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K. Alaoui Ismaili (Laboratory of Mathematics, Computing and Applications-Information Security (LabMia-SI), Department of Mathematics, Faculty of Sciences of Rabat, Mohammed V University in Rabat, Morocco),

N. Mahdou (Department of Mathematics, Faculty of Science and Technology of Fez, University S. M. Ben Abdellah Fez, Morocco),

M. A. S. Moutui[1] (University of Haute Alsace, IRIMAS, Département de Mathématiques, Mulhouse, France and Division of Science, Technology and Mathematics, American University of Afghanistan, Doha Campus, Qatar)

COMMUTATIVE RING EXTENSIONS DEFINED BY PERFECT-LIKE CONDITIONS

КОМУТАТИВНІ КІЛЬЦЕВІ РОЗШИРЕННЯ, ЩО ВИЗНАЧЕНІ ІДЕАЛЬНО ПОДІБНИМИ УМОВАМИ

In 2005, Enochs, Jenda, and López-Romos extended the notion of perfect rings to n-perfect rings such that a ring is n-perfect if every flat module has projective dimension less or equal than n. Later, Jhilal and Mahdou defined a commutative unital ring R to be strongly n-perfect if any R-module of flat dimension less or equal than n has a projective dimension less or equal than n. Recently Purkait defined a ring R to be n-semiperfect if $\overline{R} = R/\operatorname{Rad}(R)$ is semisimple and n-potents lift modulo $\operatorname{Rad}(R)$. We study of three classes of rings, namely, n-perfect, strongly n-perfect, and n-semiperfect rings. We investigate these notions in several ring-theoretic structures with an aim of construction of new original families of examples satisfying the indicated properties and subject to various ring-theoretic properties.

У 2005 році Енохс, Дженда та Лопес-Ромос розширили поняття ідеальних кілець до n-ідеальних, таких що кільце є n-ідеальним, якщо кожен плоский модуль має проєктивну розмірність меншу або рівну n. Пізніше Джилал і Махду визначили, що комутативне унітальне кільце R є сильно n-ідеальним, якщо будь-який R-модуль плоскої розмірності меншої або рівної n має проєктивну розмірність меншу або рівну n. Нещодавно Пуркайт визначив, що кільце R буде n-напівідеальним, якщо $\overline{R} = R/\operatorname{Rad}(R)$ є напівпростим, а n-потенти піднімаються по модулю $\operatorname{Rad}(R)$. Цю статтю присвячено вивченню трьох класів кілець, а саме n-ідеальних, сильно n-ідеальних і n-напівідеальних. Досліджуються ці поняття в кількох теоретико-кільцевих конструкціях з метою створення нових оригінальних сімей прикладів, що задовольняють ці властивості і підпорядковуються різним теоретико-кільцевим властивостям.

1. Introduction. All rings considered in this paper are assumed to be commutative with identity elements and all modules are unitary. Let R be a ring and let M be an R-module. We use $\operatorname{pd}_R(M)$ and $\operatorname{fd}_R(M)$ to denote, respectively, the classical projective and flat dimensions of M. gldim(R) is the classical global dimension of R. A ring R is perfect if every flat R-module is projective R-module. The pioneering work on perfect rings was done by Bass [3] and most of the principal characterizations of perfect rings are contained in Theorem P from that paper.

In 2005, Enochs, Jenda, and López-Romos extended the notion of perfect rings to n-perfect rings such that a ring is called n-perfect if every flat module has projective dimension less or equal than n [10].

In 2010, Jhilal and Mahdou defined a commutative unital ring R to be strongly n-perfect if any R-module of flat dimension less or equal than n has a projective dimension less or equal than n [12]. Observe that every strongly n-perfect ring is an n-perfect ring, and note that if $n = 0$ then the strongly 0-perfect rings are the perfect rings.

[1] Corresponding author, e-mail: abdou-salam-moutui.moutu@uha.fr.

In 1994, Costa [4] introduced a doubly filtered set of classes of rings in order to categorize the structure of non-Noetherian rings: for nonnegative integers n and d, we say that a ring R is an (n, d)-ring if $\mathrm{pd}_R(E) \leq d$ for each n-presented R-module E. An integral domain with this property will be called an (n, d)-domain. For example, the $(n, 0)$-domains are the fields, the $(0, 1)$-domains are the Dedekind domains, and the $(1, 1)$-domains are the Prüfer domains [4].

We call a commutative ring an n-Von Neumann regular ring if it is an $(n, 0)$-ring. Thus, the 1-Von Neumann regular rings are the Von Neumann regular rings [4, Theorem 1.3].

