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COMMUTATORS IN SPECIAL LINEAR GROUPS OVER CERTAIN DIVISION RINGS

КОМУТАТОРИ В СПЕЦІАЛЬНИХ ЛІНІЙНИХ ГРУПАХ НАД ДЕЯКИМИ КІЛЬЦЯМИ З ДІЛЕННЯМ

We consider the question whether an element of a special linear group $\mathrm{SL}_m(D)$ of degree $m \geq 1$ over a division ring D is a commutator. Our first aim is to show that if the division ring D is algebraically closed and finite-dimensional over its center, then every element of $\mathrm{SL}_m(D)$ is a commutator of $\mathrm{SL}_m(D)$. We also indicate that this question is related to the derived series in division rings and then describe the derived series in the Mal'cev – Neumann division rings of noncyclic free groups over fields.

Розглянуто питання, чи є елемент спеціальної лінійної групи $\mathrm{SL}_m(D)$ степеня $m \geq 1$ над кільцем з діленням D комутатором. Наша перша мета полягає в тому, щоб показати, що у випадку, коли кільце з діленням D є алгебраїчно замкненим і скінченновимірним над його центром, кожен елемент $\mathrm{SL}_m(D)$ є комутатором в $\mathrm{SL}_m(D)$. Також зазначено, що це питання пов'язане з похідним рядом у кільцях з діленням, описано похідний ряд у кільцях з діленням Мальцева – Ноймана нециклічних вільних груп над полями.

1. Introduction. Let G be a group. Let $G^{(1)} = G' = \langle aba^{-1}b^{-1} \mid a, b \in G \rangle$ be the derived subgroup of G. Inductively, for an integer $n > 1$, $G^{(n)}$ is defined to be the derived subgroup of $G^{(n-1)}$. The subgroup $G^{(n)}$ is called the *nth derived subgroup* of G. For a subgroup H of G, the element $a \in G$ is called a *commutator* of H if there exist $a_1, a_2 \in H$ such that $a = a_1 a_2 a_1^{-1} a_2^{-1}$.

The question concerning groups of which each element in the derived subgroups is a commutator or a product of commutators of certain subgroups has a long history with different types of groups (see, e.g., [3, 5, 8, 13 – 15, 17, 18]). We refer to [6, 19] for a survey of this topic. The case when G is the skew general linear group received more attention and is interesting. Let D be a division ring, m a positive integer, $\mathrm{GL}_m(D)$ the skew general linear group of degree m over D and $\mathrm{SL}_m(D) = \mathrm{GL}_m(D)'$ the skew special linear group of degree m over D. The following questions are inspired by Shoda [16] and Thompson [18].

Question 1.1. *For which division rings D, every element of $\mathrm{SL}_m(D)$ is a commutator of $\mathrm{GL}_m(D)$?*

More interestingly, the following question.

Question 1.2. *For which division rings D, every element of $\mathrm{SL}_m(D)$ is a commutator of $\mathrm{SL}_m(D)$?*

Thompson showed that for fields different from the field of two elements, the answer to Question 1.1 is positive [18, Theorems 1 and 2]. Regarding the class of fields for which the answer to Question 1.2 is positive, Shoda showed in [16] that it contains algebraically closed fields. Thompson then extended for fields in which the equation $x^2+y^2 = -1$ has a solution. Concerning to noncommutative cases, in [7], Kursov showed that for the real quaternion division ring, the answer to Question 1.2 is

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positive. Observe that in [7] there is a mistake in the proof of Lemma 2 and D. Z. Dokovic presented a new proof in [2]. It should be remarked that in case $m = 1$, the class of noncommutative division rings for which the answer to Question 1.1 is positive includes finite-dimensional division rings whose centers are p-adic number fields [12].

The first aim of this paper is to show that the answer to Question 1.2 is positive for division rings which are finite-dimensional over its center and algebraically closed. Recall that a division ring D is called *algebraically closed* if for every nonconstant polynomial $f(t) \in D[t]$, there exists $a \in D$ such that $f(a) = 0$. Trivially, algebraically closed fields are algebraically closed division rings. It is well-known that the real quaternion division ring is noncommutative algebraically closed. For a description of algebraically closed division ring, we refer [9, p. 255 – 256].

