DOI: 10.37863/umzh.v75i3.1130

UDC 512.5

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JORDAN REGULAR UNITS IN RINGS AND GROUP RINGS

РЕГУЛЯРНІ ОДИНИЦІ ЖОРДАНА В КІЛЬЦЯХ І ГРУПОВИХ КІЛЬЦЯХ

The concept of Lie regular elements and Lie regular units was defined and studied by Kanwar, Sharma and Yadav in [*Lie regular generators of general linear groups*, Comm. Algebra, **40**, № 4, 1304 – 1315 (2012)]. We introduce Jordan regular elements and Jordan regular units. It is proved that the order of the set of Jordan regular units in $M(2, Z_{2^n})$ is equal to a half of the order of $U(M(2, Z_{2^n}))$. Further, we show that the group ring KG of a group G over a field K of characteristic 2 has no Jordan regular units.

Поняття регулярних елементів Лі та регулярних одиниць Лі було введено та вивчено Канваром, Шармою та Ядавом у [*Lie regular generators of general linear groups*, Comm. Algebra, **40**, № 4, 1304 – 1315 (2012)]. Ми вводимо регулярні елементи Жордана та регулярні одиниці Жордана. Доведено, що порядок множини регулярних одиниць Жордана в $M(2, Z_{2^n})$ становить половину порядку $U(M(2, Z_{2^n}))$. Крім того, показано, що групове кільце KG групи G над полем K характеристики 2 не має регулярних одиниць Жордана.

1. Introduction. Let R be an associative unital ring. An element $a \in R$ is called regular if $a = aua$ for some element $u \in R$. If a is a unit, then a is called unit regular. An element $a \in R$ is unit regular if and only if there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = eu$. A ring R is von Neumann regular if every element of R is regular. Also, if a commutative ring R is von Neumann regular and $a \in R$, then there exist a unit $u \in R$ and an idempotent $e \in R$ such that $a = eu$. An element a of a ring R is called clean if $a = e + u$ for some idempotent $e \in R$ and some unit $u \in R$. Both unit regular and clean elements have evoked considerable interest and have been studied well. An element $a \in R$ is called Lie regular if there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = eu - ue$. A Lie regular element which is also a unit is called a Lie regular unit. Lie regular elements and Lie regular units were introduced by Kanwar, Sharma and Yadav in [1]. In this paper, we study the elements $a \in R$ for which there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = eu + ue$. We call such elements Jordan regular elements. A Jordan regular element which is also a unit is called a Jordan regular unit. As $0 = 0.1 + 1.0$ is clearly a Jordan regular element, so the set of Jordan regular elements of a ring is always non-empty.

In Section 2, we obtain some basic results on Jordan regular units and find Jordan regular units in some rings and fields. We observe that there exist rings having no Jordan regular units (see Proposition 2.1 and Examples 2.1, 2.3). There also exist rings in which every unit is a Jordan regular unit (see Propositions 2.2, 2.3, 2.4 and Example 2.1). Further, there are rings in which the set of Jordan regular units is a proper subset of the unit group of the ring. It is proved that if 2 is a unit in R, where R is a commutative ring with unity, then every unit in $M(n, R)$ is a Jordan regular unit, but if 2 is not a unit in R, then this need not be true. For even n, we find a group consisting of non Jordan regular units in $M(2, Z_n)$ and compute its order for $n = 6, 10$ and 12 (Propositions 2.13,

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2.14, 2.17, 2.18 and 2.19). Also we establish that the order of the group of non Jordan regular units in $M(2, Z_{2^n})$ is half of the order of $U(M(2, Z_{2^n}))$ (Theorem 2.4). It remains open to prove that this is also true when n is even but not a power of 2.

In Section 3, we find Jordan regular units in $GL(2, F)$, where F is a finite field of characteristic 2. It is proved that the group algebra KG, where K is any field of characteristic 2 and G is any group, has no Jordan regular units.

The unit groups of rings and group rings have always been an important object of study as is evident from a number of papers on this topic [2–4, 6–9]. The study of Jordan regular units is expected to further enrichen this area [5].

Throughout this paper, $\mathrm{Char}(R)$ denotes the characteristic and $U(R)$ denotes the unit group of the ring R. We denote by $JRU(R)$ the set of all Jordan regular units in R. We have made use of MATLAB and GAP for the general linear groups in Propositions 2.17, 2.18, and 2.19.

2. Jordan regular elements.

Definition 2.1. *Let R be an associative ring with unit element. An element $x \in R$ is called a Jordan regular element if there exist an idempotent $e \in R$ and a unit $u \in R$ such that $x = eu + ue$.*

Definition 2.2. *A Jordan regular element which is also a unit is called a Jordan regular unit.*

Definition 2.3. *A ring R is called a Jordan regular ring if every element of R is a Jordan regular element.*

Proposition 2.1. *If F is any field of characteristic 2, then F has no non-zero Jordan regular elements and, hence, no Jordan regular units.*

Proof. Since the only idempotents in F are 0 and 1 and $\mathrm{Char}(F) = 2$, so the set of Jordan regular elements $\{eu + ue | e \in \{0, 1\}, u \in F \setminus \{0\}\}$ contains only one element 0. Thus, F has no non-zero Jordan regular elements and hence no Jordan regular units.

***Corollary* 2.1.** *Integral domains of characteristic 2 do not contain Jordan regular units.*

Proposition 2.2. *Let F be a field such that $\mathrm{Char}(F) \neq 2$. Then F is a Jordan regular ring.*

Proof. Let $0 \neq u \in F$. Then $2^{-1}u \in F$ and $u = 1(2^{-1}u) + (2^{-1}u)1$. Thus every non-zero element of F is a Jordan regular element and F is a Jordan regular ring.

Proposition 2.3. *If R is an integral domain, then every non-zero Jordan regular element is a unit if and only if 2 is a unit in R.*

Proof. Let x be a non-zero Jordan regular element in R. Then $x = eu+ue$ for some idempotent e and some unit u in R. Since R is an integral domain, so $e = 0$ or $e = 1$ and $x = 2u$. Thus, x is a unit in R if and only if 2 is a unit in R.

***Example* 2.1.** Let Z_n be the ring of integers modulo n. If n is even, then any unit u in Z_n is odd and relatively prime to n. Thus, for any idempotent e and for any unit u in Z_n, $eu + ue = 2eu$ can not be a unit. But if n is odd, then 2 is a unit in Z_n. So, for any unit $u \in Z_n$, $2^{-1}u \in U(Z_n)$ and $u = 1(2^{-1}u) + (2^{-1}u)1$ is a Jordan regular unit.

***Example* 2.2.** $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is an idempotent in $M(2, \mathbb{R})$ and $u = \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}$ is a unit in $M(2, \mathbb{R})$, so $a = eu = \begin{pmatrix} 7 & 4 \\ 0 & 0 \end{pmatrix}$ is unit regular, but it is not a Jordan regular unit.

