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ON MEAN CARTAN TORSION OF FINSLER METRICS

ПРО КАРТАНОВЕ КРУЧЕННЯ МЕТРИК ФІНСЛЕРА

We prove that Finsler manifolds with unbounded mean Cartan torsion cannot be isometrically imbedded into any Minkowski space. We also study the generalized Randers metrics obtained by the Rizza structure and show that any generalized Randers metric has an unbounded mean Cartan torsion. Then generalized Randers metrics cannot be isometrically imbedded into any Minkowski space. Further, we prove that every generalized Randers metric is quasi-C-reducible. Finally, we show that every generalized Randers metric on 2-dimensional Finsler manifold has a vanishing mean Cartan torsion.

Доведено, що фінслерові многовиди з необмеженим середнім крученням Картана неможливо ізометрично вкласти в будь-який простір Мінковського. Крім того, вивчаються узагальнені метрики Рандерса, отримані за структурою Ріцца, і показано, що будь-яка узагальнена метрика Рандерса має необмежене середнє картанове кручення. Тоді узагальнені метрики Рандерса неможливо ізометрично вкласти в будь-який простір Мінковського. Також доведено, що будь-яка узагальнена метрика Рандерса є квазі-C-звідною. Насамкінець показано, що будь-яка узагальнена метрика Рандерса на 2-вимірному многовиді Фінслера має середнє картанове кручення, що зникає.

1. Introduction. There are several important and interesting non-Riemannian quantities in Finsler geometry. Let (M,F) be a Finsler manifold. The second and third order derivatives of $\dfrac{1}{2}F_x^2$ at $y\in T_xM_0$ are inner products $\mathbf{g}_y$ and symmetric trilinear forms $\mathbf{C}_y$ on $T_xM,$ respectively. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively. The Cartan torsion is one of the most important non-Riemannian quantity in Finsler geometry and it was first introduced by Finsler [8] and emphased by Cartan [5]. A Finsler metric reduces to a Riemannian metric if and only if it has a vanishing Cartan torsion. In [26], Tayebi and Sadeghi found a relation between the norm of Cartan and mean Cartan torsions of Finsler metrics defined by a Riemannian metric and a 1-form on a manifold. They obtained a subclass of these metrics which have bounded Cartan torsion. It turns out that every C-reducible Finsler metric has a bounded Cartan torsion.

Taking a trace of Cartan torsion yields the mean Cartan torsion $\mathbf{I}=\mathrm{trace}(\mathbf{C})$. In [6], Deicke proved that every Finsler metric F is Riemannian if and only if its mean Cartan torsion is vanishes, provided that the Finsler metric is positive definite. Here, we prove that a Finsler manifold with unbounded mean Cartan torsion cannot be isometrically imbedded into any Minkowski space. Thus the norm of mean Cartan torsion plays an important role for studying of immersion theory in Finsler geometry.

One of the open problems in Finsler geometry is whether or not every Finsler manifold can be isometrically immersed into a Minkowski space, which is a finite-dimensional Banach space. The answer is affirmative for Riemannian manifolds by J. Nash in [15]. He proved that any n-dimensional Riemannian manifold can be isometrically imbedded into a higher dimensional Euclidean space. However for the class of Finsler manifolds, the problem becomes very difficult. In [13], Ingarden proved that every n-dimensional Finsler manifold can be locally isometrically imbedded into a $2n$-

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dimensional “Weak” Minkowski space, i.e., a space whose indicatrix is not necessarily strongly convex. Then Burago and Ivanov showed that any compact C^r manifold $(r \geq 3)$ with a C^2 Finsler metric admits a C^r imbedding into a finite-dimensional Banach spaces [4]. Recently, Shen proved that a Finsler manifold with unbounded Cartan torsion cannot be isometrically imbedded into any Minkowski space [18]. In this paper, we prove the following theorem.

Theorem 1.1. *Suppose that a Finsler manifold (M, F) can be isometrically immersed into a Minkowski space $(V, \bar{F})$. Then the mean Cartan torsion $\mathbf{I}$ of F and the mean Cartan torsion $\bar{\mathbf{I}}$ of $\bar{F}$ satisfy*

$$\sup_{x\in M} \|\mathbf{I}\| \leq \|\bar{\mathbf{I}}\| < \infty.$$

Thus, Finsler manifolds with unbounded mean Cartan torsion cannot be isometrically imbedded into any Minkowski space.

Theorem 1.1 is an extension of Shen’s theorem about the Cartan torsion and its application in immersion theory.

In 1936, H. Whitney proved that any differentiable manifold admits a real-analytic structure [31]. In 1947, H. Hopf posed the question whether the analogous result is true for complex structures [9]. This question is answered to the negative by Hopf himself, by exhibiting infinitely many orientable even-dimensional manifolds that do not admit a complex structure, among them $\mathbb{S}^4$ and $\mathbb{S}^8$. He considered the sphere bundle over a manifold M whose fibre over a point p consists of all directions in the tangent space at p, and introduces the notion of $\mathfrak{J}$-manifold: this is a manifold whose sphere bundle admits a continuous fibre-preserving self-map for which no direction is mapped to itself or its opposite. As a complex structure on a manifold M induces a complex structure on each tangent space, it turns, in particular, M into a $\mathfrak{J}$-manifold. In [9], he derived a topological obstruction to the condition of being an $\mathfrak{J}$-manifold. In a footnote in [9], Hopf said that an alternative, but related, proof of the fact that $\mathbb{S}^4$ is not a complex manifold was communicated to him by C. Ehresmann; such a proof was written down a couple of years later in the paper [7].