Question 1.2 has a connection to the derived series of groups. To see this, let G be a group and assume that each element of G' is a commutator of G'. Then, for every a of G', there exist $a_1, a_2 \in G'$ such that $a = a_1 a_2 a_1^{-1} a_2^{-1}$. Hence, $a \in G^{(2)}$ which implies that $G^{(2)} = G'$, that is, the derived series $G' \supseteq G^{(2)} \supseteq G^{(3)} \supseteq \ldots$ stops at G'. Going back to Question 1.2, we need to find division rings D such that the derived series of $\mathrm{GL}_m(D)$ stops at $\mathrm{GL}_m(D)' = \mathrm{SL}_m(D)$. It is well-known that if $m > 1$ and D contains at least 4 elements, then $\mathrm{SL}_m(D)' = \mathrm{SL}_m(D)$. Hence, to describe the derived series of $\mathrm{GL}_m(D)$, we focus on the case $m = 1$, that is, the derived series of the unit group $D^* = D\backslash\{0\}$ of D. The following question is essentially driven by M. Mahdavi-Hezavehi [11, p. 82] (Problem 16).

Question 1.3. *For which division rings D, $D^{(2)} = D'$?*

Hence, to describe the derived series of the unit groups of division rings is itself an interesting topic. The second aim of this paper is to describe the derived series of the unit groups of the Mal'cev – Neumann division rings over fields of noncyclic free groups.

This paper is organized as follows. In Section 2, we present the first main result by showing that if D is an algebraically closed division ring which is finite-dimensional over its center, then every element in $\mathrm{SL}_m(D)$ is a commutator of $\mathrm{SL}_m(D)$ for all positive integers m. For a smooth presentation, the proof will be separated in three cases: $m = 1$, $m > 2$ and $m = 2$. In Section 3, we present a description of the derived subgroups in Mal'cev – Neumann division rings of noncyclic free groups over fields.

Our notations in this paper are standard but it is always not redundant if ambiguous notations should be explained clearly. For a division ring D, the general linear group of degree $m \geq 1$ over D is denoted by $\mathrm{GL}_m(D)$ as usual. In this paper, the notation $\mathrm{SL}_m(D)$ is for the special linear group of degree m over D, that is, $\mathrm{SL}_m(D) = \mathrm{GL}_m(D)'$. Remark that in some papers, this special linear group is denoted by $E_m(D)$ (and $\mathrm{SL}_m(D)$ for the group of matrices in $\mathrm{GL}_m(D)$ of reduced norm 1). For convenience, if D is a division ring, we write $D^{(r)}$ and D' for the rth derived subgroup $(D^*)^{(r)}$ and the first derived subgroup $(D^*)'$, respectively.

In this paper, a finite-dimensional division ring is a division ring which is finite-dimensional over its center.

2. Special linear groups over algebraically closed division rings. The aim of this section is to show that for algebraically closed finite-dimensional division rings, the answer to Question 1.2 is positive.

Theorem 2.1. *Let D be a finite-dimensional division ring and m be a positive integer. If D is algebraically closed, then each element in* $\mathrm{SL}_m(D)$ *is a commutator of* $\mathrm{SL}_m(D)$.

We begin with the case $m = 1$. To do this, we firstly present a class of division rings D for which the answer to Question 1.3 is positive, that is, $D^{(2)} = D'$.

A (multiplicative) group G is called *radicable* if for every $a \in G$ and a positive integer n, there exists an element $b \in G$ such that $b^n = a$. A division ring is called *radicable* if the unit group D^* of D is radicable (see [10]). Hence, it is trivial that algebraically closed division rings are radicable. Radical division rings have many interesting results. We refer to [10] for an investigation of radicable division rings.

We borrow the following lemma which is from [11].

