DOI: 10.37863/umzh.v75i3.6874

UDC 511.176

K. Adegoke (Obafemi Awolowo University, Nigeria),
R. Frontczak[2] (Landesbank Baden – Württemberg, Stuttgart, Germany),
T. Goy[1] (Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine)

RECIPROCAL SERIES INVOLVING HORADAM NUMBERS

РЯДИ З ОБЕРНЕНИМИ ВЕЛИЧИНАМИ ЧИСЕЛ ГОРАДАМА

We evaluate some new three-parameter families of finite reciprocal sums involving Horadam numbers. We are also able to state the results for the associated infinite sums. Some Fibonacci and Lucas sums are presented as examples.

Вивчено деякі нові трипараметричні сім'ї скінченних сум, що містять обернені величини чисел Горадама (узагальнені числа Фібоначчі). Аналогічні результати отримано для відповідних рядів з такими числами. Наведено приклади сум і рядів з числами Фібоначчі та Люка.

1. Introduction and motivation. The Horadam sequence $(w_n)_{n\geq 0} = (w_n(a,b;p,q))_{n\geq 0}$ is defined recursively by

$$w_0 = a, \quad w_1 = b, \quad w_n = pw_{n-1} - qw_{n-2}, \quad n \geq 2,$$

where a, b, p, and q are arbitrary (possibly complex) numbers [13]. The sequences $(u_n(p,q)) = (w_n(0,1;p,q))$ and $(v_n(p,q)) = (w_n(2,p;p,q))$ are the Lucas sequences of the first kind and of the second kind, respectively. The most well-known Lucas sequences are the Fibonacci numbers $F_n = u_n(1,-1)$, the Lucas numbers $L_n = v_n(1,-1)$, the Pell numbers $P_n = u_n(2,-1)$, the Pell–Lucas numbers $Q_n = v_n(2,-1)$, and the balancing numbers $B_n = u_n(6,1)$. All sequences are indexed in the On-Line Encyclopedia of Integer Sequences [28].

Denote by α and β, with $|\alpha| > |\beta|$, the distinct roots of the characteristic equation $x^2 - px + q = 0$ having discriminant $\Delta = p^2 - 4q \neq 0$.

The Binet formulas for w_n, u_n, and v_n, n a nonnegative integer, are given by

$$w_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \qquad u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad v_n = \alpha^n + \beta^n, \tag{1}$$

where $A = b - a\beta$ and $B = b - a\alpha$.

We will also need an expression for negatively subscripted Horadam numbers. For negative subscripts the sequences are given by

$$w_{-n} = \frac{av_n - w_n}{q^n}, \qquad u_{-n} = -u_n q^{-n}, \qquad v_{-n} = v_n q^{-n}. \tag{2}$$

We require the following identity [15, formula (2.12)] in the sequel:

[2] Corresponding author, e-mail: taras.goy@pnu.edu.ua.

[1] Statements and conclusions made in this paper by R. Frontczak are entirely those of the author. They do not necessarily reflect the views of LBBW.

$$w_n w_{n+r+s} - w_{n+r} w_{n+s} = e_w q^n u_r u_s, \tag{3}$$

where $e_w = -AB = pab - qa^2 - b^2$ and $n,\ r,$ and s are integers. For Fibonacci numbers $e_F = -1$, while for Lucas numbers $e_L = 5$.

The goal of this study is to evaluate a family of finite and infinite reciprocal sums involving the Horadam sequence. The interest in evaluating Fibonacci and Lucas (related) reciprocal sums in closed form is not new. The topic has challenged the mathematical community for decades. In 1974, Miller [25] proposed the problem of proving that

$$\sum_{i=0}^{\infty} \frac{1}{F_{2^i}} = \frac{7-\sqrt{5}}{2}. \tag{4}$$

Miller's proposal stimulated a great interest in the series of reciprocal Fibonacci numbers, which led to the many proofs and generalizations (see the survey paper [5] for more information and references). Note that in [25] the author's name of the problem is indicated incorrectly as Millin (see editorial note in [29, p. 92]).