In 1949, Ehresmann introduced the notion of almost complex structure [7]. The existence of an almost complex structure on a manifold M immediately turns it into a $\mathfrak{J}$-manifold. Hopf used the notion of almost complex structure in the paper [9]. However, he does not investigate in how far the notions of almost complex manifold (or $\mathfrak{J}$-manifold) and of complex manifold are distinct. In [7], Ehresmann showed that $\mathbb{S}^4$ is not a complex manifold. He proved that a 6-dimensional manifold with vanishing third integral homology carries an almost complex structure, thus proving the existence of an almost complex structure on $\mathbb{S}^6$. The existence of an almost complex structure on a manifold M^{2n} amounts to a reduction of the structure group of TM from the orthogonal group $O(2n)$ to the unitary group $U(n)$.

An almost complex structure on a smooth manifold M is a linear complex structure on each tangent space of the manifold $\mathbf{J}: TM \to TM$, such that $\mathbf{J}^2 = -1$. Then $(M, \mathbf{J})$ is called an almost complex manifold. If M admits an almost complex structure, it must be even-dimensional. Almost complex structures have important applications in symplectic geometry. For an almost complex manifold $(M, \mathbf{J})$, the Nijenhuis tensor is defined as follows:

$$N_{\mathbf{J}}(X, Y) = [X, Y] + \mathbf{J}[\mathbf{J}X, Y] + \mathbf{J}[X, \mathbf{J}Y] - [\mathbf{J}X, \mathbf{J}Y],$$

where $X, Y \in \chi(M)$. If $N_{\mathbf{J}} = 0$, then $\mathbf{J}$ is called integrable. In this case, $(M, \mathbf{J})$ is called a complex manifold. Thus every complex manifold has an almost complex structure.

Let $(M, \mathbf{g})$ be a Riemannian manifold and $\mathbf{J}$ be an almost complex structure on M. Then J is compatible wit $\mathbf{g}$ if

$$\mathbf{g}\big(\mathbf{J}(X), \mathbf{J}(Y)\big) = \mathbf{g}(X, Y).$$

In this case, the triple $(M, \mathbf{g}, \mathbf{J})$ is called an almost Hermitian manifold.

Finsler geometry is just Riemannian geometry without the quadratic restriction. Hence, the study of almost complex Finsler metrics is a natural problem. Let (M, F) be a Finsler manifold. Suppose that $J = J^i{}_j dx^j \otimes \dfrac{\partial}{\partial x^i}$ is an almost complex structure on (M, F). This is a natural question that how one can make any Minkowski space (T_xM, F_x) to a complex Banach space? Then, Ichijyō proposed a compatibility between J and F as follows:

$$F\big(x, y\cos\theta + J_x(y)\sin\theta\big) = F(x, y) \quad \forall\theta \in \mathbb{R} \quad \forall y \in T_xM. \tag{1.1}$$

Finsler manifold (M, F) with almost complex structure J which satisfies condition (1.1) is called almost Hermitian – Finsler manifold or Rizza manifold [10]. If F is Riemannian, then (M, F, J) is a Rizza manifold if and only if it is an almost Hermitian manifold. Thus, Rizza manifolds are natural extension of almost Hermitian manifolds. Also, there are some equivalent conditions for (1.1). For introducing them, we need to define some Finslerian quantities. Let (M, F) be a Finsler manifold. The second and third order derivatives of $1/2F_x^2$ at $y \in T_xM_0$ are called the fundamental form $\mathbf{g}_y$ and the Cartan torsion $\mathbf{C}_y$ on T_xM. Therefore, we can give some equivalent conditions to (1.1) as follows:

(1) $g_{ij}J^i{}_k y^k y^j = 0$,

(2) $g_{im}J^m_j + g_{jm}J^m_i + 2C_{ijm}J^m_r y^r = 0$.

In order to find the Finsler metrics compatible with Rizza structure, one can consider the simplest class of Finsler metrics, namely Randers metrics. Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric and $\beta = b_i(x)dx^i$ be a 1-form on a manifold M with $\|\beta\|_\alpha < 1$. Then $F = \alpha + \beta$ is called a Randers metric. Let a Randers metric $F = \alpha + \beta$ is compatible with the almost complex structure J. Then we have

$$F(x, y\cos\theta + J\sin\theta) = F(x, y).$$

It is equal to following:

$$\alpha(x, y\cos\theta + J\sin\theta) + \beta(x, y\cos\theta + J\sin\theta) = \alpha(x, y) + \beta(x, y) \quad \forall\theta \in \mathbb{R}.$$

Thus, for all $\theta \in \mathbb{R}$, the following hold:

$$\alpha(x, y\cos\theta + J\sin\theta) := \alpha(x, y), \tag{1.2}$$

$$\beta(x, y\cos\theta + J\sin\theta) := \beta(x, y). \tag{1.3}$$

By (1.2), it follows that (M, J, α) is a almost Hermit Riemannian structure. Let us put $\theta = \pi/2$ in (1.3). It follows that

$$\beta\big(J(y)\big) = \beta(y). \tag{1.4}$$

By (1.4), we get

$$\beta\big(J^2(y)\big) = \beta\big(J(y)\big). \tag{1.5}$$

By (1.4) and (1.5), we have $\beta = 0$, and F reduces to a Riemannian metric. It follows that a Randers metric $F = \alpha + \beta$ is compatible with the almost complex structure J if and only if it is Riemannian.