In [16], Purkait introduced the notion of n-semiperfect ring (that is a ring R in which n-potent elements lift modulo $\mathrm{Rad}(R)$ and $\overline{R} = R/\mathrm{Rad}(R)$ is semisimple, where $\mathrm{Rad}(R)$ denotes the Jacobson radical of R). He characterized strongly n-clean ring in terms of n-semiperfect ring. In addition to this, the author established some results on this ring. They proved that under certain conditions a ring is n-semiperfect if and only if it is strongly n-clean and orthogonally n-finite. Recall that an element a of a ring R is said to be n-potent if $a^n = a$ for some positive integer n. The following diagram summarizes the relationship between the notions involved in this paper:

$$\begin{array}{ccccc} \textit{strongly } n\textit{-perfect} & \Longrightarrow & n\textit{-perfect} & & \\ \Uparrow & & & & \\ \textit{perfect} & \Longrightarrow & \textit{semiperfect} & \Longrightarrow & n\textit{-semiperfect.} \end{array}$$

Notice that the above implications are not reversible in general. This paper is devoted to the study of three classes of rings, namely, n-perfect, strongly n-perfect and n-semiperfect rings. We investigate these notions in several ring-theoretic structures with an aim of construction of new original families of n-perfect rings that are not strongly n-perfect, strongly n-perfect rings that are not perfect and n-semiperfect rings which are not semiperfect. In 2006, M. D'Anna and M. Fontana [7] introduced a new construction, called amalgamated duplication of a ring A along an A-submodule E of $Q(A)$ (the total ring of fractions of A) such that $E^2 \subseteq E$. When $E^2 = \{0\}$, this construction coincides with the trivial ring extension of A by E. Motivations and more applications of the amalgamated duplication $A \bowtie E$ of A along an A-submodule E of $Q(A)$ are discussed in more details, especially in the particular case where E is an ideal of A, in recent papers, for instance, see [5–9].

In 2010, D'Anna, Finocchiaro and Fontana [8] extended the notion of amalgamated duplication construction $A \bowtie I$ of a ring A along an ideal I of A to the general context of ring homomorphism extensions as follows:

Let A and B be two rings with identity elements, J be an ideal of B and $f : A \to B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$: $A \bowtie^f J := \big\{(a, f(a) + j) \mid a \in A, j \in J\big\}$ called the amalgamation of A and B along J with respect to f. For a ring R, we denote, respectively, by $\mathrm{Max}(R)$, $\mathrm{Nil}(R)$, $\mathrm{Idem}(R)$, m-$\mathrm{potent}(R)$, $\mathrm{Rad}(R)$, the spectrum of all maximal ideals of R, the ideal of all nilpotent elements of R, the set of all idempotent elements of R, the set of all m-potent elements of R and the Jacobson radical of R.

2. On strongly n-perfect property. Our first result of this section investigates the strongly n-perfect property in the amalgamation.

Theorem 2.1. *Let $f : A \to B$ be a ring homomorphism and J be a proper ideal of B. Then*:

(1) (a) *Assume that $\mathrm{fd}_A(A \bowtie^f J) = r < \infty$. If $A \bowtie^f J$ is a strongly n-perfect ring, then A is a strongly $(n + r)$-perfect ring. In particular, if J is a flat A-module, then A is a strongly n-perfect ring if so is $A \bowtie^f J$.*

(b) *Assume that* J *is a pure ideal of* $f(A) + J$. *If* $A \bowtie^f J$ *is a strongly* n*-perfect ring, then* A *is a strongly* n*-perfect ring.*

(2) *Assume that* $f^{-1}(J)$ *is a pure ideal of* A. *If* $A \bowtie^f J$ *is a strongly* n*-perfect ring, then* $(f(A) + J)$ *is a strongly* n*-perfect ring.*

(3) *Assume that* $f^{-1}(J)$ *and* J *are pure ideals of* A *and* $(f(A) + J)$, *respectively.*

Then:

(a) $A \bowtie^f J$ *is a strongly* n*-perfect ring if and only if* A *and* $f(A) + J$ *are strongly* n*-perfect rings.*

(b) $A \bowtie^f J$ *is a strongly* n*-perfect ring and an* $(1, n)$*-ring if and only if* $\operatorname{gl}\dim(A) \leq n$ *and* $\operatorname{gl}\dim(f(A) + J) \leq n$.

The proof of Theorem 2.1 draws on the following results.

Lemma 2.1. *Let* $f : A \to B$ *be a ring homomorphism and* J *be an ideal of* B. *Then*:

(1) *The following conditions are equivalent*:

(a) J *is a pure ideal of* $(f(A) + J)$;

(b) $\{0\} \times J$ *is a pure ideal of* $A \bowtie^f J$;

(c) A *is a flat* $(A \bowtie^f J)$*-module.*

(2) *The following conditions are equivalent*:

(a) $f^{-1}(J)$ *is a pure ideal of* A;

(b) $f^{-1}\{J\} \times \{0\}$ *is a pure ideal of* $A \bowtie^f J$;

(c) $(f(A) + J)$ *is a flat* $(A \bowtie^f J)$*-module.*

Proof. (1) (a) $\Rightarrow$ (b) Assume that J is a pure ideal of $\big(f(A) + J\big)$ and let $(0, j) \in \{0\} \times J$. Then there exists $k \in J$ such that $(1 - k)j = 0$. So, $\big((1, 1) - (0, k)\big)(0, j) = (1, 1 - k)(0, j) = (0, (1 - k)j) = (0, 0)$.

(b) $\Rightarrow$ (a) Assume that $\{0\} \times J$ is a pure ideal of $A \bowtie^f J$ and let $j \in J$. Then there exists $k \in J$ such that $\big((1, 1) - (0, k)\big)(0, j) = (0, 0)$. So, $(1 - k)j = 0$.

(b) $\Leftrightarrow$ (c) Immediate from [11, Theorem 1.2.15] since $A \cong \dfrac{A \bowtie^f J}{\{0\} \times J}$.