In 1974, Good [10] showed that

$$\sum_{i=0}^{N} \frac{1}{F_{2^i}} = 3 - \frac{F_{2^N-1}}{F_{2^N}}.$$

Allowing N to approach infinity, we have (4). Hoggatt and Bicknell [11] gave eleven methods for finding the value of the sum (4). Shortly later, in [12] they proved a more general formula

$$\sum_{i=0}^{\infty} \frac{1}{F_{k2^i}} = \frac{1}{F_k} + \frac{\Phi^2+1}{\Phi(\Phi^{2k}+1)},$$

where $\Phi = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio. In 1990, André-Jeannin [2, Theorem 2] expressed the infinite reciprocal series

$$\sum_{i=1}^{\infty} \frac{1}{u_{ki}u_{k(i+1)}} \qquad \text{and} \qquad \sum_{i=1}^{\infty} \frac{1}{v_{ki}v_{k(i+1)}},$$

with k an odd positive integer, in terms of Lambert series $\sum_{n=1}^{\infty} \dfrac{x^n}{1-x^n}$, $|x| < 1$. Melham [24] considered the analogues of sequences u_n and v_n for the recurrence $w_n = pw_{n-1} - w_{n-2}$, and obtained analogues of Andre-Jeannin's results for these sequences. In 1997, André-Jeannin [3, Theorem $2'$] again studied the reciprocals of second-order recurrences and evaluated the series

$$\sum_{i=1}^{\infty} \frac{q^{mi}}{w_{mi+n}w_{m(i+k)+n}} \qquad \text{and} \qquad \sum_{i=1}^{\infty} \frac{1}{w_{mi+n}\, w_{m(i+k)+n}}$$

for integers $n \geq 0,\ m \geq 1,$ and $k \geq 1$. Some years later, Hu et al. [16, Theorem 1] obtained a general result, which contains the evaluation of the finite (and infinite) series

$$\sum_{i=1}^{N-1} \frac{q^{mi}}{w_{mi+n}w_{m(i+1)+n}},$$

where n, m, and $N \geq 2$ are integers, as a special case. Laohakosol and Kuhapatanakul [19] extended this result to reciprocal sums of second-order recurrence sequences with nonconstant coefficients.

The first named author derived in [1] a range of closed form expressions for finite and infinite Fibonacci – Lucas sums having products of Fibonacci or Lucas numbers in the denominator of the summand. His results generalize many identities, such as those from [4, 27].

More types of Fibonacci and Lucas (related) reciprocal series, both finite or infinite and alternating or nonalternating, are studied in [6, 8, 9, 14, 22, 23, 26]. For studies focusing on reciprocal sums with three and more factors we refer the reader to references [7, 18, 20, 21].

The series that are studied in the present paper are three-parameter series of the form

$$\sum_{i=1}^{N} \frac{q^{m(i-k)}}{w_{m(i-k)+n} w_{m(i+k)+n}} \qquad \text{and} \qquad \sum_{i=1}^{N} \frac{q^{m(2i-k)}}{w_{m(2i-k)}\, w_{m(2i+k)}},$$

where m, k, and n are integers. To the best of our knowledge, such types of Horadam reciprocal series have not been considered in earlier literature. For all series we provide closed forms in the finite and infinite cases using an elementary approach.

We require the following telescoping summation identities with any integers N and t:

$$\sum_{i=1}^{N} (f(i+t) - f(i)) = \sum_{i=1}^{t} (f(i+N) - f(i)) \tag{5}$$

and

$$\sum_{i=1}^{2N} (\pm 1)^i (f(i+2t) - f(i)) = \sum_{i=1}^{2t} (\pm 1)^i (f(i+2N) - f(i)). \tag{6}$$

Telescoping identities are often used to find sums of finite and infinite Fibonacci and Lucas numbers series in closed form [1, 6, 17, 23, 30].

2. New families of reciprocal Horadam series. Our first main result is the following statement.

Theorem 1. *Let m, k, and n be integers and N a natural number. Then*

$$\sum_{i=1}^{N} \frac{q^{m(i-k)}}{w_{m(i-k)+n} w_{m(i+k)+n}} = \frac{1}{e_w u_n u_{2km}} \sum_{i=1}^{2k} \left(\frac{w_{m(i-k)}}{w_{m(i-k)+n}} - \frac{w_{m(i+N-k)}}{w_{m(i+N-k)+n}} \right) \tag{7}$$

or, equivalently,

$$u_{2km} \sum_{i=1}^{N} \frac{q^{mi}}{w_{m(i-k)+n} w_{m(i+k)+n}} = u_{mN} \sum_{i=1}^{2k} \frac{q^{mi}}{w_{m(i-k)+n} w_{m(i+N-k)+n}}.$$

Proof. Writing $n - r$ for n in identity (3) gives

$$w_{n-r} w_{n+s} - w_n w_{n-r+s} = e_w q^{n-r} u_r\, u_s,$$

from which, writing $mi - km$ for n, $2km$ for s and $-n$ for r, we get

$$w_{m(i-k)+n} w_{m(i+k)} - w_{m(i-k)} w_{m(i+k)+n} = e_w q^{m(i-k)+n} u_{-n} u_{2km} = -e_w q^{m(i-k)} u_n u_{2km}, \tag{8}$$

where in the last step we used (2).