Lemma 2.1 [11, Lemma 4.3]**.** *Let D be a division ring with center F. If $\dim_F D = m^2$, then, for every $a \in D^*$, there exist $\alpha \in F^*$ and $d \in D'$ such that $a^m = \alpha d$.*

Lemma 2.2. *If D is a division ring with center F such that $\dim_F D = 4$, then $F \cap D' = \{\pm 1\}$.*

Proof. Let $a \in F \cap D'$ and $N_{F(a)/F}$ the norm of $F(a)$ to F. Since $a \in D'$, $N_{F(a)/F}(a) = 1$. Moreover, as $a \in F$, $N_{F(a)/F}(a) = a^2$. Hence, $a^2 = 1$ which implies that either $a = 1$ or $a = -1$. Thus, $F \cap D' = \{\pm 1\}$.

Proposition 2.1. *Let D be a finite-dimensional division ring. If D is radicable, then $D^{(2)} = D'$. Moreover, if an element of D' is a commutator of D^*, then it is a commutator of D'.*

Proof. To prove $D^{(2)} = D'$, it suffices to show $D' \subseteq D^{(2)}$. Assume that $\dim_F D = m^2$. Let $d = xyx^{-1}y^{-1} \in D'$. Since D is radicable, there exist $a, b \in D^*$ such that $a^m = x, b^m = y$. By Lemma 2.1, $a^m = \alpha_a d_a$ and $b^m = \alpha_b d_b$, where $\alpha_a, \alpha_b \in F$ and $d_a, d_b \in D'$. Hence,

$$d = xyx^{-1}y^{-1} = \alpha_a d_a \alpha_b d_b (\alpha_a d_a)^{-1} (\alpha_b d_b)^{-1} = d_a d_b d_a^{-1} d_b^{-1} \in D^{(2)}.$$

Therefore, $d = xyx^{-1}y^{-1}$ belongs to $D^{(2)}$. Thus, $D' \subseteq D^{(2)}$. The rest is also implies from the above arguments.

Proposition 2.1 is proved.

Recall that a field F is *real-closed* if F is not algebraically closed but the field extension $F(\sqrt{-1})$ is algebraically closed. A division ring D with center F is called an *ordinary quaternion division ring* if D has the form

$$D = \{a + bi + cj + dk \mid a, b, c, d \in F, i^2 = j^2 = k^2 = -1, ij = -ji = k\}.$$

To show the next corollary, we need the following result which is known as Baer's theorem for algebraically closed finite-dimensional division rings (see, e.g., [9, p. 255 – 256]).

Lemma 2.3. *Let D be a finite-dimensional division ring. If D is a algebraically closed, then one of the following assertions satisfies*:

(1) *D is a algebraically closed field.*

(2) *The center F of D is a real-closed field and D is the ordinary quaternion division ring.*

The following corollary is Theorem 2.1 with $m = 1$.

***Corollary* 2.1.** *Let D be a finite-dimensional division ring. If D is algebraically closed, then every element in D' is a commutator of D'.*

Proof. Assume that D is an algebraically closed division ring. Since algebraically closed division rings are radicable, by Proposition 2.1, it suffices to show that every element of D' is a commutator of D^*. According to Lemma 2.3, it suffices to consider the case when F is real-closed and

$$D = \{a + bi + cj + dk \mid a, b, c, d \in F, i^2 = j^2 = k^2 = -1, ij = -ji = k\}.$$

Let $\alpha \in D'$. If $\alpha \in F$, then either $\alpha = 1$ or $\alpha = -1$ by Lemma 2.2. Observe that $-1 = iji^{-1}j^{-1}$ and $1 = 1.1.1^{-1}1^{-1}$, so in this case, the proof is complete. Now assume that $\alpha \notin F$. Then $[F(\alpha) : F] = 2$. Since $F(\alpha)/F$ is a cyclic extension, by Hilbert's theorem 90, there exists $\alpha_2 \in F(\alpha)\backslash F$ such that $\alpha = \sigma(\alpha_2)\alpha_2^{-1}$, where σ is the nonidentity autormorphism of $F(\alpha)$ over F. By the Skolem–Noether theorem, $F(\alpha)$ contains an element α_1 such that $\sigma(\alpha_2) = \alpha_1\alpha_2\alpha_1^{-1}$. Therefore, $\alpha = \sigma(\alpha_2)\alpha_2^{-1} = \alpha_1\alpha_2\alpha_1^{-1}\alpha_2^{-1}$.

