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STOCHASTIC NAVIER – STOKES VARIATIONAL INEQUALITIES WITH UNILATERAL BOUNDARY CONDITIONS: PROBABILISTIC WEAK SOLVABILITY

СТОХАСТИЧНІ ВАРІАЦІЙНІ НЕРІВНОСТІ НАВ'Є – СТОКСА З ОДНОСТОРОННІМИ ГРАНИЧНИМИ УМОВАМИ: ЙМОВІРНІСНА СЛАБКА РОЗВ'ЯЗНІСТЬ

We initiate the investigation of stochastic Navier – Stokes variational inequalities involving unilateral boundary conditions and nonlinear forcings driven by Wiener processes for which we establish the existence of a probabilistic weak (or martingale) solution. Our approach involves an intermediate penalized problem whose weak solution is obtained by means of Galerkin's method in combination with some analytic and probabilistic compactness results. The required probabilistic weak solution of the stochastic Navier – Stokes variational inequality is consecutively obtained through the limit transition in the penalized problem. The main result is new for stochastic Navier – Stokes variational inequalities. It is a stochastic counterpart of the work of Brezis on deterministic Navier – Stokes variational inequalities and generalizes several previous results on stochastic Navier-Stokes equations to stochastic Navier – Stokes variational inequalities with unilateral boundary conditions.

У цій статті розпочато вивчення стохастичних варіаційних нерівностей Нав'є – Стокса з односторонніми граничними умовами та нелінійними впливами, що викликані вінеровськими процесами, для яких встановлюється існування ймовірнісного слабкого (або мартингального) розв'язку. Наш підхід включає проміжну штрафну задачу, слабкий розв'язок якої отримано за допомогою методу Гальоркіна в поєднанні з деякими аналітичними та ймовірнісними результатами щодо компактності. Шуканий імовірнісний слабкий розв'язок стохастичної варіаційної нерівності Нав'є – Стокса одержано в результаті граничного переходу в штрафній задачі. Отриманий основний результат є новим для стохастичних варіаційних нерівностей Нав'є – Стокса. Він є стохастичним аналогом роботи Брезіса щодо детермінованих варіаційних нерівностей Нав'є – Стокса та узагальнює кілька попередніх результатів, отриманих для стохастичних рівнянь Нав'є – Стокса, на випадок стохастичних варіаційних нерівностей Нав'є – Стокса з односторонніми граничними умовами.

1. Introductory background. Let D be a simply connected domain bounded in $\mathbb{R}^d (d = 3)$ with a sufficiently smooth boundary ∂D (at least C^2). We fix a final time $T > 0$ and denote by Q_T the cylindrical domain $(0,T) \times D$. Given a convex, lower semicontinuous function

$$\varphi : \mathbb{R}{\times}\partial D \to (-\infty, \infty],$$

and letting $u_n = u \cdot n$ be the normal component of u; n being the outward unit normal vector field to ∂D, we are interested in an hydrodynamical problem for the motion of a fluid under random fluctuations governed by the following subdifferential initial boundary value problem for the incompressible stochastic Navier – Stokes equations

$$du + \big(\nabla \times u \times u - \nu \Delta u + \nabla P\big)dt = f(t,u)dt + g(t,u)dW \quad \text{in} \quad Q_T, \tag{1}$$

$$\nabla \cdot u = 0 \quad \text{in} \quad Q_T, \tag{2}$$

$$u_\tau = 0 \quad \text{on} \quad (0,T) \times \partial D, \tag{3}$$

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$$u(0) = u_0 \quad \text{in} \quad D, \tag{4}$$

$$P(t,x) \in \partial\varphi\big(u_n(t,x),x\big) \quad \text{for all} \quad (t,x) \in (0,T)\times\partial D, \tag{5}$$

where u is the velocity of the particles of fluid, ∇ denotes the gradient operator in $\mathbb{R}^d$, $u_\tau = u-(u\cdot n)n$ is the tangential component of u, $P = p+|u|^2/2$, the total or Bernouli's pressure (p being the usual pressure), W a l-dimensional Wiener process and the right-hand side of (1) represents the force acting on the fluid and consisting of a regular part involving the 3-vector function f and a chaotic part involving the $3\times l$ matrix function g and W, the parameter ν is the viscosity of the fluid, $\partial\varphi$ denotes the subdifferential of φ, namely,

$$\partial\varphi(v,x) = \big\{w \in \mathbb{R} : \varphi(\tilde{v},x) - \varphi(v,x) \geq w(\tilde{v}-v) \text{ for all } \tilde{v} \in \mathbb{R}\big\}.$$

As example of occurrence of such a problem, we consider the in and out flow of a fluid in a domain under the following setting considered in [11, 14]. Let ∂D be partitioned in three subsets Γ_0, Γ_1 and Γ_2, with disjoint interiors. Γ_0 is the impermeable part, Γ_1 is the part through which the fluid flows into D and Γ_2 the part through which the fluid flows out of D. Choosing the function φ on $\mathbb{R}\times\partial D$ as

$$\varphi(v,x) = \begin{cases} 0, & \text{if} \quad v = 0, \quad x \in \Gamma_0, \\ v\psi(x), & \text{if either} \quad v \leq 0, \quad x \in \Gamma_1 \quad \text{or} \quad v \geq 0, \quad x \in \Gamma_2, \\ \infty & \text{otherwise,} \end{cases}$$

where ψ is a given function on ∂D, and evaluating its subdifferential, it turns out that the condition (5) is equivalent to the following unilateral boundary conditions:

$$u_n(t,x) \leq 0, \qquad P(t,x) \geq \psi(x), \qquad u_n(t,x)\big(P(t,x)-\psi(x)\big) = 0 \quad \text{on} \quad (0,T)\times\Gamma_1,$$

$$u_n(t,x) \geq 0, \qquad P(t,x) \leq \psi(x), \qquad u_n(t,x)\big(P(t,x)-\psi(x)\big) = 0 \quad \text{on} \quad (0,T)\times\Gamma_2,$$

$$u_n(t,x) = 0 \quad \text{on} \quad (0,T)\times\Gamma_0.$$

Other types of physical problems involving nonlocal conditions were also considered in [11, 14] (see also [12]) and could be modelled in terms of problem (1)–(5).

The natural framework for the study of problem (1)–(5) is that of stochastic variational partial differential inequalities, an area of research in the field of stochastic analysis which was pioneered by Rascanu in [23, 24] and further developed jointly with his coworkers (see, for instance, [6]). The theory of deterministic partial differential variational inequalities originated in the work of Signorini and the milestones in its development could be credited to the pioneering work of Lions and Stampacchia [17] and the seminal contributions of Brezis in [8] and his thesis [9], followed by large number of important works, such as Duvaut and Lions [13], Bensoussan and Lions [5] and the celebrated monograph by Lions [15].

In view of the prevalence of unilateral boundary conditions such as the ones that we discussed earlier in many important applied problems, Brezis was led to study the variational inequality for Navier–Stokes equations in [10] and thanks to the theory that he developed in [9], he established an existence result for the corresponding problem. A thorough review of the existing literature in the field of stochastic partial differential equations has revealed that the stochastic counterpart of Brezis's work has not been previously undertaken. Owing to the importance of stochastic Navier – Stokes equations in applied sciences (for instance, in the investigation of the still elusive turbulence phenomenon in fluid dynamics), we were motivated to initiate in the present work the investigation of stochastic Navier – Stokes variational inequalities. The above physically relevant unilateral boundary conditions are inspired from the corresponding deterministic cases, considered by Antontsev, Kazhikov and Monakhov in [1] (see also [16]) and Konovalova in [14] who extended Brezis's result to models that include density dependent Navier – Stokes equations.