Theorem 2.1. *Let R be a commutative ring with unity such that $2 \in U(R)$. Then R is a Jordan regular ring iff R is a von Neumann regular ring.*

Proof. Let R be a Jordan regular ring, then, for every $x \in R$, there exist an idempotent e and a unit u in R such that $x = eu + ue = 2eu = e(2u) = ev$, where $v = 2u$ is a unit. Hence R is a von Neumann regular ring.

Conversely, suppose that R is von Neumann regular ring. Let $x \in R$, then $x = eu$ for an idempotent e and a unit u in R. But then $x = e(2^{-1}u) + (2^{-1}u)e$ is a Jordan regular element and R is a Jordan regular ring.

***Example* 2.3.** $JRU(\mathbb{Z}) = \varnothing$ as 0 and 1 are the only idempotents in $\mathbb{Z}$ and $U(\mathbb{Z}) = \{-1, 1\}$.

Proposition 2.4. *For an odd integer r, $JRU(M(m, Z_r)) = U(M(m, Z_r))$.*

Proof. Since r is odd, 2 is a unit in Z_r. Let $A \in U(M(m, Z_r))$, then $A = I.2^{-1}A + 2^{-1}A.I$ is a Jordan regular unit.

From above, we conclude that in any commutative ring R with unity, if $2 \in U(R)$, then $JRU(M(m, R)) = U(M(m, R))$. Now we investigate Jordan regular units in the ring of 2×2 matrices, over commutative rings in which 2 is not a unit.

Proposition 2.5. *The trace of a Jordan regular element in $M(m, Z_n)$ is even if n is even, but it need not be even if n is odd.*

Proof. Let J be a Jordan regular element in $M(m, Z_n)$. Then $J = eu + ue$ for some idempotent e and some unit u in $M(m, Z_n)$. Therefore, $tr(J) = tr(eu + ue) = tr(eu) + tr(ue) = 2tr(eu)$, which is even if n is even, and it may be odd if n is odd.

From above we conclude that if n is even, then elements of odd trace in $U(M(m, Z_n))$, are not Jordan regular units, which implies that $JRU(M(m, Z_n)) \neq U(M(m, Z_n))$. We also observe that the trace of a Jordan regular element in $M(n, R)$, where R is a commutative ring of characteristic zero, is even.

Lemma 2.1 [3, Lemma 2.10]**.** *If R is a commutative domain, then any nontrivial idempotent in $M(2, R)$ is of the form $\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ such that $a(1-a) = bc$.*

Proposition 2.6 [3, Proposition 2.8]**.** *No nonzero idempotent in a ring is Lie regular.*

Proposition 2.7. *If $\mathrm{Char}(R) = 2$, then no nonzero idempotent is a Jordan regular element.*

Proof. If $\mathrm{Char}(R) = 2$, then every Jordan regular element is a Lie regular element.

In $M_2(Z_6)$, for the unit $u = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ and the idempotent $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $eu + ue = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$ is an idempotent.

In $M(2, R)$, where R is an integral domain of characteristic 0, trace of every nontrivial idempotent is 1. So it can not be a Jordan regular element. But if $\mathrm{Char}(R) \neq 0, 2$, then nontrivial idempotents can be Jordan regular elements. For example, in $M(2, Z_3)$, for the unit $u = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and the idempotent $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $eu + ue = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is an idempotent.

Proposition 2.8. *If R is a commutative ring, then the product of two Jordan regular units in R is again a Jordan regular unit.*

Proof. Let x and y be two Jordan regular units in R. Then there exist idempotents e_1, e_2 and units u_1, u_2 in R such that $x = e_1u_1 + u_1e_1$ and $y = e_2u_2 + u_2e_2$. Therefore,

$$\begin{aligned} xy &= (e_1u_1 + u_1e_1)(e_2u_2 + u_2e_2) \\ &= e_1u_1(e_2u_2 + u_2e_2) + u_1e_1(e_2u_2 + u_2e_2) \end{aligned}$$

$$= e_1u_1(e_2u_2 + u_2e_2) + u_1(e_2u_2 + u_2e_2)e_1$$

$$= e_1(u_1y) + (u_1y)e_1$$

is a Jordan regular unit.

***Example* 2.4.** In $M(2, Z_4)$ which is not commutative, product of two Jordan regular units need not be a Jordan regular unit. Let $e_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}$. Both e_1 and e_2 are idempotents in $M(2, Z_4)$. Let $u_1 = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$. Both u_1 and u_2 are units in $M(2, Z_4)$. Now $J_1 = e_1u_1 + u_1e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ and $J_2 = e_2u_2 + u_2e_2 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, which implies that both J_1 and J_2 are Jordan regular units, but $J_1J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is not a Jordan regular unit as its trace is odd.

Proposition 2.9. *If R is a commutative ring, then the transpose of a Jordan regular element (unit) in $M(2, R)$ is again a Jordan regular element (unit).*

Proof. In $M(2, R)$, the transpose of an idempotent matrix is an idempotent matrix and the transpose of a unit matrix is a unit matrix, therefore the transpose of a Jordan regular element (unit) is also a Jordan regular element (unit) in $M(2, R)$.

Proposition 2.10. *If R is a commutative ring, then the inverse of a Jordan regular unit is again a Jodan regular unit in $M(2, R)$.*

Proof. Let $J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Jordan regular unit in $M(2, R)$. Then there exist an idempotent $e = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$ and a unit $u = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ in $M(2, R)$ such that $J = eu + ue$. Thus,

$$a = 2e_1u_1 + e_2u_3 + e_3u_2,$$

$$b = (e_1 + e_4)u_2 + (u_1 + u_4)e_2,$$

$$c = (e_1 + e_4)u_3 + (u_1 + u_4)e_3,$$

$$d = 2e_4u_4 + e_2u_3 + e_3u_2,$$

and $J^{-1} = k\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, where $k = (ad - bc)^{-1}$. Since e is an idempotent, so

$$e_1^2 + e_2e_3 = e_1, \qquad e_4^2 + e_2e_3 = e_4, \tag{1}$$

$$(e_1 + e_4)e_2 = e_2, \qquad (e_1 + e_4)e_3 = e_3. \tag{2}$$

Let $e' = \begin{pmatrix} e_4 & -e_2 \\ -e_3 & e_1 \end{pmatrix}$. Then it is easy to see that $e'^2 = e'$. Also, $u' = k\begin{pmatrix} u_4 & -u_2 \\ -u_3 & u_1 \end{pmatrix}$ is a unit in $M(2, R)$. Therefore,

$$e'u' + u'e' = k\begin{pmatrix} u_4e_4 + u_3e_2 & -u_2e_4 - u_1e_2 \\ -u_4e_3 - u_3e_1 & u_2e_3 + u_1e_1 \end{pmatrix} + k\begin{pmatrix} u_4e_4 + u_2e_3 & -u_4e_2 - u_2e_1 \\ -u_3e_4 - u_1e_3 & u_3e_2 + u_1e_1 \end{pmatrix}$$

$$= k\begin{pmatrix} 2e_4u_4 + e_2u_3 + e_3u_2 & -(e_1+e_4)u_2 - (u_1+u_4)e_2 \\ -(e_1+e_4)u_3 - (u_1+u_4)e_3 & 2e_1u_1 + e_2u_3 + e_3u_2 \end{pmatrix}$$

$$= k\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = J^{-1}.$$

Hence, the inverse of a Jordan regular unit is a Jordan regular unit in $M(2,R)$.