To overcome to the above mentioned problem, in [11, 12], Ichijyō and Hashiguchi introduced an important class of non-Riemannian Rizza manifolds, namely (a, b, J)-manifolds. Let (M, α, J) be a $2n$-dimensional almost Hermitian manifold. For a nonvanishing 1-form $b_i(x)$ on M, we have a symmetric quadratic form

$$\beta(x, y) = \sqrt{b_{ij}(x)y^i y^j}, \qquad b_{ij} := b_i b_j + J_i J_j,$$

where $J_i := b_r J^r_{\ i}$. Indeed, J_i are the local component of the 1-form $b(J)$. Now, it is easy to see that the Finsler metric $F = \alpha + \beta$ is a typical example of Rizza manifolds [10]. In this case, (M, F, J) is called an (a, b, J)-manifold [11]. Replacing the 1-form β with a symmetric quadratic form $\beta = \sqrt{b_{ij}(x)dx^i \otimes dx^j}$ of rank $0 \leq r < n$, we get a generalized Randers metric $F = \alpha + \beta$. Every (a, b, J)-metric is a generalized Randers metric. It is well-known that every Randers metric is C-reducible [24, 26]. In this paper, we show that the Cartan and mean Cartan torsion of every generalized Randers metric satisfy in an interesting relation. More precisely, we prove the following theorem.

Theorem 1.2. *Let (M, F) be Finsler manifold. Suppose that F is a generalized Randers metric. Then the following hold*:

(i) *F has an unbounded mean Cartan torsion*; *then generalized Randers metrics cannot be isometrically imbedded into any Minkowski space*;

(ii) *F is quasi-C-reducible.*

Then, we compute the mean Cartan torsion of generalized Randers metric on a Finsler surface and prove the following theorem.

Theorem 1.3. *Every generalized Randers metric on 2-dimensional manifold has a vanishing mean Cartan torsion.*

2. Preliminaries. Let M be an n-dimensional C^∞-manifold, $TM = \bigcup_{x\in M} T_xM$ is the tangent bundle and $TM_0 := TM - \{0\}$ is the slit tangent bundle. Let (M, F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is called fundamental tensor:

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well-known that $\mathbf{C} = 0$ if and only if F is Riemannian.

Let (M, F) be a Finsler manifold. For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by

$$\mathbf{I}_y(u) = \sum_{i=1}^{n} g^{ij}(y)\mathbf{C}_y(u, \partial_i, \partial_j),$$

where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. By definition, $\mathbf{I}_y(y) = 0$ and $\mathbf{I}_{\lambda y} = \lambda^{-1}\mathbf{I}_y,\ \lambda > 0$. Therefore, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Deicke's theorem, every positive-definite Finsler metric F is Riemannian if and only if $\mathbf{I}_y = 0$ [6, 16].

A Finsler metric F on a manifold M is called quasi-C-reducible Finsler metric if its Cartan torsion satisfies the following:

$$\mathbf{C}_y(u, v, w) = \frac{1}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{A}_y(v, w) + \mathbf{I}_y(v)\mathbf{A}_y(u, w) + \mathbf{I}_y(w)\mathbf{A}_y(u, v)\Big\}, \tag{2.1}$$

where $\mathbf{A}_y(u, v) = \mathbf{A}_y(v, u)$ is a symmetric tensor. In local coordinates, (2.1) is written as follows:

$$C_{ijk} = A_{ij}I_k + A_{jk}I_i + A_{ki}I_j. \tag{2.2}$$

As a special case of quasi-C-reducible metrics, a Finsler metric F on an n-dimensional manifold M is called C-reducible Finsler metric if its Cartan torsion satisfies the following:

$$\mathbf{C}_y(u, v, w) = \frac{1}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v, w) + \mathbf{I}_y(v)\mathbf{h}_y(u, w) + \mathbf{I}_y(w)\mathbf{h}_y(u, v)\Big\},$$

where $h_{ij} = g_{ij} - F^{-2}F_{y^i}F_{y^j}$ is the angular metric.

3. Proof of Theorem 1.1. Let M be a smooth manifold, $(\bar{M}, \bar{F})$ a Finsler manifold and $f : M \to (\bar{M}, \bar{F})$ an immersion. Let

$$F := \bar{F}of_*,$$

where $f_* : TM \to T\bar{M}$ is defined by $f_*(x, y) = \big(f(x), (df)_x(y)\big)$. It follows that F is a Finsler metric on M which is called the metric induced by $\bar{F}$.

Lemma 3.1. *Let $f : (M, F) \to (\bar{M}, \bar{F})$ be an immersion. Then the following hold*:

$$g_{ab}(x, y) = \bar{g}_{ij}(\bar{x}, \bar{y})\frac{\partial f^i}{\partial x^a}\frac{\partial f^j}{\partial x^b},$$

$$I_a(x, y) = \bar{I}_i(\bar{x}, \bar{y})\frac{\partial f^i}{\partial x^a}\frac{\partial f^k}{\partial x^c},$$

where $f = \big(f^i(\bar{x}^i)\big)$, $\bar{x} = f(x)$ and $\bar{y} = (df)_x(y)$.

Proof. By a direct computations, we get the proof.

Let (M, F) be an Finsler manifold. At any point $x \in M$, the norm of $\mathbf{C}$ can be defined by

$$\|\mathbf{I}\| = \sup_{y,u\in T_xM_0} \frac{F(y)|\mathbf{I}_y(u)|}{\big[\mathbf{g}_y(u, u)\big]^{\frac{3}{2}}} = \sup_{y,u\in I_xM} \frac{|\mathbf{I}_y(u)|}{\big[\mathbf{g}_y(u, u)\big]^{\frac{3}{2}}}.$$

Here, we prove the following lemma.