Our main result is the existence of a probabilistic weak (or martingale) solution for the variational inequality version of problem (1) – (5), under appropriate conditions on the data. Our work extends previous results on martingale solutions for stochastic Navier – Stokes equations to a variational inequality setting, in particular the work of Bensoussan [4]. The study of stochastic Navier – Stokes equations was pioneered by Bensoussan and Temam [7] and has over the years been greatly enriched by important contributions of Vishik and his students [30], Mikulevicius and Rozovskii [19, 20]; just to cite a few. While there has been some previous work involving some aspects of stochastic Navier – Stokes variational inequalities, mainly for the purpose of control problems, for instance, in [2, 18], none addresses the problematic of establishing a martingale solution as investigated in the present work. Our main result is therefore new, to the best of our knowledge. The work can be generalized to cases of random forces driven by Levy processes as well as fractional Brownian motions as expounded in the monograph [21]. The present work also opens avenues to the study of variational inequalities for several important classes of stochastic models of fluids (magnetohydrodynamics, second-grade fluids, density-dependent and compressible fluids) involving unilateral boundary conditions.

This paper is organized as follows. In Section 2, we introduce needed semantics and give a variational inequality formulation of problem (1) – (5). We also introduce the definition of probabilistic weak (martingale) solutions for our problem, state the conditions on the data and formulate our main result. In Section 3, we introduce the penalized (approximation) version of the variational inequality and establish for it the existence of a martingale solution thanks to Galerkin method combined with several compactness results of analytic and probabilistic (Prokhorov and Skorokhod) nature. The closing Section 4 is devoted to the proof of our main result by means of compactness arguments and a passage to the limit in the penalized problem.

2. Variational inequality formulation and statement of main result. We start with the introduction of needed function spaces.

Let $\mathcal{D}(D)$ be the space of C^∞ functions compactly supported in D. For $1 \leq r \leq \infty$, l a nonnegative integer, we define the Sobolev spaces

$$W^l_r(D) = \big\{ v \in L^r(D) \colon D^\alpha v \in (L^r(D))^3 \text{ for } |\alpha| \leq l \big\},$$

$D^\alpha = D_1^{\alpha_1} \ldots D_3^{\alpha_3}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $D_i = \partial/\partial x_i$. We shall denote for simplicity $W^l_2(D)$ by $H^l(D)$. These spaces are endowed with their respective usual norms.

Next let

$$\mathcal{V} = \big\{ v \in \big(\mathcal{D}(D)\big)^3 \colon \nabla \cdot u = 0 \text{ in } D, v_\tau = 0 \text{ on } \partial D \big\}.$$

Denote by V the closure of $\mathcal{V}$ in $\big(H^1(D)\big)^3$ and by H the closure of $\mathcal{V}$ in $\big(L^2(D)\big)^3$. V and H are Hilbert spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_H$, respectively. We denote the Euclidean norm by $|\cdot|$.

In view of the Lipschitzity of the boundary of D the following characterization of V and H hold:

$$V = \Big\{ v \in \big(H^1(D)\big)^3 : \nabla \cdot v = 0 \text{ in } D,\ v_\tau = 0 \text{ on } \partial D \Big\}$$

and

$$H = \Big\{ v \in \big(L^2(D)\big)^3 : \nabla \cdot v = 0 \text{ in } D,\ v_\tau|_{\partial D} = 0 \Big\},$$

where $\cdot|_{\partial D}$ denotes the trace of $\cdot$ on ∂D and $v_\tau|_{\partial D}$ is defined as

$$v_\tau|_{\partial D} = v|_{\partial D} - \big(v|_{\partial D} \cdot n\big) n.$$

The inner product in H is induced by the inner product $(\cdot,\cdot)$ in $L^2(D)$. We denote by $\langle \cdot,\cdot \rangle$ the duality paring between V and V' the dual of V. V is dense in H and in view of Sobolev's embedding theorem, V is compactly embedded in H (see [14]).

We also need appropriate deterministic and probabilistic evolution spaces. For $1 \le q \le \infty$, a Banach space X, and an interval $I \subset \mathbb{R}$, $L^q(I;X)$ denote the usual Bochner space of functions defined on I with values in X endowed with the norm

$$\|v\|_{L^q(I;X)} = \begin{cases} \left(\displaystyle\int_I \|v(t)\|_X^q dt \right)^{1/q}, & \text{if} \quad q < \infty, \\ \operatorname{ess\,sup}_{t \in I} \|v(t)\|_X, & \text{if} \quad q = \infty. \end{cases}$$

We use the standard notation $L^q(a,b;X)$ for $I = [a,b]$. Let X be a Banach space, $\big(\Omega, \mathcal{F}, (F_t)_{t \in I}, P\big)$ a complete probability space with a filtration $(F_t)_{t \in I}$ $(I \subset \mathbb{R})$, and the numbers $p \in [1,\infty)$, $q \in [1,\infty]$; $L^p\big(\Omega; L^q(I;X)\big)$ denotes the space of progressively measurable processes endowed with the norm $\|u\|_{L^p(\Omega,F,P,L^q(I;X))} = \Big(E\|u\|^p_{L^q(I;X)} \Big)^{1/p}$, where E denotes the mathematical expectation with respect to the probability measure P.

Instead of problem (1)–(5), we consider the following more general stochastic Navier–Stokes variational inequality:

$$\begin{aligned} &f\big(t,u(t)\big)dt + g\big(t,u(t)\big)dW(t) \\ &\qquad \in du + A(u(t))dt + B\big(u(t)\big)dt + \partial\varphi\big(u(t)\big)dt \quad \text{for} \quad t \in (0,T), \end{aligned} \tag{6}$$

$$u(0) = u_0, \tag{7}$$

with non random initial value $u_0 \in H$, where $T > 0$, and we assume the following conditions on the data.

Let H and V denote some separable Hilbert spaces, such that V is dense in H, the embedding of V in H is compact and $V \subset H = H' \subset V'$, where H' and V' denote the dual spaces of H and V, respectively.

(i) $A : V \to V'$ is a linear continuous, symmetric operator such that, for some positive constants ν and α,

$$\langle A\phi, \phi \rangle \ge \nu \|\phi\|_V^2 + \alpha \|\phi\|_H^2 \quad \forall \phi \in V; \tag{8}$$

A plays the role of a generalized Stokes's operator.

(ii) $B\colon V\times V\to V'$ is a bilinear operator such that

$$\big\langle B(\phi,\psi),\psi\big\rangle=0 \tag{9}$$

for any $\phi,\psi\in V$. Furthermore, there exist the constants $\theta\in(0,1)$ and $C>0$ independent of $\phi,\psi\in V$ such that

$$\big|\langle B(\phi,\psi),\phi\rangle\big|\le C\|\phi\|_V^{1+\theta}\|\psi\|_H^{1-\theta}\|\phi\|_V. \tag{10}$$

We set $B(\phi)=B(\phi,\phi)$ and we assume that $B(\phi)\colon V\to V'$ is weakly continuous. Note that the estimate (10) holds for

$$\big\langle B(\phi,\psi),\chi\big\rangle=\int\limits_D(\nabla\times\phi\times\psi)\chi dx\quad\forall(\phi,\psi,\chi)\in V\times H\times V,$$

which is the case of interest to us.