Now divide through identity (8) by $w_{m(i-k)+n}w_{m(i+k)+n}$ to obtain

$$\frac{q^{m(i-k)}}{w_{m(i-k)+n}w_{m(i+k)+n}} = \frac{1}{e_w u_n u_{2km}}\left(\frac{w_{m(i-k)}}{w_{m(i-k)+n}} - \frac{w_{m(i+k)}}{w_{m(i+k)+n}}\right). \tag{9}$$

Identify $f(i) = \dfrac{w_{m(i-k)}}{w_{m(i-k)+n}}$ and $t = 2k$ and use in the summation formula (5) while noting (9).

The theorem is proved.

In particular, evaluation of (7) at $k = 1$ and $k = 2$ gives

$$\sum_{i=1}^{N} \frac{q^{mi}}{w_{m(i-1)+n}w_{m(i+1)+n}} = \frac{q^m}{e_w u_n u_{2m}}\left(\frac{w_0}{w_n} + \frac{w_m}{w_{m+n}} - \frac{w_{m(N+1)}}{w_{m(N+1)+n}} - \frac{w_{mN}}{w_{mN+n}}\right)$$

and

$$\sum_{i=1}^{N} \frac{q^{mi}}{w_{m(i-2)+n}w_{m(i+2)+n}} = \frac{q^{2m}}{e_w u_n u_{4m}}\left(\frac{w_{-m}}{w_{-m+n}} + \frac{w_0}{w_n} + \frac{w_m}{w_{m+n}} + \frac{w_{2m}}{w_{2m+n}}\right.$$
$$\left. - \frac{w_{m(N-1)}}{w_{m(N-1)+n}} - \frac{w_{mN}}{w_{mN+n}} - \frac{w_{m(N+1)}}{w_{m(N+1)+n}} - \frac{w_{m(N+2)}}{w_{m(N+2)+n}}\right).$$

Setting $n = mk$ in Theorem 1, we have the following corollary.

***Corollary* 1.** *For integers m and k, and natural number N, we have*

$$\sum_{i=1}^{N} \frac{q^{m(i-k)}}{w_{mi}w_{m(i+2k)}} = \frac{1}{e_w u_{mk} u_{2mk}} \sum_{i=1}^{2k}\left(\frac{w_{m(i-k)}}{w_{mi}} - \frac{w_{m(i+N-k)}}{w_{m(i+N)}}\right)$$

or, equivalently,

$$u_{2mk}\sum_{i=1}^{N} \frac{q^{mi}}{w_{mi}w_{m(i+2k)}} = u_{mN}\sum_{i=1}^{2k} \frac{q^{mi}}{w_{mi}w_{m(i+N)}}.$$

The associated infinite series are evaluated in the next corollary.

***Corollary* 2.** *Let m, k, and n be integers. Then*

$$\sum_{i=1}^{\infty} \frac{q^{m(i-k)}}{w_{m(i-k)+n}w_{m(i+k)+n}} = \frac{1}{e_w u_n u_{2km}}\left(\sum_{i=1}^{2k} \frac{w_{m(i-k)}}{w_{m(i-k)+n}} - \frac{2k}{\alpha^n}\right)$$

and, especially with $n = mk$,

$$\sum_{i=1}^{\infty} \frac{q^{m(i-k)}}{w_{mi}w_{m(i+2k)}} = \frac{1}{e_w u_{mk} u_{2km}}\left(\sum_{i=1}^{2k} \frac{w_{m(i-k)}}{w_{mi}} - \frac{2k}{\alpha^{mk}}\right).$$

Proof. According to (1),

$$\lim_{N\to\infty} \frac{w_N}{w_{N+r}} = \frac{1}{\alpha^r}. \tag{10}$$

Taking limits as $N \to \infty$ of both sides of identity (7), making use of (10) completes the proof.

As special cases of our results obtained so far, we have new Fibonacci and Lucas identities.