Theorem 2.2. *Let n be an even integer. Then the elements of the following forms in $U(M(2,Z_n))$:*

$$J_1 = \begin{pmatrix} 2l+1 & 2k' \\ 2k+1 & 2m+1 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 2l+1 & 2k+1 \\ 2k' & 2m+1 \end{pmatrix}, \qquad \text{and} \qquad J_3 = \begin{pmatrix} 2l & 2k'+1 \\ 2k+1 & 2m \end{pmatrix},$$

where $k,k',l,m \in \left\{0,1,2,\ldots,\dfrac{n}{2}-1,\dfrac{n}{2}\right\}$ and $2k+1$ is a unit in Z_n, are Jordan regular units.

Proof. By Proposition 2.9, it is sufficient to show that elements of the forms J_1 and J_3 in $U(M(2,Z_n))$, are Jordan regular units.

For an idempotent $e = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$ and a unit $u = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ in $M(2,Z_n),\ J_1 = eu + ue$ gives

$$2e_1u_1 + e_2u_3 + e_3u_2 = 2l+1, \tag{3}$$

$$2e_4u_4 + e_2u_3 + e_3u_2 = 2m+1, \tag{4}$$

$$(e_1+e_4)u_2 + (u_1+u_4)e_2 = 2k', \tag{5}$$

$$(e_1+e_4)u_3 + (u_1+u_4)e_3 = 2k+1. \tag{6}$$

Let $e_3 = e_4 = 0$ and $e_1 = 1$, then from (6), $u_3 = 2k+1$ is a unit in Z_n. Thus, by (4), $e_2 = (2m+1)(2k+1)^{-1}$. Then we can take $u_1 = l-m$, $u_4 = 2m+1$ and $u_2 = 2k' - (l+m+1)(2m+1)(2k+1)^{-1}$ as possible solutions of (3) and (5). Now $u_1u_4 - u_2u_3 = (l-m)(2m+1) - 2k'(2k+1) + (l+m+1)(2m+1) = (2l+1)(2m+1) - 2k'(2k+1)$ is a unit in Z_n. Therefore, J_1 is a Jordan regular unit in $M(2,Z_n)$.

Now for an idempotent $e = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$ and a unit $u = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ in $M(2,Z_n),\ J_3 = eu + ue$ gives

$$2e_1u_1 + e_2u_3 + e_3u_2 = 2l,$$

$$2e_4u_4 + e_2u_3 + e_3u_2 = 2m,$$

$$(e_1+e_4)u_2 + (u_1+u_4)e_2 = 2k'+1,$$

$$(e_1+e_4)u_3 + (u_1+u_4)e_3 = 2k+1.$$

Again let $e_3 = e_4 = 0$ and $e_1 = 1$. Then $u_3 = 2k+1,\ e_2 = 2m(2k+1)^{-1},\ u_1 = l-m,\ u_4 = 2m$ and $u_2 = 2k'+1-2m(l+m)(2k+1)^{-1}$ is a solution of the above set of equations and $u_1u_4 - u_2u_3$ is a unit in Z_n. Thus J_3 is a Jordan regular unit in $M(2,Z_n)$.

Theorem 2.2 is proved.

***Corollary* 2.2.** *The elements of the following forms in* $U(M(2, Z_{2^n}))$:

$$J_1 = \begin{pmatrix} 2l+1 & 2k' \\ 2k+1 & 2m+1 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 2l+1 & 2k+1 \\ 2k' & 2m+1 \end{pmatrix}, \qquad \textit{and} \qquad J_3 = \begin{pmatrix} 2l & 2k'+1 \\ 2k+1 & 2m \end{pmatrix},$$

where $k, k', l, m \in \{0, 1, 2, \ldots, 2^{n-1} - 1, 2^{n-1}\}$, *are Jordan regular units.*

***Example* 2.5.** In $U(M(2, Z_n))$, where $n = 2p$ and p is an odd prime, both p and $p+1$ are idempotents in Z_n. Let $J_1 = \begin{pmatrix} p+2r & 0 \\ p & p+2r \end{pmatrix}$ and $J_2 = \begin{pmatrix} p-2r & 0 \\ p & p-2r \end{pmatrix}$, where r is a natural number. Both J_1 and J_2 are units in $M(2, Z_n)$. Let $u_1 = \begin{pmatrix} p+r & p \\ p & r \end{pmatrix}$, $u_2 = \begin{pmatrix} p-r & p \\ p & -r \end{pmatrix}$, and $e = \begin{pmatrix} p+1 & p \\ 0 & 1 \end{pmatrix}$. Both u_1 and u_2 are units in $M(2, Z_n)$ and e is an idempotent in $M_2(Z_n)$ such that $J_1 = eu_1 + u_1e$ and $J_2 = eu_2 + u_2e$.

Theorem 2.3. *Elements of the form* $J = \begin{pmatrix} 2l+1 & 2k' \\ 2k & 2m+1 \end{pmatrix}$ *in* $M(2, Z_n)$, *where n is even, are not Jordan regular elements and, hence, not Jordan regular units. Here,* $k, k', l, m \in \left\{0, 1, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2}\right\}$.

Proof. Let J be a Jordan regular element in $M(2, Z_n)$. Then there exist an idempotent $e = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$ and a unit $u = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ in $M(2, Z_n)$ such that $J = eu + ue$. Therefore,

$$J_{11} = 2e_1u_1 + e_2u_3 + e_3u_2 = 2l + 1, \tag{7}$$

$$J_{22} = 2e_4u_4 + e_2u_3 + e_3u_2 = 2m + 1, \tag{8}$$

$$J_{12} = (e_1 + e_4)u_2 + (u_1 + u_4)e_2 = 2k', \tag{9}$$

$$J_{21} = (e_1 + e_4)u_3 + (u_1 + u_4)e_3 = 2k. \tag{10}$$

From (2), it is clear that if $e_1 + e_4$ is even, then both e_2 and e_3 are even. So from (7), we conclude that $e_1 + e_4$ is odd. Also, both e_2u_3 and e_3u_2 can not be even. Let e_2u_3 be odd. Then by (10), both $u_1 + u_4$ and e_3 are odd. So by (7), u_2 is even. But $u_1u_4 - u_2u_3$ is a unit in Z_n and, hence, it is odd. This yields that u_1u_4 is odd. But then $u_1 + u_4$ is even, which is a contradiction. A similar argument can be used if e_3u_2 is odd.