(iii) $\varphi\colon V\to(-\infty,\infty]$ is a proper convex lower semicontinuous function with $\partial\varphi$ its subdifferential, $\varphi(v)\ge 0$ for all $v\in V$ and $\varphi(0)=0$.

(iv) We assume that $f:[0,T]\times H\to H$ is a nonlinear mapping continuous in both its variables and such that there exists a positive constant C such that

$$\|f(t,v)\|_H\le C\big(1+\|v\|_H\big)\quad\forall t\in[0,T]. \tag{11}$$

(v) We assume that $g\colon[0,T]\times H\to H^{\times l}$ is a nonlinear mapping continuous in both its variables and such that there exists a positive constant C such that

$$\|g(t,v)\|_{H^{\times l}}\le C\big(1+\|v\|_H\big)\quad\forall t\in[0,T], \tag{12}$$

where $H^{\times l}$ denotes the product of l copies of the space H.

Following [6, 23, 24], we consider the set $\mathcal{P}_{pr}$ of all progressively measurable divergence free processes

$$v\in L^2\big(\Omega,C([0,T],H)\big)\cap L^2\big(\Omega,L^2(0,T;V)\big),$$

satisfying the representation

$$v(t)=v_0+\int\limits_0^t v^*(s)ds+\int\limits_0^t\tilde v(s)dW(s), \tag{13}$$

with the random variables $v_0\in L^2(\Omega;H)$ and processes

$$v^*\in L^2\big(\Omega;L^2(0,T;V')\big),\qquad \tilde v\in L^2\big(\Omega,L^2(0,T;H)\big).$$

***Remark* 1.** Testing equation (1) with $u-v$ over the domain D, for any $v\in V$ and integrating by parts the term involving the pressure, thanks to the divergence free condition on u and v, we get

$$\big\langle du+(\nabla\times u\times u-\nu\Delta u)dt-\big(f(t,u)dt+g(t,u)dW\big),u-v\big\rangle$$
$$=\langle\nabla P,u-v\rangle dt=\langle P,u\cdot n-v\cdot n\rangle_{\partial D},$$

where n denotes the outer normal vector field to ∂D. Under the condition (5), we see that (1)–(5) is a particular case of problem (6), (7).

We now define the type of solutions which is of interest to us.

Definition 1. *A martingale (weak probabilistic) solution of problem* (6), (7) *is a system* $\big((\Omega, \mathcal{F}, P, (F_t)_{t\geq 0}), W, u\big)$ *such that*

$\big(\Omega, \mathcal{F}, P, (F_t)_{t\geq 0}\big)$ *is filtered probability space*;

W *is* l*-dimensional Wiener process*;

u *is a progressively measurable process such that* $\varphi(u) \in L^1\big(\Omega; L^1(0,T)\big)$,

$$u(\omega) \in L^2(0,T;V) \cap L^\infty(0,T;H) \quad P\text{-a.s.},$$

$$u(\omega, t) \in \operatorname{Dom}(\varphi) \quad \text{for any} \quad t \in [0,T] \quad \text{and } P\text{-a.s.},$$

where $\operatorname{Dom}(\varphi) = \big\{v \in V : \varphi(v) < \infty\big\}$;

for any $v \in \mathcal{P}_{pr}$, *we have*

$$E\int_0^t \big\langle v^* + A(u(s)) + B(u(s)) - f(s,u(s)), v(s) - u(s)\big\rangle ds$$

$$+\frac{1}{2}E\int_0^t \big\|\tilde{v}(s) - g(s,u(s))\big\|_H^2 ds$$

$$\geq E\int_0^t \varphi(u(s))ds - E\int_0^t \varphi(v(s))ds - \frac{1}{2}E\big\|v(0) - u_0\big\|_H^2 \tag{14}$$

for any $t \in [0,T]$.

Our main result is the following theorem.

Theorem 1. *Assume that the conditions* (i) – (v) *imposed on the data hold. Then problem* (6), (7) *has a martingale solution in the sense of Definition* 1.

This result is new to the best of our knowledge, since the current treatment of stochastic Navier – Stokes inequalities has not been undertaken in previous works.

3. Penalized version of problem (6), (7). For the proof of Theorem 1, we adopt the method of penalization expounded, for instance, in [9, 13]. For that purpose, we need a smooth approximation $\varphi_\varepsilon(u)$ (ε is a small positive parameter meant to converge to zero) of the convex function $\varphi(u)$ for which it then obviously hold that $\partial\varphi_\varepsilon(u) = \nabla\varphi_\varepsilon(u)$. Our choice for $\varphi_\varepsilon(u)$ is the following:

$$\varphi_\varepsilon(u) = \inf_{v\in V}\left\{\frac{1}{2}|u-v|^2 + \varepsilon|(u)|\right\}. \tag{15}$$

Owing to the condition **(iii)** on φ, $\varphi_\varepsilon \geq 0$. The function

$$J_\varepsilon(u) = \big(\operatorname{Id} + \varepsilon\partial\varphi(u)\big)$$

(Id is the identity operator), the sole solution of the inclusion equation $u \in v + \varepsilon\partial\varphi(v)$ is related to $\varphi_\varepsilon(u)$ through

$$\partial\varphi_\varepsilon(u) = u - J_\varepsilon(u). \tag{16}$$

We have the following well-known and useful properties of φ_ε, which we borrow from [6] (they are crucial throughout the work):

$$\frac{1}{2}\|\nabla\varphi_\varepsilon(\phi)\|^2_{L^2(D)} \le \varphi_\varepsilon(\phi) \le \big(\nabla\varphi_\varepsilon(\phi),\phi\big) \le \|\phi\|^2_{(L^2(D))^d} \quad \forall\phi \in \big(L^2(D)\big)^d \cap \mathrm{Dom}(\varphi), \tag{17}$$

$$\varepsilon\varphi(J_\varepsilon(\phi)) \le \varphi_\varepsilon(\phi) \le \varepsilon\varphi(\phi) \quad \forall\phi \in (L^2(D))^d \cap \mathrm{Dom}(\varphi), \tag{18}$$

$$-(\nabla\varphi_\varepsilon(\phi),\phi-\psi) \le \varepsilon\varphi\big(J_\varepsilon(\psi)\big) + \big(\nabla\varphi_\varepsilon(\phi),\nabla\varphi_\varepsilon(\psi)\big) \quad \forall\phi,\ \psi \in (L^2(D))^d \cap \mathrm{Dom}(\varphi), \tag{19}$$

$\nabla\varphi_\varepsilon$ is Lipschitz, in the sense that

$$|\nabla\varphi_\varepsilon(\phi) - \nabla\varphi_\varepsilon(\psi)| \le |\phi-\psi| \quad \forall\phi,\psi \in \big(L^2(D)\big)^d \cap \mathrm{Dom}(\varphi), \tag{20}$$

$$\frac{1}{\varepsilon}\big(\nabla\varphi_\varepsilon(\phi),\phi-\psi\big) \le \varphi(\phi) - \varphi\big(J_\varepsilon(\psi)\big) \quad \forall\phi,\psi \in \big(L^2(D)\big)^d \cap \mathrm{Dom}(\varphi). \tag{21}$$

We consider the following penalized problem of (6), (7). Find a solution u_ε of

$$du_\varepsilon + A\big(u_\varepsilon(t)\big)dt + B\big(u_\varepsilon(t)\big)dt + \frac{1}{\varepsilon}\nabla\varphi_\varepsilon\big(u_\varepsilon(t)\big) = f\big(t,u_\varepsilon(t)\big)dt + g\big(t,u_\varepsilon(t)\big)dW(t), \quad t \in (0,T), \tag{22}$$

$$u_\varepsilon(0) = u_0. \tag{23}$$

Since the Lipschitz conditions are still violated by f and g, we understand a solution of problem (22), (23) in the martingale sense. Namely, we have the following definition.