***Corollary* 3.** *For integers m and n, and natural number N,*

$$\sum_{i=1}^{N} \frac{(-1)^{m(i-1)}}{F_{m(i-1)+n}F_{m(i+1)+n}} = \frac{1}{F_n F_{2m}}\left(\frac{F_{m(N+1)}}{F_{m(N+1)+n}} + \frac{F_{mN}}{F_{mN+n}} - \frac{F_m}{F_{m+n}}\right), \tag{11}$$

$$\sum_{i=1}^{N} \frac{(-1)^{m(i-1)}}{L_{m(i-1)+n}L_{m(i+1)+n}} = \frac{1}{5F_n F_{2m}}\left(\frac{2}{L_n} + \frac{L_m}{L_{m+n}} - \frac{L_{mN}}{L_{mN+n}} - \frac{L_{m(N+1)}}{L_{m(N+1)+n}}\right), \tag{12}$$

$$\sum_{i=1}^{\infty} \frac{(-1)^{m(i-1)}}{F_{m(i-1)+n}F_{m(i+1)+n}} = \frac{1}{F_n F_{2m}}\left(\frac{2}{\Phi^n} - \frac{F_m}{F_{m+n}}\right), \tag{13}$$

$$\sum_{i=1}^{\infty} \frac{(-1)^{m(i-1)}}{L_{m(i-1)+n}L_{m(i+1)+n}} = \frac{1}{5F_n F_{2m}}\left(\frac{2}{L_n} + \frac{L_m}{L_{m+n}} - \frac{2}{\Phi^n}\right), \tag{14}$$

where $\Phi = \dfrac{1+\sqrt{5}}{2}$.

Proof. Use Theorem 1 and Corollary 2 with $w_n = F_n$ and $w_n = L_n$, respectively, and $k = 1$. Recall that $e_F = -1$ and $e_L = 5$.

The corollary is proved.

We mention that equations (11)–(14) were discovered by the second author recently and appear in [8, Theorem 1.2].

Note that from equation (7) it is clear that it does not hold for $m,\ n,\ k = 0$. The next theorem addresses the situation of $n = 0$.

Theorem 2. *Let m and k be integers and N a natural number. Then*

$$\sum_{i=1}^{N} \frac{q^{m(i-k)}}{w_{m(i-k)}w_{m(i+k)}} = \frac{1}{e_w u_{2km}} \sum_{i=1}^{2k}\left(\frac{w_{m(i+N-k)+1}}{w_{m(i+N-k)}} - \frac{w_{m(i-k)+1}}{w_{m(i-k)}}\right) \tag{15}$$

or, equivalently,

$$u_{2km}\sum_{i=1}^{N} \frac{q^{mi}}{w_{m(i-k)}w_{m(i+k)}} = u_{mn}\sum_{i=1}^{2k} \frac{q^{mi}}{w_{m(i-k)}w_{m(i+N-k)}}.$$

Proof. Divide through identity (8) by $w_{m(i-k)}w_{m(i+k)}$ to obtain

$$\frac{e_w u_{2km} u_n q^{m(i-k)}}{w_{m(i-k)}w_{m(i+k)}} = \frac{w_{m(i+k)+n}}{w_{m(i+k)}} - \frac{w_{m(i-k)+n}}{w_{m(i-k)}},$$

where n is now arbitrary and can be set equal to unity, yielding

$$\frac{e_w u_{2km} q^{m(i-k)}}{w_{m(i-k)}w_{m(i+k)}} = \frac{w_{m(i+k)+1}}{w_{m(i+k)}} - \frac{w_{m(i-k)+1}}{w_{m(i-k)}}, \tag{16}$$

from which the result now follows upon summation over i using (5) with $f(i) = \dfrac{w_{m(i-k)+1}}{w_{m(i-k)}}$.

The theorem is proved.

Upon letting $N \to +\infty$ in (15), we obtain the following corollary.

***Corollary* 4.** *Let m and k be integers. Then*

$$\sum_{i=1}^{\infty}\frac{q^{m(i-k)}}{w_{m(i-k)}w_{m(i+k)}}=\frac{1}{e_w u_{2km}}\left(2k\alpha-\sum_{i=1}^{2k}\frac{w_{m(i-k)+1}}{w_{m(i-k)}}\right).$$

Working with Lucas numbers and $k=1$, we immediately get the next results:

$$\sum_{i=1}^{N}\frac{(-1)^{m(i-1)}}{L_{m(i-1)}L_{m(i+1)}}=\frac{1}{5F_{2m}}\left(\frac{L_{m(N+1)+1}}{L_{m(N+1)}}+\frac{L_{mN+1}}{L_{mN}}-\frac{L_{m+1}}{L_m}-\frac{1}{2}\right)$$

and

$$\sum_{i=1}^{\infty}\frac{(-1)^{m(i-1)}}{L_{m(i-1)}L_{m(i+1)}}=\frac{1}{5F_{2m}}\left(2\Phi-\frac{L_{m+1}}{L_m}-\frac{1}{2}\right).$$

The above Lucas sums are also evaluated in [8]. The results, however, are stated in a different form as follows:

$$\sum_{i=1}^{N}\frac{(-1)^{m(i-1)}}{L_{m(i-1)}L_{m(i+1)}}=\frac{1}{2F_{2m}}\left(\frac{F_{m(N+1)}}{L_{m(N+1)}}+\frac{F_{mN}}{L_{mN}}-\frac{F_m}{L_m}\right),$$

$$\sum_{i=1}^{\infty}\frac{(-1)^{m(i-1)}}{L_{m(i-1)}L_{m(i+1)}}=\frac{1}{\sqrt{5}F_{2m}}-\frac{1}{2L_m^2}.$$

The reason for the differences in expressing these sums is, that the special case of Theorem 2 with $w_n=v_n$ possesses a different expression.