Theorem 2.3 is proved.

Proposition 2.11. *The product of elements of the form* $J = \begin{pmatrix} 2l+1 & 2k' \\ 2k & 2m+1 \end{pmatrix}$ *in* $M(2, Z_n)$, *where n is even, is also an element of the same form. Also, if it is a unit, then its inverse is also of the same form. Here,* $k, k', l, m \in \left\{0, 1, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2}\right\}$.

Proof. Let $A = \begin{pmatrix} 2l+1 & 2k' \\ 2k & 2m+1 \end{pmatrix}$ and $B = \begin{pmatrix} 2l'+1 & 2p' \\ 2p & 2m'+1 \end{pmatrix}$ in $M(2, Z_n)$, where k, k', l, l', $m, m', p, p' \in \left\{0, 1, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2}\right\}$. Then

$$AB = \begin{pmatrix} 2(2ll' + 2k'p + l + l') + 1 & 2(2lp' + 2k'm' + p' + k') \\ 2(2kl' + 2mp + p + k) & 2(2p'k + 2mm' + m + m') + 1 \end{pmatrix},$$

which is of the given form.

Now, let $A \in U(M(2, Z_n))$. Then $A^{-1} = (\det(A))^{-1} \begin{pmatrix} 2m+1 & -2k' \\ -2k & 2l+1 \end{pmatrix}$ is also of the given form.

Proposition 2.12. *Elements of the form* $\begin{pmatrix} 2l+1 & 2k' \\ 2k & 2m+1 \end{pmatrix}$ *in* $U(M(2, Z_n))$, *where* n *is even and* $k, k', l, m \in \left\{0, 1, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2}\right\}$, *form a subgroup of* $U(M(2, Z_n))$.

Proof. Let $\mathfrak{S}$ be the set of all elements of the form $\begin{pmatrix} 2l+1 & 2k' \\ 2k & 2m+1 \end{pmatrix}$ in $U(M(2, Z_n))$, where $k, k', l, m \in \left\{0, 1, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2}\right\}$. Then $I \in \mathfrak{S}$, so $\mathfrak{S} \neq \varnothing$. Let $A, B \in \mathfrak{S}$, then by Proposition 2.11 $AB \in \mathfrak{S}$ and $A^{-1} \in \mathfrak{S}$. Therefore, $\mathfrak{S}$ is a subgroup of $U(M(2, Z_n))$.

Proposition 2.13. *Let* n *be an even integer*, $\mathfrak{T}$ *denote the set of elements of odd trace in* $U(M(2, Z_n))$ *and* $\mathfrak{H} = \mathfrak{T} \cup \mathfrak{S}$. *Then* $\mathfrak{H}$ *is a subgroup of* $U(M(2, Z_n))$.

Proof. Elements of odd trace in $U(M(2, Z_n))$ are of the following forms: $\begin{pmatrix} 2l+1 & 2k'+1 \\ 2k+1 & 2m \end{pmatrix}$ or $\begin{pmatrix} 2l & 2k'+1 \\ 2k+1 & 2m+1 \end{pmatrix}$, where $k, k', l, m \in \left\{0, 1, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2}\right\}$. Clearly the product of two elements of $\mathfrak{T}$ is again an element of $\mathfrak{H}$ and the inverse of an element of $\mathfrak{T}$ is again an element of $\mathfrak{T}$. Also, the product of an element of $\mathfrak{T}$ and an element of $\mathfrak{S}$ is again an element of $\mathfrak{T}$. So, by the above proposition, it is clear that $\mathfrak{H}$ is a subgroup of $U(M(2, Z_n))$.

Proposition 2.14. *For even* n, *any element of* $\mathfrak{H}$ *is a non Jordan regular unit in* $U(M(2, Z_n))$.

Proof. This is an easy consequence of Proposition 2.5 and Theorem 2.3.

For even n, Jordan regular units do not form a group in $U(M(2, Z_n))$, as I is not a Jordan regular unit. Also, the product of two Jordan regular units need not be a Jordan regular unit. We also observe that any element of $U(M(2, Z_{2^n}))$, which is not in $\mathfrak{H}$, is a Jordan regular unit. Therefore, we can say that $\mathfrak{H}$ is the group of all non Jordan regular units in $U(M(2, Z_{2^n}))$.

Proposition 2.15 [3, Proposition 3.2]**.** *For any prime* p, *the order of the group* $U(M(2, Z_{p^n}))$ *is* $p^{2n-1}(p+1)\phi(p^n))^2$.

Proposition 2.16 [3, Proposition 3.3]**.** *For any* $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, *where* $p_i's$ *are distinct primes,* $|U(M(2, Z_n))| = \prod_{i=1}^{n} |U(M(2, Z_{p_i^{\alpha_i}}))|$.

***Corollary* 2.3** [3, Proposition 3.2]. *For any two distinct primes* p *and* q, $|U(M(2, Z_{pq}))| = pq(p+1)(q+1)\phi(pq)^2$.

Theorem 2.4. *The order of the group* $\mathfrak{H}$ *of non Jordan regular units in* $M(2, Z_{2^n})$ *is* $\frac{1}{2}|U(M(2, Z_{2^n}))|$.

Proof. The set $\mathfrak{S}$ of all elements of the form $\begin{pmatrix} 2l+1 & 2k' \\ 2k & 2m+1 \end{pmatrix}$ in $U(M(2, Z_{2^n}))$ is a subgroup of $U(M(2, Z_{2^n}))$, as proved in Proposition 2.12. Also, every element of such type in $M(2, Z_{2^n})$ is a unit. There are 2^{n-1} choices for each entry in such a matrix. Therefore, $|\mathfrak{S}| = (2^{n-1})^4$. The order of $2^n - 1$ in Z_{2^n} is 2, so the order of the matrix $G = \begin{pmatrix} 0 & 1 \\ 2^n-1 & 2^n-1 \end{pmatrix}$ is 3. Let $\mathfrak{G} = \langle G \rangle$. Then $\mathfrak{G} \le \mathfrak{H}$, as trace of G, is odd. Now $\mathfrak{G} \cap \mathfrak{S} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, therefore $|\mathfrak{G}\mathfrak{S}| = 3 \times (2^{n-1})^4$. Since $\mathfrak{G}\mathfrak{S} \subseteq \mathfrak{H} \le U(M(2, Z_{2^n}))$, so $|\mathfrak{G}\mathfrak{S}| \le |\mathfrak{H}| \le |U(M(2, Z_{2^n}))|$. Therefore, $3 \times 2^{4n-4} \le |\mathfrak{H}| \le 3 \times 2^{4n-3}$. But $\mathfrak{H}$ is a proper subgroup of $U(M(2, Z_{2^n}))$. Hence, $|\mathfrak{H}| = 3 \times 2^{4n-4} = \frac{1}{2}|U(M(2, Z_{2^n}))|$.

Theorem 2.4 is proved.