Definition 2. *For each* $\varepsilon > 0$, *a martingale solution of* (22), (23) *is a probabilistic system* $(\Omega_\varepsilon, F_\varepsilon, F^t_\varepsilon, P_\varepsilon, W_\varepsilon, u_\varepsilon)$, *where*

$(\Omega_\varepsilon, F_\varepsilon, P_\varepsilon)$ *is a probability space,* F^t_ε *is a filtration on* $(\Omega_\varepsilon, F_\varepsilon, P_\varepsilon)$,

$W_\varepsilon(t)$ *is an* l*-dimensional* F^t_ε *standard Wiener process,*

$u_\varepsilon \in L^p(\Omega_\varepsilon, F_\varepsilon, P_\varepsilon, L^2(0,T;V)$ *for any* $p \ge 1$ *and* $u_\varepsilon(\omega) \in L^\infty(0,T;H)$ *for a.e.* $\omega \in \Omega_\varepsilon$,

for any $v \in V$,

$$\big(u_\varepsilon(t),v\big) + \int\limits_0^t \left\langle \left[A\big(u_\varepsilon(s)\big) + B(u_\varepsilon(s)) + \frac{1}{\varepsilon}\nabla\varphi_\varepsilon(u_\varepsilon(s))\right], v\right\rangle ds$$

$$= (u_0,v) + \int\limits_0^t \big(f(s,u_\varepsilon(s)),v\big)ds + \int\limits_0^t \big(g\big(s,u_\varepsilon(s)\big)dW_\varepsilon(s),v\big) \tag{24}$$

almost surely and for all $t \in [0,T]$.

We note a solution u_ε of problem (22), (23) additionally lies in the space $C_w([0,T];H)$ P_ε-a.s., that is,

$$\lim_{t\to t_0}\big(u_\varepsilon(t),v\big) = \big(u_\varepsilon(t_0),v\big) \quad \forall t_0 \in [0,T] \quad P_\varepsilon\text{-a.s.}$$

This follows by arguing as in [29] (Chapt. 3, Par. 3, Theorem 3.1).

Main result in this section is the following theorem.

Theorem 2. *Assume that the conditions* (i) – (v) *imposed on the data hold. Then problem* (22), (23) *has a martingale solution in the sense of Definition* 2.

For the proof of Theorem 2, we rely on Galerkin method, several compactness results and the martingale representation theorem. This will be the objects of the forthcoming subsections.

3.1. Galerkin approximation of* (22), (23) *and a priori estimates. We consider a complete system $\{e_i\}_{i\in\mathbb{N}}$ in V (for instance, the eigenfunctions of Stokes operator A with domain $\mathcal{D}(A) = V\cap(H^2)^d$) and a probability system $\left(\bar\Omega, \bar F, \bar F^t, \bar P, \bar W\right)$ ($\bar W$ is a l-dimensional standard Wiener process). Let $u^N_{0\varepsilon}$ an element of the span V^N of $\left\{e_1, \ldots, e_N\right\}$ which approximates u_0 as

$$\lim_{N\to\infty} \left\|u^N_{0\varepsilon} - u_0\right\|_H = 0. \tag{25}$$

We look for a process u^N_ε defined on the probability space $\left(\bar\Omega, \bar F, \bar P\right)$ and lying in V^N, that is,

$$u^N_\varepsilon(t) =: u^N_\varepsilon(\bar\omega, t, x) = \sum_{i=1}^N \psi^N_{\varepsilon i}(\bar\omega, t)e_i(x), \tag{26}$$

and such that it is a solution of the system of stochastic equations

$$\left(du^N_\varepsilon(t), e_i\right) + \left\langle \left[A\left(u^N_\varepsilon(t)\right) + B\left(u^N_\varepsilon(t)\right) + \frac{1}{\varepsilon}\nabla\varphi_\varepsilon\left(u^N_\varepsilon(t)\right)\right], e_i\right\rangle dt$$

$$= \left(f\left(t, u^N_\varepsilon(t)\right), e_i\right)dt + \left(g\left(t, u^N_\varepsilon(t)\right)d\bar W(t), e_i\right), \quad i = 1, \ldots, N, \tag{27}$$

$$u^N_\varepsilon(0) = u^N_{0\varepsilon}.$$

The existence of $u^N_\varepsilon(t)$ follows from the existence of the Fourier coefficients $\psi^N_{\varepsilon i}(t)$ which solve the system (27) with the initial condition $\psi^N_{\varepsilon i}(0) = \left(u^N_{0\varepsilon}, e_i\right)$, after substituting (26) in (27). The resulting system of ordinary stochastic differential equations has with probability one at least a continuous local solution, thanks to Skorokhod's existence result [28, p. 121] (Theorem 2) which holds without the Lipschitz condition on the coefficients (only linear growth is required). This guaranties that $u^N_\varepsilon(t)$ exists on a possibly short interval $[0, T_N]$ and is an element of $C\left([0, T_N]; V^N\right)$. The existence over the whole interval $[0, T]$ will follow from uniform a priori estimates that we now derive.

Multiplying (27) by $\psi^N_{\varepsilon i}(t)$, summing the resulting equations over $i = 1, \ldots, N$ and applying Itô's formula to $\left\|u^N_\varepsilon(t)\right\|^2_H$, modulo appropriate stopping times, we get

$$\frac{1}{2}d\left\|u^N_\varepsilon(t)\right\|^2_H + \left\langle A\left(u^N_\varepsilon(t)\right), u^N_\varepsilon(t)\right\rangle dt + \frac{1}{\varepsilon}\left\langle\nabla\varphi_\varepsilon\left(u^N_\varepsilon(t)\right), u^N_\varepsilon(t)\right\rangle dt$$

$$= \left(f(t, u^N_\varepsilon(t)), u^N_\varepsilon(t)\right)dt + \left(g(t, u^N_\varepsilon(t)), u^N_\varepsilon(t)\right)d\bar W(t) + \frac{1}{2}\left\|g(t, u^N_\varepsilon(t))\right\|^2_{H^l}dt, \tag{28}$$

where we have used the fact that $\left\langle B\left(u^N_\varepsilon(t)\right), u^N_\varepsilon(t)\right\rangle = 0$ thanks to (9).