Corollary 2.1 is proved.

Now we move to the case $m > 2$. In this case, our arguments depend heavily on the following two lemmas.

Lemma 2.4. *Let F be a field containing at least 4 elements and m be a positive integer.*

(1) *If $m - 2$ is not a multiple of 4, then every element in $\mathrm{SL}_m(F)$ is a commutator of $\mathrm{SL}_m(F)$.*

(2) *If F contains an element x such that $x^2 = -1$, then every element in $\mathrm{SL}_m(F)$ is a commutator of $\mathrm{SL}_m(F)$.*

Proof. This lemma is from [18, Theorems 1 and 2].

Lemma 2.5. *Let D be a finite-dimensional division ring and $m > 2$. If every element of D' is a commutator, then every noncentral element of $\mathrm{SL}_m(D)$ is a commutator of $\mathrm{SL}_m(D)$.*

Proof. This lemma is just a corollary of [4, Theorem 1].

We are ready to show Theorem 2.1 for the case $m > 2$.

Theorem 2.2. *Let D be a finite-dimensional division ring and $m > 2$. If D is algebraically closed, then every element of $\mathrm{SL}_m(D)$ is a commutator of $\mathrm{SL}_m(D)$.*

Proof. Let $A \in \mathrm{SL}_m(D)$. We show that there exist $A_1, A_2 \in \mathrm{SL}_m(D)$ such that $A = A_1A_2A_1^{-1}A_2^{-1}$. Let F be the center of D. By Lemma 2.3, we consider two cases:

Case 1: $D = F$ is an algebraically closed field. It is trivial that the equation $x^2 = -1$ has a solution in D, so by Lemma 2.4, there exist $A_1, A_2 \in \mathrm{SL}_n(F)$ such that $A = A_1A_2A_1^{-1}A_2^{-1}$. The theorem is shown in this case.

Case 2: F is real-closed and D is the ordinary quaternion division ring. Assume that

$$D = \{a + bi + cj + dk \mid a, b, c, d \in F, i^2 = j^2 = k^2 = -1, ij = -ji = k\}.$$

By Corollary 2.1, every element of D' is a commutator of D', so if A is noncentral, then there exist $A_1, A_2 \in \mathrm{SL}_m(D)$ such that $A = A_1A_2A_1^{-1}A_2^{-1}$ according to Lemma 2.5. Now assume that A is central. Then $A = \lambda \mathrm{I}_m$ where $\lambda \in F$. Since $A \in \mathrm{SL}_m(D)$, $\lambda^m \in D'$. By Lemma 2.2, $\lambda^m \in \{1, -1\}$. If $\lambda^m = 1$, then $A = \lambda I_m \in \mathrm{SL}_m(F(i))$. By Lemma 2.4 (2), there exist $A_1, A_2 \in \mathrm{SL}_m(F(i)) \subseteq \mathrm{SL}_m(D)$ such that $A = A_1A_2A_1^{-1}A_2^{-1}$. If $\lambda^m = -1$ and $m = 2m_1$ is even, then $(\lambda^{m_1})^2 = -1$ which contradicts to the fact that F is real-closed. Hence, $\lambda^m = -1$ and m is odd. Therefore, $A = \lambda I_m = (-I_m)(-\lambda)I_m$. Regarding the matrix $(-\lambda I_m)$, one has $(-\lambda)^m = (-1)^m\lambda^m = (-1)(-1) = 1$, so $-\lambda I_m = B_1B_2B_1^{-}B_2^{-1}$ where $B_1, B_2 \in \mathrm{SL}_m(F)$ (Lemma 2.4 (1)). Observe both $i\mathrm{I}_m$ and $j\mathrm{I}_m$ commute to all matrices in $\mathrm{GL}_m(F)$, so

$$(iB_1)(jB_2)(iB_1)^{-1}(jB_2)^{-1} = (iI_m)(jI_m)(iI_m)^{-1}(jI_m)^{-1}B_1B_2B_1^{-1}B_2^{-1} = (-I_m)(-\lambda I_m) = A.$$

Put $A_1 = iB_1$ and $A_2 = jB_2$. Then $A_1, A_2 \in \mathrm{SL}_m(D)$ and $A = A_1A_2A_1^{-1}A_2^{-1}$.