Lemma 3.2. *Let $f\colon (M,F)\to(\bar{M},\bar{F})$ be an isometric immersion. Then, for any point $x\in M$, the following holds*:

$$\|\mathbf{I}\|_x \le \|\bar{\mathbf{I}}\|_{f(x)}, \tag{3.1}$$

where $\mathbf{I}$ and $\bar{\mathbf{I}}$ are the mean Cartan torsion of F and $\bar{F}$, respectively.

Proof. The following holds:

$$\|\mathbf{I}\|_x = \sup_{y,u\in I_xM} \frac{F(y)|\mathbf{I}_y(u)|}{\big[\mathbf{g}_y(u,u)\big]^{\frac{3}{2}}}.$$

Since f is an isometric immersion, then, for any $y\in TM$, we have $F(y)=\bar{F}(f_*(y))$. Thus,

$$\|\mathbf{I}\|_x = \sup_{y,u\in I_xM} \frac{\bar{F}(f_*(y))\big|\bar{\mathbf{I}}_{f_*(y)}\big(f_*(u)\big)\big|}{\big[\bar{\mathbf{g}}_{f_*(y)}\big(f_*(u),f_*(u)\big)\big]^{\frac{3}{2}}},$$

which yields

$$\|\mathbf{I}\|_x \le \sup_{\bar{y},\bar{u}\in I_{f(x)}\bar{M}} \frac{\bar{F}(\bar{y})\big|\mathbf{I}_{\bar{y}}(\bar{u})\big|}{\big[\mathbf{g}_{\bar{y}}(\bar{u},\bar{u})\big]^{\frac{3}{2}}} = \|\bar{\mathbf{I}}\|_{f(x)}.$$

Lemma is proved.

***Proof of Theorem* 1.1.** Let $f\colon (M,F)\to(V,\bar{F})$ be an isometric immersion, where $\big(V,\bar{F}(y)\big)$ is a finite-dimensional Minkowski space and $F(x,y):=\bar{F}(y)$ is the metric induced by $\bar{F}$. $\|\bar{\mathbf{I}}\|_{\bar{y}}$ is independent of $\bar{y}\in V$. By (3.1), we get

$$\sup_{x\in M}\|\mathbf{I}\|_x \le \sup_{x\in M}\|\bar{\mathbf{I}}\|_{f(x)} \le \sup_{\bar{y}\in V}\|\bar{\mathbf{I}}\|_{\bar{y}} = \|\bar{\mathbf{I}}\| < \infty. \tag{3.2}$$

By (3.2), we get the proof.

4. Proof of Theorem 1.2. It is well-known that the class of Randers metrics is an special case of a general class of Finsler metrics so-called (α,β)-metrics. A Finsler metric F is called an (α,β)-metric if it can be expressed as $F=\alpha\varphi(s)$, $s=\beta/\alpha$, where $\varphi\colon(-b_0,b_0)\to\mathbb{R}$ is a positive smooth function on some symmetric open interval $I=(-b_0,b_0)$, satisfying some regularity conditions (see [1 – 3, 19 – 30]). Similarly, in order to extend the class of Rizza manifolds introduced by Ichijyō, one can define generalized (a,b,J)-metrics as follows. Consider an (a,b,J)-metric $F=\alpha+\beta$. Let $\Psi:(-b_0,b_0)\to\mathbb{R}$ be a positive smooth function. Then a Finsler metric in the form $F=\alpha\Psi(s)$, $s=\beta/\alpha$, is called a generalized (a,b,J)-metric.

One can compute the fundamental tensor of generalized (a,b,J)-metric $F=\alpha\Psi(s)$, $s:=\beta/\alpha$, as follows:

$$g_{ij} = \big[\Psi^2 - s\Psi\Psi'\big]a_{ij} + \frac{1}{s}\Psi\Psi' b_{ij} + s\Big[s(\Psi\Psi''+\Psi'\Psi') - \Psi\Psi'\Big]\alpha_i\alpha_j$$
$$+ \Big[\Psi\Psi' - s(\Psi\Psi''+\Psi'\Psi')\Big](\alpha_i\beta_j+\alpha_j\beta_i) - \frac{1}{s}\Big[\Psi\Psi' - s(\Psi\Psi''+\Psi'\Psi')\Big]\beta_i\beta_j, \tag{4.1}$$

where $\alpha_i := \partial\alpha/\partial y^i$ and $\beta_i = \partial\beta/\partial y^i$.

By (4.1), one can get the following lemma.

Lemma 4.1. *For the fundamental tensor (g_{ij}) of $F=\alpha\Psi(s)$, $s:=\beta/\alpha$, on an n-dimensional manifold M, the determinant of (g_{ij}) is given by*

$$\det(g_{ij}) = \frac{1}{s}\,\Psi^{n+1}\big(\Psi - s\Psi'\big)^{n-3}\Big[\Psi - s\Psi' + (b^2-s^2)\Psi''\Big]\Big[s\Psi + (b^2-s^2)\Psi'\Big]\det(a_{ij}).$$

One can see that for a Randers metric $\Psi = 1 + s$, we have $\Psi' = 1$ and $\Psi'' = 0$ which shows that the result is the same as the formula has been gotten in [14]. By Lemma 4.1, we get the following lemma.

Lemma 4.2. *Let* $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ *be a Riemannian metric and* β *be a symmetric quadratic form. Then* $F = \alpha\Psi(s)$, $s = \beta/\alpha$, *is a Finsler metric if and only if* $\Psi = \Psi(s)$ *satisfies the following*:

$$\Psi(s) > 0, \qquad \Psi - s\Psi' + (b^2 - s^2)\Psi'' > 0, \quad s\Psi + (b^2 - s^2)\Psi' > 0,$$

where $\Psi' = d\Psi/ds$ *and* $\Psi'' = d\Psi'/ds$.