(2) (a) $\Rightarrow$ (b) Assume that $f^{-1}(J)$ is a pure ideal of A and let $(x, 0) \in f^{-1}\{J\} \times \{0\}$. Then there exists $y \in f^{-1}(J)$ such that $(1 - y)x = 0$. So, $\big((1, 1) - (y, 0)\big)(x, 0) = (1 - y, 1)(x, 0) = ((1 - y)x, 0) = (0, 0)$.

(b) $\Rightarrow$ (a) Assume that $f^{-1}\{J\} \times \{0\}$ is a pure ideal of $A \bowtie^f J$ and let $x \in f^{-1}\{J\}$. Then there exists $y \in f^{-1}(J)$ such that $\big((1, 1) - (y, 0)\big)(x, 0) = (0, 0)$. Therefore, $(1 - y)x = 0$.

(b) $\Leftrightarrow$ (c) This follows from [11, Theorem 1.2.15] since $f(A) + J \cong \dfrac{A \bowtie^f J}{f^{-1}\{J\} \times \{0\}}$.

Lemma 2.2. *Let* $f : A \to B$ *be a ring homomorphism and* J *be an ideal of* B. *Assume that* $f^{-1}(J)$ (*resp.*, J) *is a pure ideal of* A (*resp.*, $(f(A) + J)$). *Let* M *be an* $(A \bowtie^f J)$*-module. Then*:

(1) $\operatorname{fd}_{A\bowtie^f J}(M) \leq n$ *if and only if* $\operatorname{fd}_A(M \otimes_{A\bowtie^f J} A) \leq n$ *and* $\operatorname{fd}_{(f(A)+J)}(M \otimes_{A\bowtie^f J} \big(f(A) + J)\big) \leq n$.

(2) $\operatorname{pd}_{A\bowtie^f J}(M) \leq n$ *if and only if* $\operatorname{pd}_A(M \otimes_{A\bowtie^f J} A) \leq n$ *and* $\operatorname{pd}_{(f(A)+J)}(M \otimes_{A\bowtie^f J} \big(f(A) + J)\big) \leq n$.

Proof. This follows from [12, Lemma 2.5] since $\phi\colon A \bowtie^f J \hookrightarrow A \times f(A) + J$ is an injective flat ring homomorphism, and $\{0\} \times J$ is a pure ideal of $A \bowtie^f J$ by Lemma 2.1.

***Proof of Theorem* 2.1.** (1) (a) Assume that $A \bowtie^f J$ is a strongly n-perfect ring and $\mathrm{fd}_A(A \bowtie^f J) = r < \infty$. Then A is a strongly $(n+r)$-perfect ring by [13, Theorem 3.1] since A is a module retract of $A \bowtie^f J$. If J is a flat A-module, then $A \bowtie^f J$ is a faithfully flat A-module. Therefore, A is a strongly n-perfect ring.

(b) If J is a pure ideal of $(f(A) + J)$, then, by Lemma 2.1, $\{0\} \times J$ is a pure ideal of $A \bowtie^f J$ and so $A \cong \dfrac{A \bowtie^f J}{\{0\} \times J}$ is a strongly n-perfect ring by [12, Corollary 2.2].

(2) Assume that $A \bowtie^f J$ is a strongly n-perfect ring and $f^{-1}(J)$ is a pure ideal of A. Then, by Lemma 2.1, $f^{-1}(J) \times \{0\}$ is a pure ideal of $A \bowtie^f J$, and so $f(A) + J \cong \dfrac{A \bowtie^f J}{f^{-1}(J) \times \{0\}}$ is a strongly n-perfect ring by [12, Corollary 2.2].

(3) Assume that $f^{-1}(J)$ (resp., J) is a pure ideal of A (resp., $\big(f(A) + J\big)$).

(a) If $A \bowtie^f J$ is a strongly n-perfect ring, then by assertions (1) and (2) above, A and $f(A) + J$ are strongly n-perfect rings. Conversely, assume that A and $f(A) + J$ are strongly n-perfect rings and let M be an $(A \bowtie^f J)$-module such that $\mathrm{fd}_{A\bowtie^f J}(M) \leq n$. Then $\mathrm{fd}_A(M \otimes_{A\bowtie^f J} A) \leq n$ and $\mathrm{fd}_{(f(A)+J)}(M \otimes_{A\bowtie^f J} \big(f(A) + J)\big) \leq n$ by Lemma 2.2. Thus, $\mathrm{pd}_A(M \otimes_{A\bowtie^f J} A) \leq n$ and $\mathrm{pd}_{(f(A)+J)}(M \otimes_{A\bowtie^f J} \big(f(A) + J)\big) \leq n$ since A and $f(A) + J$ are strongly n-perfect rings. Therefore, $\mathrm{pd}_{A\bowtie^f J}(M) \leq n$ by Lemma 2.2.

(b) $A \bowtie^f J$ is a strongly n-perfect ring and $(1, n)$-ring if and only if $A \bowtie^f J$ is an $(0, n)$-ring by [12, Theorem 2.7], which is equivalent to gldim($A \bowtie^f J$) $\leq n$ by [4, Theorem 1.3]. Also, it is equivalent to gldim(A) $\leq n$ and gldim($f(A) + J$) $\leq n$ by [14, Corollary 2.1].

The following corollaries are consequences of Theorem 2.1.