Theorem 2.2 is proved.

Now we focus on the case $m = 2$. Assume that a is an element of G which is a *commutator* of H, that is, $a = a_1a_2a_1^{-1}a_2^{-1}$ for some $a_1, a_2 \in H$. Then, for every $c \in G$, it is clear that $cac^{-1} = (ca_1c^{-1})(ca_2c^{-1})(ca_1c^{-1})^{-1}(ca_2c^{-1})^{-1}$. Hence, we have following lemmas.

Lemma 2.6. *If G is a group and H is a normal subgroup of G, then an element of G is a commutator of H if and only if so are all its conjugates.*

Lemma 2.7. *Let D be a finite-dimensional division ring and $A_1 = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 \\ a_2 & 1 \end{pmatrix}$ two matrices in $\mathrm{GL}_2(D)$. If $h \neq 1$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}$, then there exist $B_1 = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & 0 \\ b_2 & 1 \end{pmatrix} \in \mathrm{GL}_2(D)$ such that $A_1 = H^{-1}B_1^{-1}HB_1$ and $A_2 = B_2HB_2{}^{-1}H^{-1}$.*

Proof. This lemma is just a corollary of [4, Proposition 4.1].

Lemma 2.8. *Let D be a finite-dimensional division ring. For every noncentral matrix $A \in \mathrm{GL}_2(D)$, there exists $B \in \mathrm{GL}_2(D)$ such that BAB^{-1} has the form*

$$BAB^{-1} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

Additionally, if $A \in \mathrm{SL}_2(D)$, then $b \in D'$.

Proof. This lemma is a special case of [4, Theorem 2.1].

It is time to show Theorem 2.1 for the case $m = 2$.

Theorem 2.3. *Let D be a finite-dimensional division ring. If D is algebraically closed, then every element of $\mathrm{SL}_2(D)$ is a commutator of $\mathrm{SL}_2(D)$.*

Proof. For every element $A \in \mathrm{SL}_2(D)$, we must show that there exist $A_1, A_2 \in \mathrm{SL}_2(D)$ such that $A = A_1A_2A_1^{-1}A_2^{-1}$. By Lemma 2.3, either $D = F$ is an algebraically closed field or F is real-closed and D is the ordinary quaternion division ring. In the first case, by Lemma 2.4, there exist $A_1, A_2 \in \mathrm{SL}_2(F)$ such that $A = A_1A_2A_1^{-1}A_2^{-1}$. Now we consider the latter case, that is, F is real-closed and

$$D = \{a + bi + cj + dk \mid a, b, c, d \in F, i^2 = j^2 = k^2 = -1, ij = -ji = k\}.$$

Case 1: A is central, that is, $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ with $a \in F$ and $a^2 \in D'$. Then $a^2 \in F \cap D'$. By Lemma 2.2, either $a^2 = 1$ or $a^2 = -1$. Observe that F is real-closed, so $a^2 \neq -1$. As a corollary, $a^2 = 1$ which implies that $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathrm{SL}_2(F) \subseteq \mathrm{SL}_2(F(i))$. Hence, by Lemma 2.4 (2), there exist $A_1, A_2 \in \mathrm{SL}_2(F(i))$ such that $A = A_1A_2A_1^{-1}A_2^{-1}$.