Proposition 2.17. *The order of the group* $\mathfrak{H}$ *in* $M(2, Z_6)$ *is half of the order of* $U(M(2, Z_6))$.

Proof. Let $A = \begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}$. Then $A, B \in \mathfrak{H}$, $o(A) = 2$ and $o(B) = 8$. If $\mathrm{H}_1 = \langle A, B \rangle$, then $|\mathrm{H}_1| = 16$. Elements of H_1 are listed below:

$$\left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 4 & 5 \end{pmatrix}, \pm\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \pm\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \pm\begin{pmatrix} 1 & 4 \\ 0 & 5 \end{pmatrix}, \pm\begin{pmatrix} 1 & 4 \\ 4 & 5 \end{pmatrix}, \pm\begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}, \pm\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}\right\}.$$

Let $C = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}$. Then $C, D \in \mathfrak{H}$, $o(C) = 3$ and $o(D) = 3$. If $\mathrm{H}_2 = \langle C, D \rangle$, then $|\mathrm{H}_2| = 9$. Elements of H_2 are listed below:

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 5 \\ 3 & 1 \end{pmatrix}\right\}.$$

H_2 is a subgroup of $\mathfrak{H}$ of order 9 and $\mathrm{H}_1 \cap \mathrm{H}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore, $|\mathrm{H}_1\mathrm{H}_2| = 144$. Now $\mathrm{H}_1\mathrm{H}_2 \subseteq \mathfrak{H} \le U(M(2, Z_6))$. Thus, $|\mathrm{H}_1\mathrm{H}_2| \le |\mathfrak{H}| \le |U(M(2, Z_6))|$. But $|U(M(2, Z_6))| = 288$, therefore $144 \le |\mathfrak{H}| \le 288$. Also, $\mathfrak{H}$ is a proper subgroup of $U(M(2, Z_6))$. So, $|\mathfrak{H}| = 144 = \frac{1}{2}|U(M(2, Z_6))|$.

Proposition 2.18. *The order of the group* $\mathfrak{H}$ *in* $M(2, Z_{10})$ *is half of the order of* $U(M(2, Z_{10}))$.

Proof. Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 \\ 6 & 1 \end{pmatrix}$. Then $A, B \in \mathfrak{H}$, $o(A) = 5$ and $o(B) = 3$. If $\mathrm{H}_1 = \langle A, B \rangle$, then $|\mathrm{H}_1| = 120$. Elements of H_1 are listed below:

$$\left\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix},\right.$$

$$\pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \pm\begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}, \pm\begin{pmatrix} 1 & 2 \\ 6 & 3 \end{pmatrix}, \pm\begin{pmatrix} 1 & 2 \\ 8 & 7 \end{pmatrix},$$

$$\pm\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix}, \pm\begin{pmatrix} 1 & 4 \\ 4 & 7 \end{pmatrix}, \pm\begin{pmatrix} 1 & 4 \\ 6 & 5 \end{pmatrix}, \pm\begin{pmatrix} 1 & 4 \\ 8 & 3 \end{pmatrix},$$

$$\pm\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 6 \\ 2 & 3 \end{pmatrix}, \pm\begin{pmatrix} 1 & 6 \\ 4 & 5 \end{pmatrix}, \pm\begin{pmatrix} 1 & 6 \\ 6 & 7 \end{pmatrix}, \pm\begin{pmatrix} 1 & 6 \\ 8 & 9 \end{pmatrix},$$

$$\pm\begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 8 \\ 2 & 7 \end{pmatrix}, \pm\begin{pmatrix} 1 & 8 \\ 4 & 3 \end{pmatrix}, \pm\begin{pmatrix} 1 & 8 \\ 6 & 9 \end{pmatrix}, \pm\begin{pmatrix} 1 & 8 \\ 8 & 5 \end{pmatrix},$$

$$\pm\begin{pmatrix}3 & 0\\ 0 & 7\end{pmatrix}, \pm\begin{pmatrix}3 & 0\\ 2 & 7\end{pmatrix}, \pm\begin{pmatrix}3 & 0\\ 4 & 7\end{pmatrix}, \pm\begin{pmatrix}3 & 0\\ 6 & 7\end{pmatrix}, \pm\begin{pmatrix}3 & 0\\ 8 & 7\end{pmatrix},$$

$$\pm\begin{pmatrix}3 & 2\\ 0 & 7\end{pmatrix}, \pm\begin{pmatrix}3 & 2\\ 2 & 5\end{pmatrix}, \pm\begin{pmatrix}3 & 2\\ 4 & 3\end{pmatrix}, \pm\begin{pmatrix}3 & 2\\ 6 & 1\end{pmatrix}, \pm\begin{pmatrix}3 & 2\\ 8 & 9\end{pmatrix},$$

$$\pm\begin{pmatrix}3 & 4\\ 0 & 7\end{pmatrix}, \pm\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}, \pm\begin{pmatrix}3 & 4\\ 4 & 9\end{pmatrix}, \pm\begin{pmatrix}3 & 4\\ 6 & 5\end{pmatrix}, \pm\begin{pmatrix}3 & 4\\ 8 & 1\end{pmatrix},$$

$$\pm\begin{pmatrix}3 & 6\\ 0 & 7\end{pmatrix}, \pm\begin{pmatrix}3 & 6\\ 2 & 1\end{pmatrix}, \pm\begin{pmatrix}3 & 6\\ 4 & 5\end{pmatrix}, \pm\begin{pmatrix}3 & 6\\ 6 & 9\end{pmatrix}, \pm\begin{pmatrix}3 & 6\\ 8 & 3\end{pmatrix},$$

$$\pm\begin{pmatrix}3 & 8\\ 0 & 7\end{pmatrix}, \pm\begin{pmatrix}3 & 8\\ 2 & 9\end{pmatrix}, \pm\begin{pmatrix}3 & 8\\ 4 & 1\end{pmatrix}, \pm\begin{pmatrix}3 & 8\\ 6 & 3\end{pmatrix}, \pm\begin{pmatrix}3 & 8\\ 8 & 5\end{pmatrix},$$

$$\pm\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}, \pm\begin{pmatrix}5 & 2\\ 2 & 3\end{pmatrix}, \pm\begin{pmatrix}5 & 2\\ 2 & 5\end{pmatrix}, \pm\begin{pmatrix}5 & 2\\ 2 & 7\end{pmatrix}, \pm\begin{pmatrix}5 & 2\\ 2 & 9\end{pmatrix},$$

$$\pm\begin{pmatrix}5 & 4\\ 6 & 1\end{pmatrix}, \pm\begin{pmatrix}5 & 4\\ 6 & 3\end{pmatrix}, \pm\begin{pmatrix}5 & 4\\ 6 & 5\end{pmatrix}, \pm\begin{pmatrix}5 & 4\\ 6 & 7\end{pmatrix}, \pm\begin{pmatrix}5 & 4\\ 6 & 9\end{pmatrix}\Bigg\}.$$