***Corollary* 2.1.** *Let $f : A \to B$ be a ring homomorphism and J be a proper ideal of B. Assume that either J is generated by idempotent element or $(f(A) + J)$ is a Von Neumann regular ring, and assume that either $f^{-1}(J)$ is generated by idempotent element or A is a Von Neumann regular ring. Then*:

(1) *$A \bowtie^f J$ is a strongly n-perfect ring if and only if A and $f(A)+J$ are strongly n-perfect rings.*

(2) *$A \bowtie^f J$ is an $(0, n)$-ring if and only if A and $(f(A) + J)$ are $(0, n)$-rings.*

In particular, if A and $(f(A) + J)$ are semisimple rings, then $A \bowtie^f J$ is a strongly n-perfect ring for any ideal J and $n \geq 0$.

Proof. Follows from Theorem 2.1 since ideals generated by an idempotent element and ideals of a Von Neumann regular ring are pure ideals.

The next corollary examines the case of the amalgamated duplication.

***Corollary* 2.2.** *Let A be a ring and I be a pure ideal of A. Then*:

(1) *$A \bowtie I$ is a strongly n-perfect ring if and only if so is A.*

(2) *$A \bowtie I$ is an $(0, n)$-ring if and only if A is an $(0, n)$-ring.*

In particular, if A is semisimple ring, then $A \bowtie I$ is a strongly n-perfect ring for any ideal I of A and $n \geq 0$.

Theorem 2.1 enriches the current literature with new original examples of strongly n-perfect rings.

***Example* 2.1.** *Let D be an integral domain such that $\operatorname{gl\,dim}(D) = n$, $K = qf(D)$ and $n \geq 2$. Consider the quotient ring* $S := \dfrac{K[X]}{(X^n - X)} = K + \bar{X}K[\bar{X}]$. *Set* $I := \bar{X}K[\bar{X}]$ *and* $R := D + I$. *Let $f : R \to R \times S$ be a ring homomorphism* (*given by* $f(x) := (x, 0)$). *Then*:

(1) *$R \bowtie I$ and $S \bowtie I$ are strongly n-perfect rings.*

(2) *$(R \times S) \bowtie (I \times I)$, $(R \times S) \bowtie (I \times \{0\})$, and $(R \times S) \bowtie (\{0\} \times I)$ are strongly n-perfect rings.*

(3) *$R \bowtie^f (I \times \{0\})$ is a strongly n-perfect ring.*

Proof. S and R are strongly n-perfect rings and I is a pure ideal of R by [12, Example 2.6]. So:

(1) $R \bowtie I$ and $S \bowtie I$ are strongly n-perfect rings by Corollary 2.2.

(2) $(R \times S) \bowtie (I \times I)$, $(R \times S) \bowtie (I \times \{0\})$, and $(R \times S) \bowtie (\{0\} \times I)$ are strongly n-perfect rings by Corollary 2.2 since $R \times S$ is a strongly n-perfect ring by [12, Theorem 2.16].

(3) $R \bowtie^f (I \times \{0\})$ is a strongly n-perfect ring by Theorem 2.1 since R and $f(R) + (I \times \{0\}) = R \times \{0\}$ are strongly n-perfect rings, and $I \times \{0\}$ and $f^{-1}(I \times \{0\}) = I$ are pure ideals of $R \times \{0\}$ and R, respectively.

***Example* 2.2.** *Let A be a Von Neumann regular ring such that* $\operatorname{gl\,dim}(A) \leq d$ (*see, for instance*, [4, *Example* 2.7]). *Let I and K two proper ideals of A such that $I \subset K$. Let $f : A \to B$ be a ring homomorphism*, $B := \dfrac{A}{I}$, *and* $J := \dfrac{K}{I}$. *Then*:

(1) *$A \bowtie I$, $A \bowtie K$, and $B \bowtie J$ are strongly d-perfect rings.*

(2) *$A \bowtie^f J$ is a strongly d-perfect ring.*

Theorem 2.2. *Let (A, M) be a local Noetherian regular ring of Krull dimension d. Then*:

(1) (a) *A is a strongly d-perfect ring.*

(b) *A_P is a strongly $\dim(A_P)$-perfect ring, for all $P \in \operatorname{Spec}(A)$.*

(2) *Let $f : A \to B$ be a ring homomorphism and J be a proper ideal of B such that $J \subseteq \operatorname{Rad}(B)$. Assume that at least one of the following conditions hold*:

(a) *f is a finite homomorphism*;

(b) *J is a finitely generated A-module and either $J \subseteq \operatorname{Nil}(B)$ or $\dim\big(f(A) + J)\big) \leq d$*;

(c) *$f(A) + J$ is Noetherian as A-module and either $J \subseteq \operatorname{Nil}(B)$ or $\dim\big(f(A) + J)\big) \leq d$.*

Then $A \bowtie^f J$ is a strongly d-perfect ring.

Proof. (1) (a) A is a local Noetherian regular ring, then $\operatorname{gl\,dim}(A) = \dim(A) = d$ by [2, Theorems 3.2 and Theorem 4.1]. Hence, A is a strongly d-perfect.

(b) A_P is a local Noetherian ring and it is a regular ring for all $P \in \operatorname{Spec}(A)$ by [2, Corollary 4.4]. So, A_P is a strongly $\dim(A_P)$-perfect ring by (1) (a).

(2) $A \bowtie^f J$ is a local ring by [1, Remark 2.1], $A \bowtie^f J$ is a Noetherian ring by [8, Proposition 5.7], and $A \bowtie^f J$ is a regular ring since $\dim(A \bowtie^f J) = \dim(A)$ by [9, Proposition 4.1], that is, the minimal number of generators of $M \bowtie^f J$. Therefore, $A \bowtie^f J$ is a strongly d-perfect ring by (1) (a).

