra-type integral equation with delay.

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Received 06.09.21