For $p \geq 1$, thanks to Itô's formula applied to (28), we derive the estimate

$$\bar E \sup_{s\in[0,t]} \|u^N_\varepsilon(s)\|^{2p}_H + 2p\bar E \int_0^t \left\|u^N_\varepsilon(s)\right\|^{2(p-1)}_H \left\langle A\left(u^N_\varepsilon(s)\right), u^N_\varepsilon(s)\right\rangle ds$$

$$+\frac{2p}{\varepsilon}\bar{E}\int\limits_0^t \left\|u_\varepsilon^N(s)\right\|_H^{2(p-1)}\big\langle\nabla\varphi_\varepsilon(u_\varepsilon^N(s)),u_\varepsilon^N(s)\big\rangle ds$$

$$=\left\|u_\varepsilon^N(0)\right\|_H^{2p}+2p\bar{E}\int\limits_0^t \left\|u_\varepsilon^N(s)\right\|_H^{2(p-1)}\big(f\big(s,u_\varepsilon^N(s)\big),u_\varepsilon^N(s)\big)ds$$

$$+2p\bar{E}\int\limits_0^t \left\|u_\varepsilon^N(s)\right\|_H^{2(p-1)}\big(g(s,u_\varepsilon^N(s)),u_\varepsilon^N(s)\big)d\bar{W}(s)$$

$$+p\bar{E}\int\limits_0^t \left\|u_\varepsilon^N(s)\right\|_H^{2(p-1)}\left\|g(s,u_\varepsilon^N(s))\right\|_{H^l}^2 ds$$

$$+p(p-1)\bar{E}\int\limits_0^t \left\|u_\varepsilon^N(s)\right\|_H^{2(p-2)}\big(g\big(s,u_\varepsilon^N(s)\big),u_\varepsilon^N(s)\big)^2 ds. \tag{29}$$

Thanks to the conditions (17), (11), (12) and a combination of Cauchy – Schwarz's, Young's and Burkholder – Davis – Gundy's inequalities and standard arguments, we get from (29) the following estimate:

$$\bar{E}\sup_{s\in[0,t]}\left\|u_\varepsilon^N(s)\right\|_H^{2p}+2p\bar{E}\int\limits_0^t \left\|u_\varepsilon^N(s)\right\|_H^{2(p-1)}\Big(\nu\left\|u_\varepsilon^N(s)\right\|_V^2+\alpha\left\|u_\varepsilon^N(s)\right\|_H^2\Big)ds$$

$$+\frac{p}{\varepsilon}\bar{E}\int\limits_0^t \left\|u_\varepsilon^N(s)\right\|_H^{2(p-1)}\left\|\nabla\varphi_\varepsilon(u_\varepsilon^N(s))\right\|_{L^2(D)}^2 ds$$

$$\le\left\|u_\varepsilon^N(0)\right\|_H^{2p}+C_\kappa\bar{E}\int\limits_0^t\Big(1+\left\|u_\varepsilon^N(s)\right\|_H^{2p}\Big)ds,$$

which implies thanks to Gronwall's lemma that, for all $t\le T_N$,

$$\bar{E}\sup_{s\in[0,t]}\left\|u_\varepsilon^N(s)\right\|_H^{2p}\le C. \tag{30}$$

The estimate (30) being uniform with respect to N, we conclude that u_ε^N exists on the whole interval $[0,T]$ and (30) holds for $t\in[0,T]$. Integrating (28) over the interval $[0,T]$, raising the resulting relation to the power $p\ge 1$, and proceeding as we did for the derivation of (30), we get the estimates

$$\bar{E}\left(\int\limits_0^T\Big(\nu\left\|\nabla u_\varepsilon^N(s)\right\|_{L^2(D)}^2+\alpha\left\|u_\varepsilon^N(s)\right\|_H^2\Big)ds\right)^p\le C, \tag{31}$$

$$\bar{E}\left(\int\limits_0^T\Big\langle\nabla\varphi_\varepsilon\Big(u_\varepsilon^N(s)\Big),u_\varepsilon^N(s)\Big\rangle ds\right)^p\le C\varepsilon^p. \tag{32}$$

From the last estimate and the relations (16)–(18), we get, in particular, for $p \geq 1$,

$$\bar{E}\left(\int_0^T \varphi_\varepsilon\Big(u_\varepsilon^N(s)\Big)ds\right)^p \leq C\varepsilon^p, \qquad \bar{E}\left(\int_0^T \big\|\nabla\varphi_\varepsilon(u_\varepsilon^N(s))\big\|_{L^2(D)}^2 ds\right)^p \leq C\varepsilon^p, \tag{33}$$

$$\bar{E}\left(\int_0^T \varphi\Big(J_\varepsilon\Big(u_\varepsilon^N(s)\Big)\Big)ds\right)^p \leq C. \tag{34}$$

We summarize our findings in the following lemma.

Lemma 1. *Under the assumptions on the data of problem* (27), *the Galerkin sequence* $\big(u_\varepsilon^N\big)$ *satisfies the estimates* (30)–(34) *uniformly with respect to* N.

Our next task is to establish a crucial estimate of the finite difference of u_ε^N. It follows from (27), Itô's formula, the condition (8) on A and the vanishing property (9) of B that for $r > 0$ fixed and u_ε^N extended by zero outside $[0, T]$,

$$\begin{aligned}
&\sup_{0<h\leq r}\int_0^T \big\|u_\varepsilon^N(s+h) - u_\varepsilon^N(s)\big\|_H^2 ds \\
&\leq 2\sup_{0<h\leq r}\int_0^T\int_s^{s+h}\big\langle A\big(u_\varepsilon^N(\tau)\big) + B\big(u_\varepsilon^N(\tau)\big), u_\varepsilon^N(s)\big\rangle d\tau ds \\
&\quad - \frac{2}{\varepsilon}\sup_{0<h\leq r}\int_0^T\int_s^{s+h}\big\langle\nabla\varphi_\varepsilon\big(u_\varepsilon^N(\tau)\big), u_\varepsilon^N(\tau) - u_\varepsilon^N(s)\big\rangle d\tau ds \\
&\quad + 2\sup_{0<h\leq r}\int_0^T\int_s^{s+h}\big(f\big(\tau, u_\varepsilon^N(\tau)\big), u_\varepsilon^N(\tau) - u_\varepsilon^N(s)\big)d\tau ds \\
&\quad + 2\sup_{0<h\leq r}\int_0^T\int_s^{s+h}\big(g\big(\tau, u_\varepsilon^N(\tau)\big), u_\varepsilon^N(\tau) - u_\varepsilon^N(s)\big)d\bar{W}(\tau) ds \\
&\quad + \sup_{0<h\leq r}\int_0^T\int_s^{s+h}\big\|g\big(\tau, u_\varepsilon^N(\tau)\big)\big\|_{H^l}^2 d\tau ds.
\end{aligned} \tag{35}$$

We illustrate the idea of how to estimate the terms on the right-hand side of (35) by limiting ourselves essentially to the term involving the operator B and the one involving the stochastic integral.