***Example* 2.3.** *Let (A, M) be a principal local ring. Then A, A_M, and $A \bowtie M$ are strongly 1-perfect rings.*

Proof. Follows from Theorem 2.2 since A is a local Noetherian regular ring and $\dim(A) = 1$.

3. On n-perfect property. In this section, we investigate the transfer of n-perfect property in amalgamated algebra.

Theorem 3.1. *Let $f\colon A \to B$ be a ring homomorphism and J be a proper ideal of B. Assume that $f^{-1}(J)$ and J are pure ideals of A and $(f(A)+J)$, respectively. If A and $f(A)+J$ are n-perfect rings, then $A \bowtie^f J$ is an n-perfect ring.*

The proof of the previous theorem requires the following lemma.

Lemma 3.1. *Let $(A_i)_{i=1,\ldots,n}$ be a family of rings. Then $\prod_{i=1}^{n} A_i$ is an n-perfect ring if and only if A_i is an n-perfect ring for each $i = 1,\ldots,n$.*

Proof. By induction on n, it suffices to prove the assertion for $n = 2$. Let A_1 and A_2 be two rings such that $A_1 \times A_2$ is a n-perfect ring and let M_1 be a flat A_1-module and M_2 be a flat A_2-module. So $M_1 \times M_2$ is a flat $(A_1 \times A_2)$-module. Hence, $\operatorname{pd}_{A_1\times A_2}(M_1 \times M_2) \le n$ since $A_1 \times A_2$ is a n-perfect ring. So $\operatorname{pd}_{A_1}(M_1) \le n$ and $\operatorname{pd}_{A_2}(M_2) \le n$ since $\operatorname{pd}_{A_1\times A_2}(M_1 \times M_2) = \sup\{\operatorname{pd}_{A_1}(M_1), \operatorname{pd}_{A_2}(M_2)\}$ by [15, Lemma 2.5 (2)]. Therefore, A_1 and A_2 are n-perfect rings. Conversely, assume that A_1 and A_2 are n-perfect rings. Let $M_1 \times M_2$ be a flat $(A_1 \times A_2)$-module. Then M_1 is a flat A_1-module and M_2 is a flat A_2-module. Thus, $\operatorname{pd}_{A_1}(M_1) \le n$ and $\operatorname{pd}_{A_2}(M_2) \le n$ since A_1 and A_2 are n-perfect rings. Therefore $\operatorname{pd}_{A_1\times A_2}(M_1 \times M_2) \le n$ by [15, Lemma 2.5(2)] and so $A_1 \times A_2$ is a n-perfect ring.

Proof of Theorem **3.1.** Assume that $f^{-1}(J)$ and J are pure ideals of A and $(f(A)+J)$, respectively. Then A and $f(A)+J$ are flat $(A \bowtie^f J)$-modules by Lemma 2.1. So $\phi\colon A \bowtie^f J \hookrightarrow A \times f(A)+J$ is an injective flat ring homomorphism. Therefore, $A \bowtie^f J$ is a n-perfect ring by [12, Proposition 2.12] since $\dfrac{A \bowtie^f J}{\{0\} \times J} \cong A$ is a n-perfect ring and $A \times f(A)+J$ is a n-perfect ring by Lemma 3.1.

The following corollaries are immediate consequences of Theorems 2.1 and 3.1.

Corollary **3.1.** *Let $f\colon A \to B$ be a ring homomorphism and J be a proper ideal of B. Assume that $f^{-1}(J)$ and J are pure ideals of A and $(f(A)+J)$, respectively. If A and $f(A)+J$ are n-perfect rings and A or $f(A)+J$ is not a strongly n-perfect ring, then $A \bowtie^f J$ is an n-perfect ring that is not a strongly n-perfect ring.*

Corollary **3.2.** *Let A be a ring and I be a pure ideal of A. If A is an n-perfect ring and it is not a strongly n-perfect ring, then $A \bowtie I$ is an n-perfect ring that is not a strongly n-perfect ring.*

Example **3.1.** *Let A be Von Neumann regular hereditary ring that is not a semisimple ring* (*see, for example*, [4, *Example* 2.7]). *Let I be an ideal of A. Then $A \bowtie I$ is a strongly* 1*-perfect ring that is not a perfect ring.*

Proof. Follows from Corollary 3.2 since A is a strongly 1-perfect ring that is not a perfect ring by [12, Theorem 2.7].

4. On n-semiperfect property. Our first result of this section gives a characterization of n-semiperfect in the case $\operatorname{Rad}(R)$ is prime.

Proposition 4.1. *Let R be a ring such that* $\operatorname{Rad}(R)$ *is prime. Then R is n-semiperfect if and only if R is local with unique maximal ideal* $\operatorname{Rad}(R)$.