Now, we can compute the mean Cartan torsion of a generalized (a, b, J)-metric.

Lemma 4.3. *The mean Cartan torsion of a generalized* (a, b, J)*-metric* $F = \alpha\Psi(s)$, $s = \beta/\alpha$, *is given by*

$$I_i = \frac{1}{2\alpha}\left\{(n+1)\frac{\Psi'}{\Psi} - (n-3)\frac{s\Psi''}{\Psi - s\Psi'} + \frac{(b^2 - s^2)\Psi''' - 3s\Psi''}{(\Psi - s\Psi') + (b^2 - s^2)\Psi''} + \frac{s(b^2 - s^2)\Psi'' - b^2\Psi'}{s(b^2 - s^2)\Psi' + s^2\Psi}\right\}h_i,$$

where

$$h_i := \frac{b_{ki}y^k}{\beta} - \frac{sy_i}{\alpha} = \beta_i - \alpha^{-1}sy_i$$

and $b^2 := g^{ij}b_ib_j = b^jb_j$.

Proof. By definition, we have

$$I_i = g^{jk}C_{ijk} = \frac{\partial}{\partial y^i}\left(\ln\sqrt{\det(g_{jk})}\right).$$

A direct computation shows that

$$\frac{\partial s}{\partial y^i} = \frac{h_i}{\alpha}.$$

Thus,

$$I_i = \frac{1}{2}\frac{\partial}{\partial y^i}\Bigg\{(n+1)\ln\Psi + (n-3)\ln(\Psi - s\Psi') + \ln\left[(\Psi - s\Psi') + (b^2 - s^2)\Psi''\right]$$
$$+ \ln\left[\Psi + \frac{1}{s}(b^2 - s^2)\Psi'\right] + \ln\big(\det(a_{ij})\big)\Bigg\}.$$

Finally, the desired result is obtained.

Let us define

$$T := \frac{1}{2\alpha}\left\{(n+1)\frac{\Psi'}{\Psi} - (n-3)\frac{s\Psi''}{\Psi - s\Psi'} + \frac{(b^2 - s^2)\Psi''' - 3s\Psi''}{\Psi - s\Psi' + (b^2 - s^2)\Psi''} + \frac{s(b^2 - s^2)\Psi'' - b^2\Psi'}{s(b^2 - s^2)\Psi' + s^2\Psi}\right\}.$$

Then we have the following lemma.

Lemma 4.4. *The norm of mean Cartan torsion of a generalized* (a, b, J)*-metric* $F = \alpha\Psi(s)$, $s = \beta/\alpha$, *is given by following*:

$$||\mathbf{I}||^2 := I_iI^i = \rho^{-1}T^2\Bigg[-\frac{(\alpha b - \beta)(\alpha b + \beta)(\tau\alpha^2b^2 - \tau\beta^2 - \alpha^2)}{\alpha^4}$$
$$- \sigma''\left\{\frac{(\alpha b - \beta)^2(\alpha b + \beta)^2(\varepsilon\tau\alpha b^2b_0^2 + \tau\beta b_0^2 - \varepsilon\alpha\beta^2)^2}{\alpha^6\beta^4}\right\}\Bigg].$$

Proof. Put $\alpha^i := a^{ir}\alpha_r,\ \beta^i := a^{ir}\beta_r,\ b^i := a^{ij}b_j,\ J^i := a^{ij}J_j$. Then the following relationships hold:

$$\alpha^i = a^{ir}\frac{y_r}{\alpha} = \frac{y^i}{\alpha},$$

$$\beta^i = \frac{a^{ir}b_{rs}y^s}{\beta} = \frac{b^i{}_s y^s}{\beta},$$

$$\alpha_i\alpha^i = \frac{y_i}{\alpha}\frac{y^i}{\alpha} = 1,$$

$$\alpha^i\beta_i = \frac{y^i}{\alpha}\frac{b_{ij}y^j}{\beta} = \frac{b_{ij}y^iy^j}{\alpha\beta} = \frac{\beta^2}{\alpha\beta} = \frac{\beta}{\alpha},$$

$$\alpha_i\beta^i = \frac{y_i}{\alpha}\frac{a^{ir}b_{rs}y^s}{\beta} = \frac{b_{rs}y^ry^s}{\alpha\beta} = \frac{\beta^2}{\alpha\beta} = \frac{\beta}{\alpha}.$$

Also,

$$J_r = b_iJ^i_r, \qquad b^{ij}b_{ij} = 2b^4,$$

$$b^i{}_j = a^{ir}b_{rj}, \qquad b^iJ_i = b_iJ^i = 0,$$

$$J^jJ_j = b^2, \qquad b^{ij} = a^{jr}b^i{}_r,$$

$$b^sb_s = a^{rs}b_rb_s = b^2, \qquad b^i{}_j = b^ib_j + J^iJ_j,$$

$$y_iy^i = a_{ij}y^iy^j = \alpha^2, \qquad a^{ij}b_{ij} = 2b^2,$$

$$b^{is}b_{js} = b^2b^i{}_j, \qquad b^{ij} = b^ib^j + J^iJ^j.$$

Let us put

$$\rho := \Psi(\Psi - s\Psi'),$$

$$\rho_1 := -s(-s(\Psi\Psi'' + \Psi'\Psi') + \Psi\Psi'),$$

$$\rho_2 := -s(\Psi\Psi'' + \Psi'\Psi') + \Psi\Psi',$$

$$\rho_3 := -\frac{1}{s}\Psi\Psi' + \Psi'\Psi' + \Psi\Psi'',$$

$$\mu := \frac{\rho_1}{\rho} = \frac{s\{s(\Psi\Psi'' + \Psi'\Psi') - \Psi\Psi'\}}{\Psi(\Psi - s\Psi')}.$$