As the family of series involving the terms of Lucas sequences is interesting on its own we give an expression involving the terms of Lucas sequences of each kind in a separate theorem.

Theorem 3. *For integers m and k, we have the following identities:*

$$\sum_{i=1}^{N}\frac{q^{m(i-k)}}{v_{m(i-k)}v_{m(i+k)}}=\frac{1}{2u_{2km}}\sum_{i=1}^{2k}\left(\frac{u_{m(i+N-k)}}{v_{m(i+N-k)}}-\frac{u_{m(i-k)}}{v_{m(i-k)}}\right)$$
$$=\frac{u_{mN}}{u_{2km}}\sum_{i=1}^{2k}\frac{q^{m(i-k)}}{v_{m(i-k)}v_{m(i+N-k)}} \qquad (17)$$

and

$$\sum_{i=1}^{N}\frac{q^{mi}}{u_{mi}u_{m(i+2k)}}=\frac{1}{2u_{2km}}\sum_{i=1}^{2k}\left(\frac{v_{mi}}{u_{mi}}-\frac{v_{m(i+N)}}{u_{m(i+N)}}\right)=\frac{u_{mN}}{u_{2km}}\sum_{i=1}^{2k}\frac{q^{mi}}{u_{mi}u_{m(i+N)}}. \qquad (18)$$

Proof. The proof is similar to that of Theorem 1. Here we use

$$u_{m(i+k)}v_{m(i-k)}-u_{m(i-k)}v_{m(i+k)}=2q^{m(i-k)}u_{2mk}, \qquad (19)$$

which is obtained by setting $s=m(i+k)$ and $t=m(i-k)$ in the identity

$$u_s v_t - v_s u_t = -2q^s u_{t-s}.$$

Dividing through identity (19) by $v_{m(i-k)}v_{m(i+k)}$ gives

$$\frac{2u_{2km}q^{m(i-k)}}{v_{m(i-k)}v_{m(i+k)}}=\frac{u_{m(i+k)}}{v_{m(i+k)}}-\frac{u_{m(i-k)}}{v_{m(i-k)}}, \tag{20}$$

while dividing through the same identity (19) by $u_{m(i-k)}u_{m(i+k)}$ and shifting the index i gives

$$\frac{2u_{2km}q^{mi}}{u_{mi}u_{m(i+2k)}}=\frac{v_{mi}}{u_{mi}}-\frac{v_{m(i+2k)}}{u_{m(i+2k)}}. \tag{21}$$

Identities (17) and (18) now follow by summing, over i, both sides of each of (20) and (21); noting that the sum on the right-hand side, in each case, telescopes according to the telescoping summation formula (6).

As a by-product from Theorems 2 and 3 we obtain the following relation involving Lucas sequences of the first and second kind:

$$\sum_{i=1}^{2k}\left(\frac{u_{m(i+N-k)}}{v_{m(i+N-k)}}-\frac{u_{m(i-k)}}{v_{m(i-k)}}\right)=\frac{2}{\Delta}\sum_{i=1}^{2k}\left(\frac{v_{m(i+N-k)+1}}{v_{m(i+N-k)}}-\frac{v_{m(i-k)+1}}{v_{m(i-k)}}\right).$$

Similarly, comparing the second part of Theorem 3 with Corollary 1 $(w_n=u_n)$, we get the interesting relation

$$\sum_{i=1}^{2k}\left(\frac{v_{mi}}{u_{mi}}-\frac{v_{m(i+N)}}{u_{m(i+N)}}\right)=\frac{2q^{mk}}{u_{mk}}\sum_{i=1}^{2k}\left(\frac{u_{m(i+N-k)+1}}{u_{m(i+N)}}-\frac{u_{m(i-k)}}{u_{mi}}\right).$$

We conclude this section with the observation, that the identity of Theorem 2 will crash in general for sequences with $w_0=a=0$, such as the Lucas sequence of the first kind. We now give a nonsingular version of the theorem. The proof is similar to that of Theorem 1 and hence it is omitted.