Thanks to (10), Cauchy–Schwarz's inequality and repeated use of Hölder's inequality, we have

$$\int_0^T\left(\int_s^{s+h}\big\langle B\big(u_\varepsilon^N(\tau)\big), u_\varepsilon^N(s)\big\rangle d\tau\right)ds$$

$$\leq C\left(\sup_{\tau\in[0,T]}\|u_\varepsilon^N(\tau)\|_H^2\right)^{\frac{1-\theta}{2}}\left(\left(\left(\int\limits_0^T\left\|u_\varepsilon^N(s)\right\|_V h^{\frac{1-\theta}{2}}\left(\int\limits_0^T\|u_\varepsilon^N(\tau)\|_V^2 d\tau\right)^{\frac{1+\theta}{2}} ds\right)^{\frac{2}{1+\theta}}\right)\right)^{\frac{1+\theta}{2}}$$

$$\leq Ch^{(1-\theta)/2}\left(\sup_{\tau\in[0,T]}\|u_\varepsilon^N(\tau)\|_H^2\right)^{(1-\theta)/2}\left(\int\limits_0^T\|u_\varepsilon^N(\tau)\|_V^2 d\tau\right)^{(2+\theta)/2}.$$

We infer from this estimate and Hölder's inequality that

$$\bar{E}\sup_{0<h\leq r}\int\limits_0^T\left(\int\limits_s^{s+h}\left\langle B\big(u_\varepsilon^N(\tau)\big),u_\varepsilon^N(s)\right\rangle d\tau\right)ds$$

$$\leq Cr^{(1-\theta)/2}\left(\bar{E}\sup_{\tau\in[0,T]}\left\|u_\varepsilon^N(\tau)\right\|_H^2\right)^{(1-\theta)/2}\left(\bar{E}\int\limits_0^T\left\|u_\varepsilon^N(\tau)\right\|_V^2 d\tau\right)^{(2+\theta)/2}.$$

Then owing to (30) and (31), we get

$$\bar{E}\sup_{0<h\leq r}\int\limits_0^T\left(\int\limits_s^{s+h}\left\langle B\big(u_\varepsilon^N(\tau)\big),u_\varepsilon^N(s)\right\rangle d\tau\right)ds\leq Cr^{(1-\theta)/2}.$$

For the stochastic integral in (35), it readily follows from Fubini's theorem, Burkholder – Davis – Gundy's inequality, Cauchy – Schwarz's inequality, the assumption (12) on g and the estimate (30) that

$$\bar{E}\sup_{0<h\leq r}\int\limits_0^T\int\limits_s^{s+h}\big(g\big(\tau,u_\varepsilon^N(\tau)\big),u_\varepsilon^N(\tau)-u_\varepsilon^N(s)\big)d\bar{W}(\tau)ds$$

$$\leq\int\limits_0^T\bar{E}\sup_{0<h\leq r}\left|\int\limits_s^{s+h}\big(g\big(\tau,u_\varepsilon^N(\tau)\big),u_\varepsilon^N(\tau)-u_\varepsilon^N(s)\big)d\bar{W}(\tau)\right|ds$$

$$\leq C\left(\int\limits_0^T\bar{E}\left(\int\limits_s^{s+r}\big(g\big(\tau,u_\varepsilon^N(\tau)\big),u_\varepsilon^N(\tau)-u_\varepsilon^N(s)\big)^2 d\tau\right)^{\frac{1}{2}}ds\right)$$

$$\leq C(T)r^{1/2}\left(1+\bar{E}\sup_{\tau\in[0,T]}\left\|u_\varepsilon^N(\tau)\right\|^2\right)\leq C(T)r^{1/2}.$$

The estimate of other terms in (35) is carried out similarly and for $h<0$, the corresponding estimates also hold. These findings imply the following lemma.

Lemma 2. *Under the assumptions on the data of problem* (27) *and for each fixed* $\varepsilon>0$, *there exists a number* $\vartheta\in(0,1)$ *such that the sequence* $\left(u_\varepsilon^N\right)$ *satisfies the estimate*

$$\sup_{N\in\mathbb{N}} \bar{E} \sup_{0<|h|\le r} \int_0^T \left\| u_\varepsilon^N(s+h) - u_\varepsilon^N(s) \right\|_H^2 ds \le C r^{\vartheta}.$$

***3.2. Probabilistic compactness results and proof of Theorem* 2.** Following [4], for any sequences (μ_n), (ν_n) such that $\mu_n, \nu_n \ge 0$ and $\mu_n, \nu_n \longrightarrow 0$ as $n \longrightarrow \infty$, we define the space U_{μ_n,ν_n} of functions $\varphi \in L^2(0,T;V) \cap L^\infty(0,T;H)$ such that

$$\sup_n \frac{1}{\nu_n} \sup_{|h|\le \mu_n} \left(\int_0^T \left\| \varphi(t+h) - \varphi(t) \right\|_H^2 dt \right)^{\frac{1}{2}} < \infty.$$

U_{μ_n,ν_n} is a Banach space, when it is endowed with the norm

$$\|\varphi\|_{U_{\mu_n,\nu_n}} = \sup_{0\le t\le T} \|\varphi(t)\|_H + \left(\int_0^T \|\varphi(t)\|_V^2 dt \right)^{\frac{1}{2}} + \sup_n \frac{1}{\nu_n} \left(\sup_{|h|\le\mu_n} \int_0^T \|\varphi(t+h) - \varphi(t)\|_H^2 dt \right)^{\frac{1}{2}}.$$

Due to the compact embedding of V in H, we have the following compactness result from [3].

Lemma 3. *The space U_{μ_n,ν_n} is compactly embedded in $L^2(0,T;H)$.*

Next for $2 \le p < \infty$, we consider the space $\mathcal{U}_{\mu_n,\nu_n}$ consisting of random variables φ on $(\bar{\Omega}, \bar{F}, \bar{P})$ with finite norms

$$\|\varphi\|_{\mathcal{U}_{\mu_n,\nu_n}} = \left(\bar{E} \sup_{0\le t\le T} \|\varphi(t)\|_H^2 \right)^{\frac{1}{2}} + \left(\bar{E} \int_0^T \|\varphi(t)\|_V^2 dt \right)^{\frac{1}{2}}$$

$$+ \bar{E} \sup_n \frac{1}{\nu_n} \left(\sup_{|\theta|\le\mu_n} \int_0^T \left\| \varphi(t+\theta) - \varphi(t) \right\|_H^2 dt \right).$$

$\mathcal{U}_{\mu_n,\nu_n}$ is a Banach space.

The a priori estimates established in the previous lemmas allow us to assert that, for any $p \ge 2$ and for μ_n, ν_n such that the series $\sum_{n=1}^\infty \frac{(\mu_n)^{1/2}}{\nu_n}$ converges, the sequence $\left(u_\varepsilon^N\right)$ remains in a bounded subset of $\mathcal{U}_{\mu_n,\nu_n}$.

Let us consider the space $\mathcal{S} = C\big([0,T];\mathbb{R}^l\big) \times L^2(0,T;H)$ and $\mathcal{B}(\mathcal{S})$ the σ-algebra of the Borel sets of $\mathcal{S}$. We fix ε and let the index N vary on the set of natural numbers. For each N, let Φ^N be the map defined by

$$\Phi_\varepsilon^N : \big(\bar{\Omega}, \bar{F}, \bar{P}\big) \to \mathcal{S} : \bar{\omega} \mapsto \big(\bar{W}(\bar{\omega}, \cdot), u_\varepsilon^N(\bar{\omega}, \cdot)\big).$$

We introduce the sequence of probability measures π_ε^N on $\big(\mathcal{S}, \mathcal{B}(\mathcal{S})\big)$ given by

$$\pi_\varepsilon^N(A) = \bar{P}\big((\Phi_\varepsilon^N)^{-1}(A)\big) \quad \text{for all} \quad A \in \mathcal{B}(\mathcal{S}).$$

Using Lemma 3 combined with the fact that $\left(u_\varepsilon^N\right)$ is a bounded sequence in $\mathcal{U}_{\mu_n,\nu_n}$ and proceeding as in [3, 4] and [25, 26], we readily get the following tighness result.