Then the inverse of fundamental tensor of generalized (a, b, J)-metric is given by following:

$$g^{ij} = \rho^{-1}\Big\{a^{ij} - \tau b^{ij} - \sigma\beta^i\beta^j - \sigma''Y^iY^j\Big\},$$

where

$$Y_i := \alpha_i + \varepsilon\beta_i,$$

$$Y^i := \alpha^i + \varepsilon\beta^i - \tau b^i b^j \alpha_j - \tau\varepsilon b^i b^j \beta_j,$$

$$\varepsilon := \frac{\rho_2}{\rho_1},$$

$$\delta := \frac{\rho_3 - \varepsilon^2\rho_1}{\rho},$$

$$\sigma := \frac{\delta(1-\tau b^2)^2}{1+\delta(1-\tau b^2)b^2},$$

$$\tau := \frac{\Psi'}{s\Psi + (b^2-s^2)\Psi'},$$

$$\sigma' := -(\tau+\sigma)(s+\varepsilon b^2) + \varepsilon,$$

$$\sigma'' := \frac{s\mu}{s-\mu(b^2-s^2)\sigma'},$$

$$\beta^i = \frac{b^i{}_s\, y^s}{\beta}.$$

Then

$$g^{ij} I_i = \rho^{-1} T (a^{ij} - \tau b^{ij} - \sigma\beta^i\beta^j - \sigma'' Y^i Y^j)(\beta_i - \alpha^{-1} s y_i),$$

which yields

$$I^j = \frac{T}{\rho}\left\{\beta^j - \frac{1}{\alpha}s\bar{y}^j - \tau b^{ij}\Big(\beta_i - \frac{1}{\alpha}sy_i\Big) - \sigma\beta^i\beta^j\Big(\beta_i - \frac{1}{\alpha}sy_i\Big) - \sigma'' Y^i Y^j\Big(\beta_i - \frac{1}{\alpha}sy_i\Big)\right\}.$$

Thus, we get

$$I^i = \rho^{-1} T\left[\frac{b^i b_0 + J^i J_0}{\beta} - \frac{\beta}{\alpha^2}y^i - \tau\frac{b^2(b^i b_0 + J^i J_0)}{\beta} - \frac{\beta(b^i b_0 + J^i J_0)}{\alpha^2}\right.$$
$$\left. - \sigma''\left[A\left(\frac{y^i}{\alpha} + \frac{\varepsilon(b^i b_0 + J^i J_0)}{\beta} + \frac{\tau b^i b_0}{\alpha} - \frac{\tau\varepsilon b^2 b^i b_0}{\beta}\right)\right]\right],$$

where $b_0 = b^i y_i,\ J_0 = J^i y_i$ and

$$A := \varepsilon b^2 - \frac{\tau b^2 b_0^2}{\alpha\beta} - \frac{\varepsilon\beta^2}{\alpha^2} - \frac{\tau\varepsilon b^4 b_0^2}{\beta^2} + \frac{\tau\varepsilon b^2 b_0^2}{\alpha^2} + \frac{\tau\beta b_0^2}{\alpha^3}.$$

Lemma 4.4 is proved.

Proposition 4.1. *The norm of mean Cartan torsion of a generalized Randers metric $F = \alpha + \beta$ on an n-dimensional manifolds M is given by following*:

$$\|\mathbf{I}\|^2 = \frac{b^2 - s^2}{4(1+s)^2}\left[(n+1) - \frac{b^2(1+s)}{(b^2+s)s}\right]^2. \tag{4.2}$$

Then F has an unbounded mean Cartan torsion. It follows that generalized Randers metrics cannot be isometrically imbedded into any Minkowski space.

Proof. For a generalized Randers metric, the equation (4.2) reduces to following:

$$I^i = \mu\left[\frac{\alpha}{F}a^{ij}(\alpha\beta_j - \beta\alpha_j) - \frac{\alpha^2}{(b^2\alpha+\beta)F}b^{ij}(\alpha\beta_j - \beta\alpha_j) + \frac{(b^2\alpha+\beta)\alpha^2}{F^3}\alpha^i\alpha^j(\alpha\beta_j - \beta\alpha_j)\right.$$
$$\left. - \frac{\alpha}{F^2\beta}(\alpha^i\beta^j + \beta^i\alpha^j)(\alpha\beta_j - \beta\alpha_j) + \frac{\alpha^2}{(b^2\alpha+\beta)F}\beta^i\beta^j(\alpha\beta_j - \beta\alpha_j)\right],$$

which yields

$$\|\mathbf{I}\|^2 = I_iI^i = \mu^2\frac{\alpha}{F}\big(\alpha\beta^i - \beta\alpha^i\big) + \frac{\alpha^2}{\gamma F}\left(\alpha\beta^i b^2 - \frac{\beta^2}{\alpha}\beta^i - \frac{\alpha}{\beta}b^2 b^i_k y^k + \beta b^{ij}\alpha_j\right)$$
$$- \frac{\alpha}{F^2\beta}(\alpha\beta_i - \beta\alpha_i)\left(\alpha b^2\alpha^i - \frac{\beta^2}{\alpha}\alpha^i\right),$$

where $b_0 := b_iy^i$ and $J_0 := J_iy^i$. By using $b^iJ_i = b_iJ^i = 0,\ J^iJ_i = b^2$, we get