Theorem 4. *Let m and k be integers and N a natural number. Then*

$$\sum_{i=1}^{N}\frac{q^{mi}}{w_{mi}w_{m(i+2k)}}=\frac{1}{e_w u_{2km}}\sum_{i=1}^{2k}\left(\frac{w_{m(i+N)+1}}{w_{m(i+N)}}-\frac{w_{mi+1}}{w_{mi}}\right)=\frac{u_{mN}}{u_{2km}}\sum_{i=1}^{2k}\frac{q^{mi}}{w_{mi}w_{m(i+N)}},$$

as well as

$$\sum_{i=1}^{\infty}\frac{q^{mi}}{w_{mi}w_{m(i+2k)}}=\frac{1}{e_w u_{2km}}\left(2k\alpha-\sum_{i=1}^{2k}\frac{w_{mi+1}}{w_{mi}}\right).$$

3. Still other Horadam series. The next achievement of this paper is the following theorem.

Theorem 5. *Let m, k, and n be integers and N a natural number. Then*

$$\sum_{i=1}^{2N}\frac{(\pm1)^i q^{m(i-k)}}{w_{m(i-k)+n}w_{m(i+k)+n}}=\frac{1}{e_w u_n u_{2km}}\sum_{i=1}^{2k}(\pm1)^i\left(\frac{w_{m(i-k)}}{w_{m(i-k)+n}}-\frac{w_{m(i+2N-k)}}{w_{m(i+2N-k)+n}}\right) \tag{22}$$

or, equivalently,

$$u_{2km}\sum_{i=1}^{2N}\frac{(\pm1)^i q^{mi}}{w_{m(i-k)+n}w_{m(i+k)+n}}=u_{2mN}\sum_{i=1}^{2k}\frac{(\pm1)^i q^{mi}}{w_{m(i-k)+n}w_{m(i+2N-k)+n}}.$$

Proof. Use $f(i) = \dfrac{w_{m(i-k)}}{w_{m(i-k)+n}}$ and $t = k$ in (6) to obtain, by (9),

$$f(i+2k) - f(i) = \frac{w_{m(i+k)}}{w_{m(i+k)+n}} - \frac{w_{m(i-k)}}{w_{m(i-k)+n}} = -\frac{e_w u_n u_{2km} q^{m(i-k)}}{w_{m(i-k)+n} w_{m(i+k)+n}}$$

and

$$f(i+2N) - f(i) = \frac{w_{m(i+2N-k)}}{w_{m(i+2N-k)+n}} - \frac{w_{m(i-k)}}{w_{m(i-k)+n}}.$$

Putting these values into the summation formula (6) produces the stated identity (22).

The theorem is proved.

Letting N approach infinity we immediate obtain from (22) the following corollary.

***Corollary* 5.** *Let m, k, and n be integers. Then*

$$\sum_{i=1}^{\infty} \frac{(-1)^i q^{m(i-k)}}{w_{m(i-k)+n} w_{m(i+k)+n}} = \frac{1}{e_w u_n u_{2km}} \sum_{i=1}^{2k} \frac{(-1)^i w_{m(i-k)}}{w_{m(i-k)+n}}.$$

Theorem 6. *Let m, k, and n be integers and N a natural number. Then*

$$\sum_{i=1}^{N} \frac{q^{m(2i-k)}}{w_{m(2i-k)+n} w_{m(2i+k)+n}} = \frac{1}{e_w u_n u_{2km}} \sum_{i=1}^{k} \left(\frac{w_{m(2i-k)}}{w_{m(2i-k)+n}} - \frac{w_{m(2(i+N)-k)}}{w_{m(2(i+N)-k)+n}} \right)$$

or, equivalently,

$$u_{2km} \sum_{i=1}^{N} \frac{q^{2mi}}{w_{m(2i-k)+n} w_{m(2i+k)+n}} = u_{2mN} \sum_{i=1}^{k} \frac{q^{2mi}}{w_{m(2i-k)+n} w_{m(2(i+N)-k)+n}}.$$

Proof. Write $2i$ for i in (9) to obtain

$$\frac{q^{m(2i-k)}}{w_{m(2i-k)+n} w_{m(2i+k)+n}} = \frac{1}{e_w u_n u_{2km}} \left(\frac{w_{m(2i-k)}}{w_{m(2i-k)+n}} - \frac{w_{m(2i+k)}}{w_{m(2i+k)+n}} \right).$$

Use $f(i) = \dfrac{w_{m(2i-k)}}{w_{m(2i-k)+n}}$ and $t = k$ in (5).

The theorem is proved.

In the limit as N approaches infinity in Theorem 6, we have the following result.