Let $C = \begin{pmatrix}2 & 5\\ 5 & 7\end{pmatrix}$. Then $C \in \mathfrak{H}$, $o(C) = 12$ and C is a central element of $\mathfrak{H}$. If $\mathrm{H}_2 = \langle C\rangle$, then $\mathrm{K} = \mathrm{H}_1\mathrm{H}_2$ is a subgroup of $\mathfrak{H}$ and $\mathrm{H}_1 \cap \mathrm{H}_2 = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}9 & 0\\ 0 & 9\end{pmatrix}\right\}$. Thus, $|\mathrm{K}| = 720$. Let $T = \begin{pmatrix}1 & 4\\ 4 & 9\end{pmatrix}$, then $T \in \mathfrak{H}$, $o(T) = 8$ and $T \notin \mathrm{K}$. Let $\mathrm{T} = \langle T\rangle$, then $\mathrm{K} \cap \mathrm{T} = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}, \begin{pmatrix}7 & 0\\ 0 & 7\end{pmatrix}, \begin{pmatrix}9 & 0\\ 0 & 9\end{pmatrix}\right\}$. Therefore, $|\mathrm{KT}| = 1440$. Also $\mathrm{KT} \subseteq \mathfrak{H} \leq U(M(2, Z_{10}))$. Hence, $|\mathrm{KT}| \leq |\mathfrak{H}| \leq |U(M(2, Z_{10}))|$. This implies that $1440 \leq |\mathfrak{H})| \leq 2880$. Also, $\mathfrak{H}$ is a proper subgroup of $U(M(2, (Z_{10}))$. Thus, $|\mathfrak{H}| = 1440 = \frac{1}{2}|U(M(2, Z_{10}))|$.

Proposition 2.19. *The order of the group* $\mathfrak{H}$ *in* $M(2, Z_{12})$ *is half of the order of* $U(M(2, Z_{12}))$.

Proof. Let

$$P = \begin{pmatrix}1 & 6\\ 10 & 11\end{pmatrix}, \qquad Q = \begin{pmatrix}3 & 4\\ 2 & 9\end{pmatrix}, \qquad R = \begin{pmatrix}3 & 4\\ 10 & 7\end{pmatrix},$$

$$S = \begin{pmatrix}3 & 10\\ 8 & 3\end{pmatrix}, \qquad T = \begin{pmatrix}5 & 0\\ 2 & 1\end{pmatrix}, \qquad U = \begin{pmatrix}5 & 10\\ 4 & 3\end{pmatrix}.$$

Then $P, Q, R, S, T, U \in \mathfrak{H}$, $o(P) = 2$, $o(Q) = 4$, $o(R) = 8$, $o(S) = 4$, $o(T) = 2$ and $o(U) = 8$. Let $\mathrm{H}_1 = \langle P, Q, R, S, T, U\rangle$. Then $|\mathrm{H}_1| = 256$ and elements of H_1 are listed below:

$$\Bigg\{\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\ 0 & 7\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\ 4 & 5\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\ 4 & 11\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\ 6 & 1\end{pmatrix},$$

$$\pm\begin{pmatrix}1&0\\6&7\end{pmatrix},\ \pm\begin{pmatrix}1&0\\10&5\end{pmatrix},\ \pm\begin{pmatrix}1&0\\10&11\end{pmatrix},\ \pm\begin{pmatrix}1&2\\2&3\end{pmatrix},\ \pm\begin{pmatrix}1&2\\2&5\end{pmatrix},$$

$$\pm\begin{pmatrix}1&2\\2&9\end{pmatrix},\ \pm\begin{pmatrix}1&2\\2&11\end{pmatrix},\ \pm\begin{pmatrix}1&2\\8&3\end{pmatrix},\ \pm\begin{pmatrix}1&2\\8&5\end{pmatrix},\ \pm\begin{pmatrix}1&2\\8&9\end{pmatrix},$$

$$\pm\begin{pmatrix}1&2\\8&11\end{pmatrix},\ \pm\begin{pmatrix}1&4\\0&5\end{pmatrix},\ \pm\begin{pmatrix}1&4\\0&11\end{pmatrix},\ \pm\begin{pmatrix}1&4\\4&5\end{pmatrix},\ \pm\begin{pmatrix}1&4\\4&11\end{pmatrix},$$

$$\pm\begin{pmatrix}1&4\\6&5\end{pmatrix},\ \pm\begin{pmatrix}1&4\\6&11\end{pmatrix},\ \pm\begin{pmatrix}1&4\\10&5\end{pmatrix},\ \pm\begin{pmatrix}1&4\\10&11\end{pmatrix},\ \pm\begin{pmatrix}1&6\\0&1\end{pmatrix},$$

$$\pm\begin{pmatrix}1&6\\0&7\end{pmatrix},\ \pm\begin{pmatrix}1&6\\4&5\end{pmatrix},\ \pm\begin{pmatrix}1&6\\4&11\end{pmatrix},\ \pm\begin{pmatrix}1&6\\6&1\end{pmatrix},\ \pm\begin{pmatrix}1&6\\6&7\end{pmatrix},$$

$$\pm\begin{pmatrix}1&6\\10&5\end{pmatrix},\ \pm\begin{pmatrix}1&6\\10&11\end{pmatrix},\ \pm\begin{pmatrix}1&8\\2&3\end{pmatrix},\ \pm\begin{pmatrix}1&8\\2&5\end{pmatrix},\ \pm\begin{pmatrix}1&8\\2&9\end{pmatrix},$$

$$\pm\begin{pmatrix}1&8\\2&11\end{pmatrix},\ \pm\begin{pmatrix}1&8\\8&3\end{pmatrix},\ \pm\begin{pmatrix}1&8\\8&5\end{pmatrix},\ \pm\begin{pmatrix}1&8\\8&9\end{pmatrix},\ \pm\begin{pmatrix}1&8\\8&11\end{pmatrix},$$

$$\pm\begin{pmatrix}1&10\\0&5\end{pmatrix},\ \pm\begin{pmatrix}1&10\\0&11\end{pmatrix},\ \pm\begin{pmatrix}1&10\\4&5\end{pmatrix},\ \pm\begin{pmatrix}1&10\\4&11\end{pmatrix},\ \pm\begin{pmatrix}1&10\\6&5\end{pmatrix},$$

$$\pm\begin{pmatrix}1&10\\6&11\end{pmatrix},\ \pm\begin{pmatrix}1&10\\10&5\end{pmatrix},\ \pm\begin{pmatrix}1&10\\10&11\end{pmatrix},\ \pm\begin{pmatrix}3&2\\2&5\end{pmatrix},\ \pm\begin{pmatrix}3&2\\2&11\end{pmatrix},$$