$$\beta_i\beta^i = \frac{b_{ij}y^j}{\beta}\frac{a^{ir}b_{rs}y^s}{\beta} = \frac{b^2 b_{js}y^jy^s}{\beta^2} = \frac{b^2\beta^2}{\beta^2} = b^2,$$

$$b^{ij}\beta_i\beta_j = a^{jr}b^i_r\beta_i\beta_j = b^i_r\beta_i\beta^r = b^i_r\frac{b_{ip}y^p}{\beta}\frac{b^r_sy^s}{\beta} = \frac{b^2b^i_sb_{ip}y^sy^p}{\beta^2} = \frac{b^2(b^2b_{sp}y^sy^p)}{\beta^2} = b^4,$$

$$b^{ij}\alpha_i\alpha_j = a^{jr}b^i_r\frac{y_i}{\alpha}\frac{y_j}{\alpha} = \frac{b^r_iy_iy^r}{\alpha^2} = \frac{\beta^2}{\alpha^2},$$

$$b^{ij}\alpha_i\beta_j = b_{ij}\frac{y_i}{\alpha}\frac{b_{js}y^s}{\beta} = \frac{a^{jr}b^i_rb_{js}y_iy^s}{\alpha\beta} = \frac{b^i_rb^r_sy_iy^s}{\alpha\beta} = \frac{b^2b^i_sa_{pi}y^py^s}{\alpha\beta} = \frac{b^2b_{sp}y^sy^p}{\alpha\beta} = \frac{b^2\beta}{\alpha},$$

where $J^2 := J_iJ^i$. The following hold: $\alpha_i\beta^i = \alpha^i\beta_i = \dfrac{\beta}{\alpha},\ \alpha_i\alpha^i = 1$.

Let $F = \alpha(1+s),\ s = \beta/\alpha,\ |s| < 1$. Then we have

$$\|\mathbf{I}\|^2 = \mu^2\left[\frac{\alpha}{F}\big(\alpha^2\beta^i\beta_i - \alpha\beta\alpha^i\beta_i - \alpha\beta\alpha_i\beta^i + \beta^2\alpha^i\alpha_i\big) + \frac{\alpha^2}{\gamma F}\Big(\alpha^2b^2\beta^i\beta_i - \beta^2\beta^i\beta_i\right.$$
$$\left. - \frac{\alpha^2}{\beta}b^2b^i_ky^k\beta_i + \alpha\beta b^{ij}\alpha_j\beta_i\Big) - \frac{\alpha}{F^2\beta}\left(\alpha^2b^2\alpha^i\beta_i - \beta^2\alpha^i\beta_i - \alpha\beta b^2\alpha^i\alpha_i + \frac{\beta^3}{\alpha}\alpha^i\alpha_i\right)\right], \tag{4.3}$$

where $\gamma := \beta + b^2\alpha$. (4.3) yields

$$\|\mathbf{I}\|^2 = \mu^2\Big[\frac{\alpha}{F}\big(\alpha^2b^2 - \beta^2\big)\Big]. \tag{4.4}$$

Let $s = \beta/\alpha$. The range of s is $[-b, b]$. Assume that y is a unit vector, i.e., $F(y) = \alpha(y) + \beta(y) = 1$. Then, by (4.4) we get (4.2). It is easy to see that if $s \to 0$, then $\|\mathbf{I}\| \to \infty$.

Proposition 4.1 is proved.

Proposition 4.2. *Every generalized Randers metric is quasi-C-reducible.*

Proof. For the case of $\psi = 1 + s$, we get the generalized Randers metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = \sqrt{b_{ij}(x)dx^i \otimes dx^j}$ are a Riemannian metric and a 2-form on a manifold M with $b_{ij} = b_i b_j + J_i J_j$ and $\|\beta\|_\alpha < 1$. We obtain

$$g_{ij} = \frac{F}{\alpha}a_{ij} + \frac{F}{\beta}b_i b_j + \frac{F}{\beta}J_i J_j + F_i F_j - \frac{F}{\alpha}\alpha_i\alpha_j - \frac{F}{\beta}\beta_i\beta_j,$$

$$g^{ij} = \frac{\alpha}{F}a^{ij} - \frac{\alpha^2}{(b^2\alpha+\beta)F}b^{ij} + \frac{(b^2\alpha+\beta)\alpha^2}{F^3}\alpha^i\alpha^j - \frac{\alpha}{F^2\beta}(\alpha^i\beta^j + \beta^i\alpha^j) + \frac{\alpha^2}{(b^2\alpha+\beta)F}\beta^i\beta^j,$$

where $\alpha_i = \dfrac{\partial\alpha}{\partial y^i}$, $\beta_i = \dfrac{\partial\beta}{\partial y^i}$, $F_i = \alpha_i + \beta_i$. We get

$$C_{ijk} := \frac{1}{4}(F^2)_{y^i y^j y^k} = \frac{1}{2}(g_{ij})_{y^k} = \frac{1}{2}\sigma_{(i,j,k)}\left\{\left(\frac{\alpha_{ij}}{\alpha} - \frac{\beta_{ij}}{\beta}\right)(\alpha\beta_k - \beta\alpha_k)\right\}. \tag{4.5}$$

Here, the notation $\sigma_{(i,j,k)}$ denotes the summation of the cyclic permutation of indices i, j and k. By a simple calculation, we have

$$I_i = \mu(\alpha\beta_i - \beta\alpha_i), \tag{4.6}$$

where

$$\mu := \frac{1}{2}\left(\frac{n+1}{\alpha} - \frac{b^2 F}{(b^2\alpha+\beta)\beta}\right). \tag{4.7}$$

By considering (2.2), let us put

$$A_{ij} := \frac{1}{2\mu}\left(\frac{\alpha_{ij}}{\alpha} - \frac{\beta_{ij}}{\beta}\right).$$

Then, by (4.5), (4.6) and (4.7), we get the proof.