Theorem 3. *The family of measures $\left\{\pi_\varepsilon^N : N \in \mathbb{N}\right\}$ is tight in $\mathcal{S}$ for each $\varepsilon > 0$.*

Thus by Prokhorov's compactness theorem [22] the sequence $\left(\pi_\varepsilon^N\right)$ (up to a subfamily denoted by the same symbol) weakly converges in the sense of measures to a measure π_ε and subsequently Skorokhod's theorem [27] provides a probability space $(\Omega_\varepsilon, F_\varepsilon, P_\varepsilon)$ and $\mathcal{S}$-valued processes $\left(\tilde{W}_\varepsilon^N, \tilde{u}_\varepsilon^N\right)$ and $(W_\varepsilon, u_\varepsilon)$ defined on $(\Omega_\varepsilon, F_\varepsilon, P_\varepsilon)$ such that the law of $\left(\tilde{W}_\varepsilon^N, \tilde{u}_\varepsilon^N\right)$ is π_ε^N and that of $(W_\varepsilon, u_\varepsilon)$ is π_ε. Furthermore, we have the convergence

$$\left(\tilde{W}_\varepsilon^N(\tilde{\omega}, \cdot), \tilde{u}_\varepsilon^N(\tilde{\omega})\right) \to \left(W_\varepsilon(\tilde{\omega}, \cdot), u_\varepsilon(\tilde{\omega}, \cdot)\right) \quad \text{strongly in} \quad \mathcal{S} \quad P_\varepsilon\text{-a.s.} \tag{36}$$

Setting $F_{\varepsilon t} = \sigma\left\{\left(W_\varepsilon(s), u_\varepsilon(s)\right) : s \in [0, t]\right\}$, it readily follows that W_ε is an $F_{\varepsilon t}$-standard l-dimensional Wiener process from the convergence (36) and the fact that $\tilde{W}_\varepsilon^N$ is an l-dimensional Wiener process. Furthermore, the relation

$$\tilde{u}_\varepsilon^N(t) = u_0 - \int_0^t \left[A\left(\tilde{u}_\varepsilon^N(s)\right) + B\left(\tilde{u}_\varepsilon^N(s)\right) + \frac{1}{\varepsilon}\nabla\varphi_\varepsilon\left(\tilde{u}_\varepsilon^N(s)\right)\right] ds$$

$$+ \int_0^t f\left(s, \tilde{u}_\varepsilon^N(s)\right) ds + \int_0^t g\left(s, \tilde{u}_\varepsilon^N(s)\right) d\tilde{W}_\varepsilon^N(s) \tag{37}$$

holds in V' for any $t \in [0, T]$. For a proof of these claims we refer, for instance, to [4, 25, 26].

Owing to (37) we have that the estimates of u_ε^N in Lemmas 1 and 2 hold for the sequence $\left(\tilde{u}_\varepsilon^N\right)$, namely, for any $t \in [0, T]$ and any $p \in [1, \infty)$,

$$\tilde{E} \sup_{s \in [0, t]} \left\|\tilde{u}_\varepsilon^N(s)\right\|_H^{2p} \le C, \tag{38}$$

$$\tilde{E}\left(\int_0^t \left(\left\|\nabla\tilde{u}_\varepsilon^N(s)\right\|_{L^2(D)}^2 + \alpha\left\|\tilde{u}_\varepsilon^N(s)\right\|_H^2\right) ds\right)^p \le C, \tag{39}$$

$$\tilde{E}\left(\int_0^t \left\|\nabla\varphi_\varepsilon(\tilde{u}_\varepsilon^N(s))\right\|_{L^2(D)}^2 ds\right)^p \le C\varepsilon^p, \tag{40}$$

and we have the following convergences:

$$\tilde{u}_\varepsilon^N(\tilde{\omega}, \cdot) \to u_\varepsilon(\tilde{\omega}, \cdot) \quad \text{weakly-star in} \quad L^\infty(0, T; H) \quad \text{for a.e.} \quad \tilde{\omega} \in \tilde{\Omega},$$

$$\tilde{u}_\varepsilon^N \to u_\varepsilon \quad \text{weakly} \quad L^r\left(\tilde{\Omega}; L^q(0, T; H)\right) \quad \text{for any} \quad r, q \in [1, \infty),$$

$$\tilde{u}_\varepsilon^N \to u_\varepsilon \quad \text{weakly} \quad L^2\left(\tilde{\Omega}; L^2(0, T; V)\right).$$

These weak convergences imply the membership of u_ε to the corresponding spaces in which the convergences occur.

Next, the estimate (38) implies that $\left\|\tilde{u}_\varepsilon^N(s)\right\|_H^2$ is equiintegrable in $L^1(\tilde{\Omega})$, and from the convergence (36) follows the almost everywhere convergence of $\left\|\tilde{u}_\varepsilon^N(s)\right\|_H$ to $\left\|u_\varepsilon(s)\right\|_H$. Thus, Vitaly's convergence theorem implies that

$$\tilde{u}_\varepsilon^N \to u_\varepsilon \quad \text{strongly} \quad L^2\big(\tilde{\Omega}; L^2(0,T;H)\big) \tag{41}$$

and modulo the extraction of a new subsequence and for almost every $\big(\tilde{\omega}, t\big)$ with respect to the measure $d\tilde{P} \times dt$

$$\tilde{u}_\varepsilon^N \to u_\varepsilon \quad \text{strongly in} \quad H. \tag{42}$$

It remains to establish the integral identity (24). It follows by passage to the limit in the weak formulation

$$\big(\tilde{u}_\varepsilon^N(t), v\big) + \int_0^t \left\langle \left[A\big(\tilde{u}_\varepsilon^N(s)\big) + B\big(\tilde{u}_\varepsilon^N(s)\big) + \frac{1}{\varepsilon}\nabla\varphi_\varepsilon\big(\tilde{u}_\varepsilon^N(s)\big)\right], v \right\rangle ds$$

$$= (u_0, v) + \int_0^t \big(f\big(s, \tilde{u}_\varepsilon^N(s)\big), v\big) ds + \int_0^t \big(g\big(s, \tilde{u}_\varepsilon^N(s)\big) dW_\varepsilon^N(s), v\big),$$

of equation (37) for any $v \in V$. We use for that purpose the following convergences. Let Π_N denote the projection onto $V^N =$:span $\{e_1, \ldots, e_N\}$.