Proof. Assume that R is n-semiperfect. Then $\overline{R} = R/\operatorname{Rad}(R)$ is semisimple domain. So, $\overline{R}$ is Von Neumann integral domain. Therefore, $\overline{R}$ is a field. And so $\operatorname{Rad}(R)$ is a maximal ideal R. On the other hand, $\operatorname{Rad}(R) = \bigcap_{M_i\in \operatorname{Max}(R)} M_i$. Since $\operatorname{Rad}(R) = \bigcap_{M_i\in \operatorname{Max}(R)} M_i \subseteq M_i$ for every maximal ideal M_i and $\operatorname{Rad}(R)$ is a maximal ideal, then it follows that $\operatorname{Rad}(R) = M_i$. Hence, R is local with unique maximal ideal $\operatorname{Rad}(R)$. Conversely, assume that R is local with maximal ideal

$\operatorname{Rad}(R)$. Then $\overline{R} = R/\operatorname{Rad}(R)$ is a field and so is semisimple. It remains to show that n-potent lift modulo $\operatorname{Rad}(R)$. Let $x \in R$ such that $x - x^n \in \operatorname{Rad}(R)$. Two cases are then possible:

Case 1: $x \in \operatorname{Rad}(R)$. Then $0 - x \in \operatorname{Rad}(R)$ with $0^n = 0$ for every positive integer $n \geq 2$.

Case 2: $x \notin \operatorname{Rad}(R)$. Then x is a unit. We claim that $1 - x$ is not a unit. Deny. It follows that $x \in \operatorname{Rad}(R)$, which is a contradiction. So, $1 - x \in \operatorname{Rad}(R)$, with 1 an n-potent element for every $n \geq 2$.

Hence, in all cases, it follows that n-potents lift modulo $\operatorname{Rad}(R)$. Thus, R is n-semiperfect, as desired.

Our next result study the n-semiperfect ring property to homomorphic image.

Proposition 4.2. *Let R be a ring and I be an ideal of R such that $I \subseteq \operatorname{Rad}(R)$. If R is n-semiperfect, then R/I is n-semiperfect. The converse holds if n-potents lift modulo I.*

Proof. First observe that $\operatorname{Rad}(R/I) = \operatorname{Rad}(R)/I$ (as $I \subseteq \operatorname{Rad}(R)$). Assume that R is n-semiperfect. We need to show that $\overline{R/I} = (R/I)/\operatorname{Rad}(R/I)$ is semisimple and n-potents lift modulo $\operatorname{Rad}(R/I)$. We have $\overline{R/I} = (R/I)/\operatorname{Rad}(R/I) = (R/I)/(\operatorname{Rad}(R)/I) \simeq R/\operatorname{Rad}(R) = \overline{R}$. Since R is n-semiperfect, $\overline{R} = R/\operatorname{Rad}(R)$ is semisimple and therefore $\overline{R/I}$ is semisimple. Next, let $\bar{x} \in R/I$ such that $\bar{x} - \bar{x}^n \in \operatorname{Rad}(R/I) = \operatorname{Rad}(R)/I$. Then $\overline{x - x^n} \in \operatorname{Rad}(R)/I$ and so $(x - x^n) + I \in \operatorname{Rad}(R)/I$. Consequently, $x - x^n \in I$. From assumption, there exists an n-potent e in R such that $e - x \in I$ with $e^n = e$. And so $e^n + I = e + I$ and $e - x + I \in \operatorname{Rad}(R)/I$. Therefore, there exists an n-potent $\bar{e}$ in R/I such that $\overline{e - x} \in \operatorname{Rad}(R/I) = \operatorname{Rad}(R)/I$. Hence, R/I is n-semiperfect. Conversely, assume that R/I is n-semiperfect and n-potents lift modulo I. We claim that R is n-semiperfect. Since $(R/I)/\operatorname{Rad}(R/I) \simeq R/\operatorname{Rad}(R)$, then it follows that $\overline{R}$ is semisimple. Now, let $x \in R$ such that $x - x^n \in \operatorname{Rad}(R)$. Then $(x - x^n) + I \in \operatorname{Rad}(R)/I = \operatorname{Rad}(R/I)$. The fact that n-potents lift modulo $\operatorname{Rad}(R/I)$, then there exists an n-potent $\bar{e}$ in R/I such that $\overline{e - x} \in \operatorname{Rad}(R/I)$. So, $e - x + I \in \operatorname{Rad}(R/I) = \operatorname{Rad}(R)/I$, and therefore, $e - x \in I \subseteq \operatorname{Rad}(R)$. Since $\bar{e}$ is n-potent, then $e^n - e \in I$ which n-potent lift modulo I. And so there exists h n-potent in R such that $h - e \in I$ with $h^n = h$. On the other hand, $(h - e) + I \in R/I$. Then $h + I = (e + I) \in R/I$. So, $\bar{h} = \bar{e}$ with $h \in R$ such $h^n = h$. Consequently, $\overline{e - x} = \bar{e} - \bar{x} = \bar{h} - \bar{x} \in \operatorname{Rad}(R)/I$ and so $h - x + I \in \operatorname{Rad}(R)/I$ and, therefore, $h - x \in \operatorname{Rad}(R)$ with $h \in R$ such $h^n = h$. Hence, n-potent lift modulo $\operatorname{Rad}(R)$. Thus, R is n-semiperfect, as desired.

Now, we examine the stability of n-semiperfect rings under direct product. Observe that, for two rings A_1 and A_2, the Jacobson radical of the product $A_1 \times A_2$ is $\operatorname{Rad}(A_1 \times A_2) = \operatorname{Rad}(A_1) \times \operatorname{Rad}(A_2)$.