Case 2: A is noncentral. By Lemmas 2.6 and 2.8, it suffices to consider A of the form $A = UHV$, where $U = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$, $V = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}$ for some $u, v \in D$ and $h \in D'$. By Corollary 2.1, $h = sts^{-1}t^{-1}$ for some $s, t \in D'$. If $h = 1$, then $h = sss^{-1}s^{-1}$ for every $s \in D'$, so without loss of generality, we assume that $s \neq 1$. Put $H_1 = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}$ and $H_2 = \begin{pmatrix} 1 & 0 \\ 0 & ts^{-1}t^{-1} \end{pmatrix}$. Then $H = H_1H_2$. According to Lemma 2.7, there exist $U_1 = \begin{pmatrix} 1 & 0 \\ u_1 & 1 \end{pmatrix}$ and $V_1 = \begin{pmatrix} 1 & v_1 \\ 0 & 1 \end{pmatrix}$ such that $U = U_1H_1U_1^{-1}H_1^{-1}$ and $V = H_2^{-1}V_1^{-1}H_2V_1$. Hence,

$$A = UHV = U_1H_1U_1^{-1}H_1^{-1}H_1H_2H_2^{-1}V_1^{-1}H_2V_1 = U_1H_1U_1^{-1}V_1^{-1}H_2V_1.$$

If $T = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$, $A_1 = U_1 H_1 U_1^{-1}$ and $A_2 = V_1^{-1} T U_1^{-1}$, then we may check that $A = A_1 A_2 A_1^{-1} A_2^{-1}$. Moreover, since $U_1, H_1, T, V_1 \in \mathrm{SL}_m(D)$, one has $A_1, A_2 \in \mathrm{SL}_m(D)$.

Theorem 2.3 is proved.

***Proof of Theorem* 2.1.** The theorem follows from Corollary 2.1, Theorems 2.2 and 2.3.

3. Derived series of Mal'cev – Neumann division rings of free groups. The aim of this section is to describe the derived subgroups of Mal'cev – Neumann division rings of noncyclic free groups over fields. As a corollary, it is shown that if D is the Mal'cev – Neumann division ring of a noncyclic free group over a field, then the derived series

$$D' \supsetneqq D^{(2)} \supsetneqq \cdots$$

never stops. In particular, for the class of these division rings, the answer to Question 1.3 is negative.

We recall briefly the definition of Mal'cev – Neumann division rings of totally ordered groups. Recall that a group G with total order $\preceq$ is called a *totally ordered group* if $a \preceq b$, then $ca \preceq cb$ and $ac \preceq bc$ for every $a, b, c \in G$. Assume that S is a subset of the totally ordered group G. We say that S is *well-ordered* (or *WO* for short) if every nonempty subset of S has a least element. We denote $\min(S)$ the least element of a WO subset S.

Let F be a field and G a totally ordered group. For a formal sum $\alpha = \sum_{g\in G} a_g g$, where $a_g \in F$, put $\mathrm{supp}(\alpha) = \{g \in G \mid a_g \neq 0\}$ and call it the *support* of α. Put

$$F((G)) = \left\{ \alpha = \sum_{g\in G} a_g g \mid \mathrm{supp}(\alpha) \text{ is WO} \right\}.$$

For every $\alpha = \sum_{g\in G} a_g g$, $\beta = \sum_{g\in G} b_g g \in F((G))$, we define

$$\alpha + \beta = \sum_{g\in G} (a_g + b_g) g$$

and

$$\alpha\beta = \sum_{t\in G} \left(\sum_{gh=t} a_g b_h \right) t.$$

The above operators are well-defined [9] and moreover $F((G))$ is a division ring called the *Mal'cev – Neumann division ring* of G over F [9, Theorem 14.21].

Lemma 3.1. *Let F be a field, G be a totally ordered group and $F((G))$ be the Mal'cev – Neumann division ring of G over F. If $F^* \cdot G$ denotes $\{\alpha g \mid \alpha \in F^*, g \in G\}$, then the map*

$$\sigma\colon F((G))^* \to F^* \cdot G, \quad \alpha = \sum_{g\in G} \alpha_g g \mapsto \alpha_{\min(\mathrm{supp}(\alpha))} \min(\mathrm{supp}(\alpha))$$

is a surjective group morphism.