***Proof of Theorem* 1.2.** Follows from Propositions 4.1 and 4.2.

5. Proof of Theorem 1.3. Now, we are going to find the norm of Cartan torsion of a generalized Randers metric in the case of $\dim(M) = 2$. Assume that $\dim(M) = 2$. Let us remark the Lemma 1.2.2 of [17].

Lemma 5.1. *Let (V, F) be a Minkowski plane. For a vector $y \in V$ with $F(y) \neq 0$, there exists a vector $y^\perp \in V - \{0\}$ such that*

$$\mathbf{g}_y(y, y^\perp) = 0, \qquad \mathbf{g}_y(y^\perp, y^\perp) = F^2(y).$$

By using Lemma 5.1, we prove Theorem 1.3.

***Proof of Theorem* 1.3.** The unit circle $S = F^{-1}(1)$ is a simple closed curve around the origin. For a unit vector $y \in S$, there is a vector $y^\perp \in V$ satisfying

$$\mathbf{g}_y(y, y^\perp) = 0, \qquad \mathbf{g}_y(y^\perp, y^\perp) = 1.$$

The set $\{y, y^\perp\}$ is called the Berwald basis at y. Define

$$I(y) := \mathbf{C}_y\left(y^\perp, y^\perp, y^\perp\right), \quad y \in S.$$

We call I the main scalar. Note that $I = 0$ if and only if $\mathbf{C} = 0$. We have $\|\mathbf{C}\| = \sup_{y\in S} |I(y)|$. Take an oriented basis $\{e_1, e_2\}$ for V which determines a global coordinate system (u, v) in V. Parameterize S by a map $c(t) := u(t)e_1 + v(t)e_2$. Then

$$\sigma(t) := \mathbf{g}_{c(t)}\big(\dot{c}(t), \dot{c}(t)\big) = \frac{u'(t)v''(t) - u''(t)v'(t)}{u(t)v'(t) - u'(t)v(t)} > 0.$$

For the vector $y = c(t) \in S$, define the vector $y^{\perp} \in V$ as follows:

$$y^{\perp} := \frac{1}{\sqrt{\sigma(t)}}\, \dot{c}(t).$$

The scalar $I(t) := I(c(t))$ is given by

$$I(t) = \frac{1}{\sqrt{\sigma(t)}} \frac{d}{dt}\left[\ln \frac{\sqrt{\sigma(t)}}{u(t)v'(t) - u'(t)v(t)}\right]. \tag{5.1}$$

Now, we consider a generalized Randers norm $F = \alpha + \beta$ in V, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = \sqrt{b_{ij}(x)y^i y^j}$ is a quadratic form for a nonvanishing 1-form $b_i(x)$ on V, where $b_{ij} = b_i b_j + J_i J_j$ and $J_i = b_r J_i^r$. Indeed, J_i are the local component of the 1-form $b\,o\,J$. Take an orthonormal basis $\{e_1, e_2\}$ for (V, α) such that $b_i(ue_1 + ve_2) = bu$, where $b = \|\beta\| := \sup_{\alpha(y)=1} \beta(y) < 1$. This means that $b_1 = b$, $b_2 = 0$. We have $\beta = \sqrt{b_{11}u^2 + b_{12}uv + b_{12}vu + b_{22}v^2}$ and

$$J = \begin{pmatrix} J_1^1 & J_2^1 \\ J_1^2 & J_2^2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Using $b_2 = 0$, we get the following:

$$J_1 = b_r J_1^r = b_1 J_1^1 + b_2 J_1^2 = 0.$$

Thus,

$$b_{11} = b_1 b_1 + J_1 J_1 = b^2,$$

$$b_{12} = b_{21} = b_1 b_2 + J_1 J_2 = 0.$$

Also, we have

$$J_2 = b_r J_2^r = b_1 J_2^1 + b_2 J_2^2 = -b_1 = -b,$$

which yields

$$b_{22} = b_2 b_2 + J_2 J_2 = (J_2)^2 = b^2.$$

Then

$$F(ue_1 + ve_2) = \sqrt{u^2 + v^2} + \sqrt{b^2(u^2 + v^2)}.$$

The indicatrix $S = F^{-1}(1)$ is a circle determined by the following equation:

$$u^2 + v^2 = \frac{1}{(1 + b)^2}.$$

Parameterize S by $c(t) = u(t)e_1 + v(t)e_2$, where

$$u(t) = \frac{1}{1+b}\cos(t), \qquad v(t) = \frac{1}{1+b}\sin(t). \tag{5.2}$$

Plugging (5.2) into (5.1), we obtain

$$I(t) = 0.$$

Theorem 1.3 is proved.

By Theorem 1.3, we conclude the following corollary.

***Corollary* 5.1.** *Every 2-dimensional generalized Randers metric is Riemannian.*

Proof. The Cartan torsion of every 2-dimensional Finsler manifold satisfies the following:

$$C_{ijk} = \frac{1}{3}\Big\{h_{ij}I_k + h_{jk}I_i + h_{ik}I_j\Big\}. \tag{5.3}$$

Indeed, every Finsler surface is C-reducible, namely, satisfies (5.3). By Theorem 1.3, F satisfies $\mathbf{I} = 0$. Putting it in (5.3) implies that $\mathbf{C} = 0$. Then F is Riemannian.

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