Proposition 4.3. $A = \prod_{i=1}^{n} A_i$ *is n-semiperfect ring if and only if so is A_i, $i = 1, 2, \ldots, n$.*

Proof. The proof is done by induction on n and it suffices to check it for $n = 2$. Assume that $A = A_1 \times A_2$ is n-semiperfect. Then $\overline{A_1 \times A_2} = (A_1 \times A_2)/(\operatorname{Rad}(A_1 \times A_2))$ is semisimple. Since $(A_1 \times A_2)/(\operatorname{Rad}(A_1 \times A_2)) = (A_1 \times A_2)/(\operatorname{Rad}(A_1) \times \operatorname{Rad}(A_2)) \simeq (A_1/\operatorname{Rad}(A_1)) \times \big(A_2/\operatorname{Rad}(A_2))\big)$ which is semisimple, then $A_1/\operatorname{Rad}(A_1) \simeq \dfrac{(A_1/\operatorname{Rad}(A_1)) \times (A_2/\operatorname{Rad}(A_2))}{0 \times \big(A_2/\operatorname{Rad}(A_2)\big)}$ is semisimple (as semisimple rings are stable under factor ring). Next, we prove that n-potents lift modulo $\operatorname{Rad}(A_1)$. Let $x_1 \in A_1$ such that $x_1 - x_1^n \in \operatorname{Rad}(A_1)$. Then $(x_1, 0) \in A_1 \times A_2$ and $(x_1 - x_1^n, 0) \in \operatorname{Rad}(A_1 \times A_2)$. Since n-potents lift modulo $\operatorname{Rad}(A_1 \times A_2)$, then there exists (e_1, e_2) n-potent in $A_1 \times A_2$ such that $(e_1, e_2) - (x_1, 0) \in \operatorname{Rad}(A_1 \times A_2)$. Therefore, there exists n-potent e_1 in A_1 such that $e_1 - x_1 \in A_1$. Hence, A_1 is n-semiperfect. Likewise, we show that A_2 is n-semiperfect. Conversely, assume that A_1 and A_2 are n-semiperfect rings. Then:

Claim 1: $\overline{A_1 \times A_2}$ is semisimple. Observe that $\overline{A_1 \times A_2} \simeq (A_1/\operatorname{Rad}(A_1)) \times \big(A_2/\operatorname{Rad}(A_2)\big)$. Since $\overline{A_1}$ and $\overline{A_2}$ are semisimple, then we claim that $\overline{A_1} \times \overline{A_2}$ is semisimple. Indeed, any ideal of $\overline{A_1} \times \overline{A_2}$ has the form $\overline{I_1} \times \overline{I_2}$ with $\overline{I_1}$ (resp., $\overline{I_2}$) is an ideal of $\overline{A_1}$ (resp., $\overline{A_2}$). Since $\overline{A_1}$ and $\overline{A_2}$ are semisimple, then $\overline{I_1}$ and $\overline{I_2}$ are both sum of submodules, and so it follows that $\overline{I_1} \times \overline{I_2}$ is a sum of submodules of $A_1 \times A_2$, making $\overline{A_1 \times A_2}$ is semisimple as module. Hence, $\overline{A_1 \times A_2}$ is semisimple.

Claim 2: n-potent lift modulo $\operatorname{Rad}(A_1 \times A_2)$. Let $(x_1, x_2) \in A_1 \times A_2$ such that $(x_1, x_2) - (x_1, x_2)^n \in \operatorname{Rad}(A_1 \times A_2)$. Then $(x_1 - x_1^n, x_2 - x_2^n) \in \operatorname{Rad}(A_1 \times A_2) = \operatorname{Rad}(A_1) \times \operatorname{Rad}(A_2)$. So, $x_1 - x_1^n \in \operatorname{Rad}(A_1)$ and $x_2 - x_2^n \in \operatorname{Rad}(A_2)$. Therefore, there exist e n-potent of A_1 and f n-potent of A_2 such that $e - x_1 \in \operatorname{Rad}(A_1)$ and $f - x_2 \in \operatorname{Rad}(A_2)$. Consequently, there exists (e, f) n-potent of $A_1 \times A_2$ such that $(e, f) - (x_1, x_2) \in \operatorname{Rad}(A_1 \times A_2)$. Hence, n-potent lift modulo $\operatorname{Rad}(A_1 \times A_2)$.

Finally, $A_1 \times A_2$ is a n-semiperfect ring, as desired.

Our next theorem studies the n-semiperfect ring property into amalgamated algebra.

Theorem 4.1. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Assume that $J \subseteq \operatorname{Rad}(B)$. Then $A \bowtie^f J$ is n-semiperfect if and only if so is A.*

The proof of the previous theorem requires the following lemma. For a ring A, we denote by $\operatorname{Max}(A)$, the set of all maximal ideals of A.

Lemma 4.1. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B such that $J \subseteq \operatorname{Rad}(B)$. Then $\operatorname{Rad}(A \bowtie^f J) = \operatorname{Rad}(A) \bowtie^f J$.*

Proof. Recall that from [9, Proposition 2.6], $\operatorname{Max}(A \bowtie^f J) = \big\{P \bowtie^f J / P \in \operatorname{Max}(A)\big\} \cup \big\{\overline{Q}^f / Q \in \operatorname{Max}(B) - V(J)\big\}$. Since $J \subseteq \operatorname{Rad}(B)$, then J is contained in every maximal ideal of B and therefore $\big\{\overline{Q}^f / Q \in \operatorname{Max}(B) - V(J)\big\}$ is an empty set. Consequently, $\operatorname{Max}(A \bowtie^f J) = \big\{P \bowtie^f J \,/ P \in \operatorname{Max}(A)\big\}$. Hence, $\operatorname{Rad}(A \bowtie^f J) = \cap_{P \in \operatorname{Max}(A)} P \bowtie^f J = (\cap_{P \in \operatorname{Max}(A)} P) \bowtie^f J = \operatorname{Rad}(A) \bowtie^f J$.