Proof. First note that $F^* \cdot G$ is a subgroup of the multiplicative group $F((G))^*$. If $\alpha \in F((G))^*$ then $\mathrm{supp}(\alpha)$ is a nonempty subset of G and $\min(\mathrm{supp}(\alpha))$ exists. Thus σ is well-defined. Moreover, σ is trivially surjective since the restriction of σ to $F^* \cdot G$ is the identity map. To conclude our arguments, we now show that σ preserves the multiplication in $F((G))^*$. Indeed, if $\alpha = \sum_{g\in G} \alpha_g g$ and $\beta = \sum_{g\in G} \beta_g g$ from $F((G))^*$ with $x = \min(\text{supp}(\alpha))$ and $y = \min(\text{supp}(\beta))$, then $\sigma(\alpha) = \alpha_x x$ and $\sigma(\beta) = \beta_y y$. Recall that $\alpha\beta$ can be written in the form

$$\alpha\beta = \sum_{u\in G}\left(\sum_{(g,h)\in S_u} \alpha_g\beta_h\right)u,$$

where $S_u = \{(g,h) \in \text{supp}(\alpha) \times \text{supp}(\beta) : gh = u\}$. For every $(g,h) \in \text{supp}(\alpha) \times \text{supp}(\beta) \setminus (x,y)$, we have $xy < gh$. Therefore, $xy = \min(\text{supp}(\alpha\beta))$ and $S_{xy} = \{(x,y)\}$, which clearly imply the following:

$$\sigma(\alpha\beta) = \left(\sum_{(g,h)\in S_{xy}} \alpha_g\beta_h\right)(xy) = (\alpha_x\beta_y)(xy) = (\alpha_x x)(\beta_y y) = \sigma(\alpha)\sigma(\beta).$$

Lemma 3.1 is proved.

Let G be a free group. It is known that G always has the lexicographic order which is defined by the Magnus automorphism. With this order, G is a totally ordered group.

Lemma 3.2. *Let G be a free group with lexicographic order and F be a field. For every $\alpha \in F((G))$ with $y = \min(\text{supp}(\alpha)) > 1$, there exists $\beta \in F((G))^*$ such that $\beta\alpha\beta^{-1} = \sum_{i=n}^{\infty} a_i y^i$, where $n \in \mathbb{Z}$ and $a_i \in F$ for every $i \geq n$.*

Proof. This follows from [1, Lemma 4.2].

Now we show the main result of this section.

Theorem 3.1. *Let G be a noncyclic free group, F be a field and $D = F((G))$ be the Mal'cev–Neumann division ring of G over F. Then*:

(1) *If $\sigma : D \to G \cdot F^*$ is the map defined in Lemma* 3.1, *then $\sigma\big(D^{(r)}\big) = G^{(r)}$ for every positive integer r.*

(2) *For every positive integer r, $D^{(r)}$ is the normal closure in D^* of the set*

$$\left\{\alpha = 1 + \sum_{i=1}^{\infty} a_i y^i : \alpha \in D^{(r)}, y > 1\right\} \cup G^{(r)}.$$

In particular, $D' \supsetneq D^{(2)} \supsetneq D^{(3)} \supsetneq \cdots$ is a strictly descending sequence.

Proof. 1. We shall show by induction on r that $\sigma\big(D^{(r)}\big) \subseteq G^{(r)}$. For $r = 1$, consider the generator $\alpha\beta\alpha^{-1}\beta^{-1}$ of D' where $\alpha, \beta \in D^*$ and put $x = \min(\text{supp}(\alpha))$, $y = \min(\text{supp}(\beta))$. Then

$$\sigma\big(\alpha\beta\alpha^{-1}\beta^{-1}\big) = \sigma(\alpha)\sigma(\beta)(\sigma(\alpha))^{-1}(\sigma(\beta))^{-1} = xyx^{-1}y^{-1} \in G^{(1)}.$$