***Corollary* 6.** *Let m, k, and n be integers. Then*

$$\sum_{i=1}^{\infty} \frac{q^{m(2i-k)}}{w_{m(2i-k)+n} w_{m(2i+k)+n}} = \frac{1}{e_w u_n u_{2km}} \left(\sum_{i=1}^{k} \frac{w_{m(2i-k)}}{w_{m(2i-k)+n}} - \frac{k}{\alpha^n} \right).$$

Now we list some Fibonacci and Lucas series which follow from Corollary 6:

$$\sum_{i=1}^{\infty} \frac{1}{F_{m(2i-1)+n} F_{m(2i+1)+n}} = \frac{(-1)^m}{F_n F_{2m}} \left(\frac{1}{\Phi^n} - \frac{F_m}{F_{m+n}} \right),$$

$$\sum_{i=1}^{\infty} \frac{1}{F_{2m(i-1)+n} F_{2m(i+1)+n}} = \frac{1}{F_n F_{4m}} \left(\frac{2}{\Phi^n} - \frac{F_{2m}}{F_{2m+n}} \right),$$

$$\sum_{i=1}^{\infty} \frac{1}{L_{m(2i-1)+n}L_{m(2i+1)+n}} = \frac{(-1)^m}{5F_nF_{2m}}\left(\frac{L_m}{L_{m+n}} - \frac{1}{\Phi^n}\right),$$

$$\sum_{i=1}^{\infty} \frac{1}{L_{2m(i-1)+n}L_{2m(i+1)+n}} = \frac{1}{5F_nF_{4m}}\left(\frac{2}{L_n} + \frac{L_{2m}}{L_{2m+n}} - \frac{2}{\Phi^n}\right).$$

The next theorem is a nonsingular version of the first identity from Theorem 6 and Corollary 6 in case $n = 0$.

Theorem 7. *Let m and k be integers and N a natural number. Then*

$$\sum_{i=1}^{N} \frac{q^{m(2i-k)}}{w_{m(2i-k)}w_{m(2i+k)}} = \frac{1}{e_w u_{2km}} \sum_{i=1}^{k}\left(\frac{w_{m(2(i+N)-k)+1}}{w_{m(2(i+N)-k)}} - \frac{w_{m(2i-k)+1}}{w_{m(2i-k)}}\right)$$

$$= \frac{u_{2mN}}{u_{2km}} \sum_{i=1}^{k} \frac{q^{m(2i-k)}}{w_{m(2i-k)}w_{m(2(i+N)-k)}}$$

and

$$\sum_{i=1}^{\infty} \frac{q^{m(2i-k)}}{w_{m(2i-k)}w_{m(2i+k)}} = \frac{1}{e_w u_{2km}}\left(k\alpha - \sum_{i=1}^{k} \frac{w_{m(2i-k)+1}}{w_{m(2i-k)}}\right). \qquad (23)$$

Proof. Write $2i$ for i in identity (16) to obtain

$$-\frac{e_w u_{2km} q^{m(2i-k)}}{w_{m(2i-k)}w_{m(2i+k)}} = \frac{w_{m(2i-k)+1}}{w_{m(2i-k)}} - \frac{w_{m(2i+k)+1}}{w_{m(2i+k)}},$$

from which the result now follows upon summation over i using (5) with $f(i) = \dfrac{w_{m(2i-k)+1}}{w_{m(2i-k)}}$ and $t = k$. Taking limit as $N \to \infty$, we obtain (23).

4. Conclusion. We have evaluated some new three-parameter families of reciprocal Horadam sums in closed form. The approach is elementary and is based on clever telescoping. It seems possible to extend the results of the present paper to reciprocal sums involving three and four Horadam numbers as factors in the denominator. This will be explored further in a future project.

Acknowledgments. We thank the referee for spending time for a careful review and for offering very useful suggestions which have helped to improve the quality of our paper.

References

1. K. Adegoke, *Generalizations of reciprocal Fibonacci – Lucas sums of Brousseau*, J. Integer Seq., **21**, Article 18.1.6 (2018).
2. R. André-Jeannin, *Lambert series and the summation of reciprocals in certain Fibonacci – Lucas-type sequences*, Fibonacci Quart., **28**, № 3, 223 – 226 (1990).
3. R. André-Jeannin, *Summation of reciprocals in certain second-order recurring sequences*, Fibonacci Quart., **35**, № 1, 68 – 74 (1997).
4. P. S. Bruckman, I. J. Good, *A generalization of a series of de Morgan, with applications of Fibonacci type*, Fibonacci Quart., **14**, № 3, 193 – 196 (1976).
5. D. Duverney, I. Shiokawa, *On series involving Fibonacci and Lucas numbers. I*, Proc. DARF Conf. 2007, AIP Conf. Proc., **976**, 62 – 76 (2008).