***Proof of Theorem* 4.1.** Assume that $J \subseteq \operatorname{Rad}(B)$. Then, by Lemma 4.1, $\operatorname{Rad}(A \bowtie^f J) = \operatorname{Rad}(A) \bowtie^f J$.

Suppose that $A \bowtie^f J$. Recall that from [8, Proposition 5.1(3)], $A \simeq \dfrac{A \bowtie^f J}{(\{0\} \times J)}$. Since the ideal $\{0\} \times J \subseteq \operatorname{Rad}(A) \bowtie^f J = \operatorname{Rad}(A \bowtie^f J)$, then, by Proposition 4.2, A is n-semiperfect. Conversely, assume that A is n-semiperfect. Then $A/\operatorname{Rad}(A)$ is semisimple. Since $A \bowtie^f J/\operatorname{Rad}(A \bowtie^f J) = A \bowtie^f J/\operatorname{Rad}(A) \bowtie^f J \simeq A/\operatorname{Rad}(A)$, then it follows that $A \bowtie^f J/\operatorname{Rad}(A \bowtie^f J)$ is semisimple. Next, let $(x, f(x) + j) \in A \bowtie^f J$ such that $(x, f(x) + j) - (x, f(x) + j)^n \in \operatorname{Rad}(A) \bowtie^f J$. Then $x - x^n \in \operatorname{Rad}(A)$ and so there exists an n-potent element e such that $e - x \in \operatorname{Rad}(A)$. So, $f(e - x) = f(e) - f(x)$. Therefore, $(e, f(e))$ is an n-potent element of $A \bowtie^f J$ and one can easily check that $(e, f(e)) - (x, f(x) + j) = (e - x, f(e - x) + j) \in \operatorname{Rad}(A) \bowtie^f J = \operatorname{Rad}(A \bowtie^f J)$. Hence, it follows that n-potents lift modulo $\operatorname{Rad}(A \bowtie^f J)$. Thus, $A \bowtie^f J$ is n-semiperfect, as desired.

For the special case of trivial ring extension, we have the following corollary.

***Corollary* 4.1.** *Let A be a ring, E be an A-module and $R := A \propto E$ be the trivial ring extension of A by E. Then R is n-semiperfect if and only if so is A.*

Proof. Consider $f : A \hookrightarrow B$ the injective ring homomorphism defined by $f(a) = (a, 0)$ for every $a \in A$, $J := 0 \propto E$ be an ideal of B. Clearly, $f^{-1}(J) = 0$. Therefore, by [8, Proposition 5.1 (3)], $f(A) + J = A \propto 0 + 0 \propto E = A \propto E = B \simeq A \bowtie^f J$. On the other hand, $J := 0 \propto E \subseteq \operatorname{Rad}(B)$ and so by application to Theorem 4.1, we have the desired result.

As an application of Theorem 4.1, we give a characterization for the power series ring to inherit the n-semiperfect ring property.

***Corollary* 4.2.** *Let R be a ring. Then $R[[X]]$ is n-semiperfect if and only if so is R.*

Proof. Take $A := R,\ B := R[[X]],\ f : A \hookrightarrow B$ be the canonical injection and $J := (X)$ is a maximal ideal of B. Observe that $f(A) + J = R + XR[[X]] = R[[X]]$ and $f(A) \cap J = (0)$ and so, by [8, Proposition 5.1(3)], $A \bowtie^f J \simeq f(A) + J = R[[X]]$. On the other hand, it is well-known that $\mathrm{Max}(B) = \{M + (X) \text{ such that } M \in \mathrm{Max}(A)\}$. Clearly, $J \subseteq \mathrm{Rad}(B)$. Hence, by application of Theorem 4.1, we obtain the desired result.

It is worthwhile noting that every semiperfect ring is 2-semiperfect. However, an n-semiperfect ring need not be a semiperfect ring. The next example illustrates Theorem 4.1 by providing new original classes of 3-semiperfect rings that are not semiperfect.

***Example* 4.1.** *Let B be a 3-semiperfect ring that is semilocal with two maximal ideals m_1 and m_2 (for instance take $B := \mathbb{Z}_6$). Clearly B is not semiperfect. Consider $A := B[[X]]$ the power series ring, $f : A \to B$ the canonical surjection and $J := \mathrm{Rad}(B) = m_1 \cap m_2$ is an ideal of B. Then:*

(1) *$A \bowtie^f J$ is 3-semiperfect;*

(2) *$A \bowtie^f J$ is not semiperfect.*

Proof. (1) By Corollary 4.2, A is 3-semiperfect as B is 3-semiperfect. By Theorem 4.1, $A \bowtie^f J$ is 3-semiperfect.

(2) $A \bowtie^f J$ is not semiperfect since $f(A) + J \simeq \dfrac{A \bowtie^f J}{f^{-1}(J) \times \{0\}} = B$ is not semiperfect.

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