$$\pm\begin{pmatrix}3&2\\4&3\end{pmatrix},\ \pm\begin{pmatrix}3&2\\4&9\end{pmatrix},\ \pm\begin{pmatrix}3&2\\8&5\end{pmatrix},\ \pm\begin{pmatrix}3&2\\8&11\end{pmatrix},\ \pm\begin{pmatrix}3&2\\10&3\end{pmatrix},$$

$$\pm\begin{pmatrix}3&2\\10&9\end{pmatrix},\ \pm\begin{pmatrix}3&4\\2&3\end{pmatrix},\ \pm\begin{pmatrix}3&4\\2&9\end{pmatrix},\ \pm\begin{pmatrix}3&4\\4&1\end{pmatrix},\ \pm\begin{pmatrix}3&4\\4&7\end{pmatrix},$$

$$\pm\begin{pmatrix}3&4\\8&3\end{pmatrix},\ \pm\begin{pmatrix}3&4\\8&9\end{pmatrix},\ \pm\begin{pmatrix}3&4\\10&1\end{pmatrix},\ \pm\begin{pmatrix}3&4\\10&7\end{pmatrix},\ \pm\begin{pmatrix}3&8\\2&5\end{pmatrix},$$

$$\pm\begin{pmatrix}3&8\\2&11\end{pmatrix},\ \pm\begin{pmatrix}3&8\\4&3\end{pmatrix},\ \pm\begin{pmatrix}3&8\\4&9\end{pmatrix},\ \pm\begin{pmatrix}3&8\\8&5\end{pmatrix},\ \pm\begin{pmatrix}3&8\\8&11\end{pmatrix},$$

$$\pm\begin{pmatrix}3&8\\10&3\end{pmatrix},\ \pm\begin{pmatrix}3&8\\10&9\end{pmatrix},\ \pm\begin{pmatrix}3&10\\2&3\end{pmatrix},\ \pm\begin{pmatrix}3&10\\2&9\end{pmatrix},\ \pm\begin{pmatrix}3&10\\4&1\end{pmatrix},$$

$$\pm\begin{pmatrix}3&10\\4&7\end{pmatrix},\ \pm\begin{pmatrix}3&10\\8&3\end{pmatrix},\ \pm\begin{pmatrix}3&10\\8&9\end{pmatrix},\ \pm\begin{pmatrix}3&10\\10&1\end{pmatrix},\ \pm\begin{pmatrix}3&10\\10&7\end{pmatrix},$$
$$\pm\begin{pmatrix}5&0\\0&5\end{pmatrix},\ \pm\begin{pmatrix}5&0\\0&11\end{pmatrix},\ \pm\begin{pmatrix}5&0\\2&1\end{pmatrix},\ \pm\begin{pmatrix}5&0\\2&7\end{pmatrix},\ \pm\begin{pmatrix}5&0\\6&5\end{pmatrix},$$
$$\pm\begin{pmatrix}5&0\\6&11\end{pmatrix},\ \pm\begin{pmatrix}5&0\\8&1\end{pmatrix},\ \pm\begin{pmatrix}5&0\\8&7\end{pmatrix},\ \pm\begin{pmatrix}5&2\\0&1\end{pmatrix},\ \pm\begin{pmatrix}5&2\\0&7\end{pmatrix},$$
$$\pm\begin{pmatrix}5&2\\2&1\end{pmatrix},\ \pm\begin{pmatrix}5&2\\2&7\end{pmatrix},\ \pm\begin{pmatrix}5&2\\6&1\end{pmatrix},\ \pm\begin{pmatrix}5&2\\6&7\end{pmatrix},\ \pm\begin{pmatrix}5&2\\8&1\end{pmatrix},$$
$$\pm\begin{pmatrix}5&2\\8&7\end{pmatrix},\ \pm\begin{pmatrix}5&4\\4&1\end{pmatrix},\ \pm\begin{pmatrix}5&4\\4&3\end{pmatrix},\ \pm\begin{pmatrix}5&4\\4&7\end{pmatrix},\ \pm\begin{pmatrix}5&4\\4&9\end{pmatrix},$$
$$\pm\begin{pmatrix}5&4\\10&1\end{pmatrix},\ \pm\begin{pmatrix}5&4\\10&3\end{pmatrix},\ \pm\begin{pmatrix}5&4\\10&7\end{pmatrix},\ \pm\begin{pmatrix}5&4\\10&9\end{pmatrix},\ \pm\begin{pmatrix}5&6\\0&5\end{pmatrix},$$
$$\pm\begin{pmatrix}5&6\\0&11\end{pmatrix},\ \pm\begin{pmatrix}5&6\\2&1\end{pmatrix},\ \pm\begin{pmatrix}5&6\\2&7\end{pmatrix},\ \pm\begin{pmatrix}5&6\\6&5\end{pmatrix},\ \pm\begin{pmatrix}5&6\\6&11\end{pmatrix},$$
$$\pm\begin{pmatrix}5&6\\8&1\end{pmatrix},\ \pm\begin{pmatrix}5&6\\8&7\end{pmatrix},\ \pm\begin{pmatrix}5&8\\0&1\end{pmatrix},\ \pm\begin{pmatrix}5&8\\0&7\end{pmatrix},\ \pm\begin{pmatrix}5&8\\2&1\end{pmatrix},$$
$$\pm\begin{pmatrix}5&8\\2&7\end{pmatrix},\ \pm\begin{pmatrix}5&8\\6&1\end{pmatrix},\ \pm\begin{pmatrix}5&8\\6&7\end{pmatrix},\ \pm\begin{pmatrix}5&8\\8&1\end{pmatrix},\ \pm\begin{pmatrix}5&8\\8&7\end{pmatrix},$$
$$\pm\begin{pmatrix}5&10\\4&1\end{pmatrix},\ \pm\begin{pmatrix}5&10\\4&3\end{pmatrix},\ \pm\begin{pmatrix}5&10\\4&7\end{pmatrix},\ \pm\begin{pmatrix}5&10\\4&9\end{pmatrix},$$
$$\left.\pm\begin{pmatrix}5&10\\10&1\end{pmatrix},\ \pm\begin{pmatrix}5&10\\10&3\end{pmatrix},\ \pm\begin{pmatrix}5&10\\10&7\end{pmatrix},\ \pm\begin{pmatrix}5&10\\10&9\end{pmatrix}\right\}.$$

Let $X=\begin{pmatrix}1&4\\0&1\end{pmatrix}, Y=\begin{pmatrix}10&3\\3&1\end{pmatrix}$. Then $X,Y\in\mathfrak{H}$, $o(X)=o(Y)=3$. Let $\mathrm{H}_2=\langle X,Y\rangle$. Then $|\mathrm{H}_2|=9$. Elements of H_2 are listed below:

$$\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \begin{pmatrix}1&1\\9&10\end{pmatrix},\ \begin{pmatrix}1&4\\0&1\end{pmatrix},\ \begin{pmatrix}1&5\\9&10\end{pmatrix},\ \begin{pmatrix}1&8\\0&1\end{pmatrix},\begin{pmatrix}1&9\\9&10\end{pmatrix},\ \begin{pmatrix}10&3\\3&1\end{pmatrix},\ \begin{pmatrix}10&7\\3&1\end{pmatrix},\ \begin{pmatrix}10&11\\3&1\end{pmatrix}\right\}.$$

Now $|\mathrm{H}_1\mathrm{H}_2|=2304$. Also $\mathrm{H}_1\mathrm{H}_2\subseteq\mathfrak{H}\le U(M(2,Z_{12}))$. Therefore, $2304\le|\mathfrak{H}|\le 4608$. Since $\mathfrak{H}$ is a proper subgroup of $U(M(2,Z_{12}))$, so $|\mathfrak{H}|=2304=\dfrac{1}{2}|U(M(2,Z_{12}))|$.