6. B. Farhi, *Summation of certain infinite Lucas-related series*, J. Integer Seq., **22**, Article 19.1.6 (2019).
7. R. Frontczak, *Closed-form evaluations of Fibonacci – Lucas reciprocal sums with three factors*, Notes Number Theory Discrete Math., **23**, № 2, 104 – 116 (2017).
8. R. Frontczak, *Further results on arctangent sums with applications to generalized Fibonacci numbers*, Notes Number Theory Discrete Math., **23**, № 1, 39 – 53 (2017).
9. R. Frontczak, *On two new classes of Fibonacci and Lucas reciprocal sums with subscripts in arithmetic progression*, Appl. Math. Sci., **11**, № 25, 1211 – 1219 (2017).
10. I. J. Good, *A reciprocal series of Fibonacci numbers*, Fibonacci Quart., **12**, № 4, 346 (1974).
11. V. E. Hoggatt (Jr.), M. Bicknell, *A primer for the Fibonacci numbers, Part XV. Variations on summing a series of reciprocals of Fibonacci numbers*, Fibonacci Quart., **14**, № 3, 272 – 276 (1976).
12. V. E. Hoggatt (Jr.), M. Bicknell, *A reciprocal series of Fibonacci numbers with subscripts* $2^n k$, Fibonacci Quart., **14**, № 5, 453 – 455 (1976).
13. A. F. Horadam, *Basic properties of a certain generalized sequence of numbers*, Fibonacci Quart., **3**, № 3, 161 – 176 (1965).
14. A. F. Horadam, *Elliptic functions and Lambert series in the summation of reciprocals in certain recurrence-generated sequences*, Fibonacci Quart., **26**, № 2, 98 – 114 (1988).
15. A. F. Horadam, *Generalization of a result of Morgado*, Port. Math., **44**, № 2, 131 – 136 (1987).
16. H. Hu, Z.-W. Sun, J.-X. Liu, *Reciprocal sums of second-order recurrent sequences*, Fibonacci Quart., **39**, № 3, 214 – 220 (2001).
17. E. Kılıç, D. Ersanli, *New reciprocal sums involving finite products of second order recursions*, Miskolc Math. Notes, **20**, № 2, 1039 – 1050 (2019).
18. E. Kılıç, H. Prodinger, *Closed form evaluation of Melham's reciprocal sums*, Miskolc Math. Notes, **18**, № 1, 251 – 264 (2017).
19. V. Laohakosol, K. Kuhapatanakul, *Reciprocal sums of generalized second order recurrence sequences*, Fibonacci Quart., **46/47**, № 4, 316 – 325 (2008/2009).
20. R. S. Melham, *Finite reciprocal sums involving summands that are balanced products of generalized Fibonacci numbers*, J. Integer Seq., **17**, Article 14.6.5 (2014).
21. R. S. Melham, *More on finite sums that involve reciprocals of products of generalized Fibonacci numbers*, Integers, **14**, #A4 (2014).
22. R. S. Melham, *On certain families of finite reciprocal sums that involve generalized Fibonacci numbers*, Fibonacci Quart., **53**, № 4, 323 – 334 (2015).
23. R. S. Melham, *Reciprocal series of squares of Fibonacci related sequences with subscripts in arithmetic progression*, J. Integer Seq., **18**, Article 15.8.7 (2015).
24. R. S. Melham, A. G. Shannon, *On reciprocal sums of Chebyshev related sequences*, Fibonacci Quart., **33**, № 3, 194 – 202 (1995).
25. D. A. Miller, *Problem H-237*, Fibonacci Quart., **12**, № 3, 309 (1974).
26. S. Rabinowitz, *Algorithmic simplification of reciprocals sums*, F. T. Howard (ed.). Applications of Fibonacci Numbers, Springer, Dordrecht (1999), p. 277 – 292.
27. S. Rabinowitz, *Algorithmic summation of reciprocals of products of Fibonacci numbers*, Fibonacci Quart., **37**, № 2, 122 – 127 (1999).
28. N. J. A. Sloane, ed., *The On-Line Encyclopedia of Integer Sequences*; https://oeis.org.
29. R. E. Whitney, ed., *Advanced problem and solutions*, Fibonacci Quart., **14**, № 1, 88 – 92 (1976).
30. F.-Z. Zhao, *Notes on reciprocal series related to Fibonacci and Lucas numbers*, Fibonacci Quart., **37**, № 3, 254 – 257 (1999).

Received 17.08.21