This means that $\sigma(D') \subseteq G^{(1)}$. For $r \geq 2$, let $\alpha\beta\alpha^{-1}\beta^{-1}$ be a generator of $D^{(r)}$, where $\alpha, \beta \in D^{(r-1)}$. Using the inductive hypothesis, we have $\sigma(\alpha), \sigma(\beta) \in G^{(r-1)}$, which implies that

$$\sigma\big(\alpha\beta\alpha^{-1}\beta^{-1}\big) = \min(\text{supp}(\alpha))\sigma(\beta)(\min(\text{supp}(\alpha)))^{-1}(\sigma(\beta))^{-1} \in G^{(r)}.$$

Therefore, $\sigma\big(D^{(r)}\big) \subseteq G^{(r)}$. Moreover, $G^{(r)} \subseteq D^{(r)}$ yields

$$G^{(r)} = \sigma\Big(G^{(r)}\Big) \subseteq \sigma\Big(D^{(r)}\Big) \subseteq G^{(r)}.$$

Hence, $\sigma\big(D^{(r)}\big) = G^{(r)}$ for all positive integer r.

2. Fix a positive integer r and put $S_r = \Big\{\alpha = 1 + \sum_{i=1}^{\infty} a_i y^i : \alpha \in D^{(r)}, y > 1\Big\}$. Let N be the normal closure in D^* of $S_r \cup G^{(r)}$. It is obvious that $D^{(r)}$ is a normal subgroup of D^* containing $S_r \cup G^{(r)}$, so $N \subseteq D^{(r)}$. To show the reverse inclusion, we may assume that α is an element of $D^{(r)}$. Let $y = \min(\text{supp}(\alpha))$.

Case 1: $y > 1$. By Lemma 3.2, there exists $\beta \in D^*$ such that $\beta\alpha\beta^{-1} = \sum_{i=n}^{\infty} a_i y^i$, where $n \in \mathbb{Z}$ and $a_i \in F$ for every $i > n$, $a_n \in F^*$. Then

$$\sigma\left(\beta^{-1}\left(\sum_{i=n}^{\infty} a_i y^i\right)\beta\right) = \sigma(\alpha) \in \sigma(D^{(r)}) = G^{(r)}$$

implies that

$$\min(\text{supp}(\beta))^{-1} a_n y^n \min(\text{supp}(\beta)) \in G^{(r)}.$$

Since $G^{(r)}$ is normal in G, we have $a_n y^n \in \min(\text{supp}(\beta)) G^{(r)} \min(\text{supp}(\beta))^{-1} = G^{(r)}$, so $a_n = 1$ and $y^n \in G^{(r)}$. Therefore,

$$\big(\beta\alpha\beta^{-1}\big) y^{-n} = 1 + \sum_{i=1}^{\infty} a'_i y^i \in D^{(r)},$$

where $a'_i \in F$ for every $i \geq 1$. This shows that $\big(\beta\alpha\beta^{-1}\big)y^{-n}$ is contained in S_r and therefore it is contained in N. Hence,

$$\alpha \in \beta^{-1}(N y^n)\beta \subseteq \beta^{-1}(N \cdot G^{(r)})\beta \subseteq \beta^{-1} N \beta = N.$$

Case 2: $y < 1$. Then $\min\big(\text{supp}(\alpha^{-1})\big) > 1$. Repeating the arguments in the proof of Case 1 for α^{-1}, one has $\alpha^{-1} \in N$, which also deduces that $\alpha \in N$.

Case 3: $y = 1$. Then $\alpha g \in D^{(r)}$ and $\min(\text{supp}(\alpha g)) \neq 1$, where $g \in G^{(r)} \backslash \{1\}$. According to the above two cases, $\alpha g \in N$, so $\alpha = (\alpha g) g^{-1} \in N$.

The three cases considered above prove that $N = D^{(r)}$, as desired.

Now by the fact that G is a noncyclic free group, one has $G^{(r)} \supsetneqq G^{(r+1)}$, and we can easily deduce that $D' \supsetneqq D^{(2)} \supsetneqq D^{(3)} \supsetneqq \cdots$ is a strictly descending sequence.

Theorem 3.1 is proved.

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