3. Jordan regular units over field of characteristic 2. Throughout this section, F denotes a finite field of characteristic 2 containing 2^n elements.

Theorem 3.1. *In $GL(2,F)$, every non scalar matrix having same diagonal elements is a Jordan regular unit.*

Proof. Let $A \in M(2,F)$ be a Jordan regular element. Then there exist a unit matrix $u = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ and an idempotent matrix $e = \begin{pmatrix} e_1 & e_2 \\ e_3 & 1-e_1 \end{pmatrix}$ in $M_2(F)$ such that $A = eu + ue$. Thus,

$$A = \begin{pmatrix} 2e_1u_1 + e_2u_3 + e_3u_2 & (u_1+u_4)e_2 + u_2 \\ (u_1+u_4)e_3 + u_3 & e_2u_3 + e_3u_2 + 2e_1u_4 \end{pmatrix} = \begin{pmatrix} e_2u_3 + e_3u_2 & (u_1+u_4)e_2 + u_2 \\ (u_1+u_4)e_3 + u_3 & e_2u_3 + e_3u_2 \end{pmatrix}.$$

Hence, the diagonal elements of a Jordan regular element in $M(2,F)$ are same. Now suppose that $A \in GL(2,F)$ such that the diagonal elements of A are same. Since $F^* = F \setminus \{0\}$ is a cyclic group, let $F^* = \langle \alpha \rangle$.

***Case* I.** If both the diagonal elements of A are zero, then $A = \begin{pmatrix} 0 & \alpha^i \\ \alpha^j & 0 \end{pmatrix}$. For the unit $u = \begin{pmatrix} 0 & \alpha^i \\ \alpha^j & 0 \end{pmatrix}$, and the idempotent $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A = eu + ue$.

***Case* II.** If both the diagonal elements of A are non-zero, then $A = \begin{pmatrix} \alpha^i & \alpha^j \\ \alpha^k & \alpha^i \end{pmatrix}$ such that $2i \neq (j+k) \mod (2)$. First let $j + k \neq 2^n - 1 \mod (2)$. Then, for the unit $u = \begin{pmatrix} 1 & \alpha^j \\ \alpha^k & 1 \end{pmatrix}$ and the idempotent $e = \begin{pmatrix} 1 & \alpha^{i-k} \\ 0 & 0 \end{pmatrix}$, $A = eu + ue$. If $j + k = 2^n - 1$, then $\alpha^{2i} \neq 1$ and, for the unit $u = \begin{pmatrix} 1 & \alpha^j - \alpha^{i-k} \\ \alpha^k & 0 \end{pmatrix}$ and the idempotent $e = \begin{pmatrix} 1 & \alpha^{i-k} \\ 0 & 0 \end{pmatrix}$, $A = eu + ue$.

***Case* III.** If $A = \begin{pmatrix} \alpha^i & \alpha^j \\ 0 & \alpha^i \end{pmatrix}$, then, for the unit $u = \begin{pmatrix} 1 & \alpha^j \\ 0 & 1 \end{pmatrix}$ and the idempotent $e = \begin{pmatrix} 1 & 0 \\ \alpha^{i-j} & 0 \end{pmatrix}$, $A = eu + ue$.

***Case* IV.** If $A = \begin{pmatrix} \alpha^i & 0 \\ \alpha^k & \alpha^i \end{pmatrix}$, then, by Proposition 2.9 and Case III, A is a Jordan regular unit.

Theorem 3.1 is proved.

Proposition 3.1. *Scalar matrices in $GL(2,F)$ are not Jordan regular units.*

Proof. Let $F^* = F \setminus \{0\} = \langle \alpha \rangle$ and $A = \begin{pmatrix} \alpha^i & 0 \\ 0 & \alpha^i \end{pmatrix} \in GL(2,F)$ be a Jordan regular unit. Then there exist a unit matrix $u = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ and an idempotent matrix $e = \begin{pmatrix} e_1 & e_2 \\ e_3 & 1-e_1 \end{pmatrix}$ in $M(2,F)$ such that $A = eu + ue$, This yields

$$e_2u_3 + e_3u_2 = \alpha^i, \tag{11}$$

$$(u_1+u_4)e_2 + u_2 = 0, \tag{12}$$

$$(u_1+u_4)e_3 + u_3 = 0. \tag{13}$$

From (12) and (13), $u_2 = (u_1 + u_4)e_2,\ u_3 = (u_1 + u_4)e_3$. Substituting these in (11), we get $\alpha^i = 2(u_1 + u_4)e_2e_3 = 0$, which is a contradiction. Hence, scalar matrices are not Jordan regular units.

Also, $\begin{pmatrix}\alpha^2 & \alpha^2\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & \alpha^2\\ 0 & \alpha^2\end{pmatrix} = \begin{pmatrix}\alpha^2 & 0\\ 1 & \alpha^2\end{pmatrix}$, which shows that the product of two non Jordan regular units in $GL(2,F)$, is a Jordan regular unit. Therefore, non Jordan regular units in $GL(2,F)$ do not form a group.

Proposition 3.2. *The image of a Jordan regular element* (*unit*) *under an isomorphism of rings a Jordan regular element* (*unit*).

Proof. As the image of an idempotent is an idempotent and the image of a unit is a unit under isomorphism of rings, so the image of a Jordan regular element (unit) is a Jordan regular element (unit).

Proposition 3.3. *Let G be a group. The group algebra KG of the group G over a field K of characteristic 2 has no non-zero Jordan regular element and therefore $JRU(KG) = \varnothing$.*

Proof. The augmentation map, $\varepsilon\colon KG \to K$ given by $\varepsilon\left(\sum_{g\in G}\alpha_g g\right) = \sum_{g\in G}\alpha_g$, is an epimorphism of rings. Let $x \in KG$ be a Jordan regular element, then $x = eu+ue$ for some idempotent e and some unit u in KG. Then $\varepsilon(x) = \varepsilon(eu + ue) = \varepsilon(eu) + \varepsilon(ue) = \varepsilon(e)\varepsilon(u) + \varepsilon(u)\varepsilon(e) = 2(\varepsilon(e)\varepsilon(u)) = 0$ and so KG has does not have Jordan regular units.

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Received 15.11.19