The convergence (42), together with the conditions on f, the estimate (38) and Vitali's theorem give

$$\Pi_N f\big(\tilde{u}_\varepsilon^N(\cdot), \cdot\big) \to f(u_\varepsilon(\cdot), \cdot) \quad \text{strongly in} \quad L^2\Big(\tilde{\Omega}; L^2(0,T;H)\Big).$$

Similarly, owing to the conditions on g

$$\Pi_N g\big(\tilde{u}_\varepsilon^N(\cdot), \cdot\big) \to g(u_\varepsilon(\cdot), \cdot) \quad \text{strongly in} \quad L^2\Big(\tilde{\Omega}; L^2\big(0,T;H^{\times l}\big)\Big).$$

Thanks to the Lipschitzity of $\nabla\varphi_\varepsilon$ (see (20)) and the estimate (41),

$$\nabla\varphi_\varepsilon\big(\tilde{u}_\varepsilon^N(\cdot)\big) \to \nabla\varphi_\varepsilon(u_\varepsilon(\cdot)) \quad \text{strongly in} \quad L^2\Big(\tilde{\Omega}, L^2\big(0,T,L^2(D)\big)\Big).$$

The convergence of the terms involving $A\big(\tilde{u}_\varepsilon^N(s)\big) + B\big(\tilde{u}_\varepsilon^N(s)\big)$ and the stochastic integral follows the same lines in [4, 25, 26]. It therefore follows that u_ε satisfies the integral identity

$$\big(u_\varepsilon(t), v\big) + \int_0^t \left\langle \left[A(u_\varepsilon(s)) + B(u_\varepsilon(s)) + \frac{1}{\varepsilon}\nabla\varphi_\varepsilon(u_\varepsilon(s))\right], v \right\rangle ds$$

$$= (u_0, v) + \int_0^t (f(s, u_\varepsilon(s)), v) ds + \int_0^t \big(g\big(s, u_\varepsilon(s)\big) dW_\varepsilon(s), v\big) \quad P_\varepsilon\text{-a.s.} \tag{43}$$

for all $v \in V$ and any $t \in [0,T]$.

Theorem 2 is proved.

4. Proof of Theorem 1. We are now in the position to prove our main result stated in Theorem 1. We essentially use the same ideas as in the proof of Theorem 2 in addition to aspects relevant to variational inequalities. Note that for any $p \in [1, \infty)$, u_ε as a solution of (43) defined on the probability space $(\Omega_\varepsilon, F_\varepsilon, P_\varepsilon)$ satisfies the following uniform (with respect to ε) estimates established in Lemmas 1 and 2. Namely,

$$E_\varepsilon \sup_{t\in[0,T]} \|u_\varepsilon(s)\|_H^{2p} \le C, \qquad E_\varepsilon \left(\int_0^T \|\nabla u_\varepsilon(s)\|^2_{L^2(D)} ds \right)^p \le C, \tag{44}$$

$$E_\varepsilon \left(\int_0^T \big\|\nabla \varphi_\varepsilon(u_\varepsilon(s))\big\|^2_{L^2(D)} ds \right)^p \le C\varepsilon^p, \qquad E_\varepsilon \left(\int_0^T \varphi\big(J_\varepsilon\big(u_\varepsilon(s)\big)\big) ds \right)^p \le C, \tag{45}$$

$$E_\varepsilon \sup_{0<|h|\le r} \int_0^T \big\|u_\varepsilon(s+h) - u_\varepsilon(s)\big\|^2_H ds \le C r^\vartheta, \quad \vartheta \in (0,1). \tag{46}$$

We consider the space $\tilde S = C\big([0,T];\mathbb{R}^l\big) \times L^2(0,T;H)$ and $\mathcal{B}(\tilde S)$ its Borel σ-algebra. The family of probability measures generated by $\big(W_\varepsilon(\omega,\cdot), u_\varepsilon(\omega,\cdot)\big)$ on $\big(\tilde S, \mathcal{B}(\tilde S)\big)$ turns out to be uniformly tight and thanks to Prokhorov and Skorokhod's compactness results used in the previous section, we get a probability space (Ω, F, P) and $\tilde S$-valued processes $\big(\bar W_\varepsilon, \bar u_\varepsilon\big)$ and (W,u) defined on (Ω, F, P) such that the law of $\big(\bar W_\varepsilon, \bar u_\varepsilon\big)$ is identical to that of $(W_\varepsilon, u_\varepsilon)$ and

$$\big(\bar W_\varepsilon(\omega,\cdot), \bar u_\varepsilon(\omega,\cdot)\big) \to (W(\omega,\cdot), u(\omega,\cdot)) \quad \text{strongly in} \quad \tilde S \quad P\text{-a.s.} \tag{47}$$

Endowed with the filtration $F_t = \sigma\big\{(W(s), u(s)) : s\in[0,t]\big\}$, W is an F_t-standard l-dimensional Wiener process. Furthermore, the relation

$$\bar u_\varepsilon(t) = u_0 - \int_0^t \left[A\big(\bar u_\varepsilon(s)\big) + B(\bar u_\varepsilon(s)) + \frac{1}{\varepsilon}\nabla\varphi_\varepsilon\big(\bar u_\varepsilon(t)\big)\right] ds$$

$$+ \int_0^t f\big(s, \bar u_\varepsilon(s)\big) ds + \int_0^t g\big(s, \bar u_\varepsilon(s)\big) d\bar W_\varepsilon(s) \tag{48}$$

holds in V' for any $t \in [0,T]$.

Therefore, we have that $\bar u_\varepsilon$ satisfies the estimates (44) – (46) and subsequently the following convergences (up to the extraction of a suitable subsequence) of $\bar u_\varepsilon$, which are obtained as in the case of u_ε^N in the previous section hold:

$$\bar u_\varepsilon \to u \quad \text{strongly} \quad L^2\big(\Omega; L^2(0,T;H)\big) \tag{49}$$

and, for almost every (ω, t) with respect to the measure $dP \times dt$,

$$\bar u_\varepsilon \to u \quad \text{strongly in} \quad H, \tag{50}$$

$$f\big(\bar u_\varepsilon(\cdot), \cdot\big) \to f\big(u(\cdot), \cdot\big) \quad \text{strongly in} \quad L^2\big(\Omega, L^2(0,T;H)\big), \tag{51}$$

$$g\big(\bar u_\varepsilon(\cdot), \cdot\big) \to g\big(u(\cdot), \cdot\big) \quad \text{strongly in} \quad L^2\big(\Omega; L^2(0,T;H^{\times l})\big). \tag{52}$$

It now remains to prove estimate (14). Let v be any element of the set $\mathcal{P}_{pr}$ and represented in the form (13), namely,

$$v(t) = v_0 + \int_0^t v^*(s)ds + \int_0^t \tilde{v}(s)dW(s)$$

with $v_0 \in L^2(\Omega; H),\ v^* \in L^2\big(\Omega; L^2(0,T;V')\big),\ \tilde{v} \in L^2(0,T;H)$.

By Itô's formula,

$$\begin{aligned}
&\big\|v(t) - \bar{u}_\varepsilon(t)\big\|_H^2 \\
&\quad = \|v_0 - u_0\|_H^2 + 2\int_0^t \langle v^*(s) - f(s, \bar{u}_\varepsilon(s)), v(s) - \bar{u}_\varepsilon(s)\rangle ds \\
&\qquad + 2\int_0^t \Big\langle v^*(s) + A(\bar{u}_\varepsilon(s)) + B(\bar{u}_\varepsilon(s)) + \frac{1}{\varepsilon}\nabla\varphi_\varepsilon\big(\bar{u}_\varepsilon(s)\big), v(s) - \bar{u}_\varepsilon(s)\Big\rangle ds \\
&\qquad + 2\int_0^t (g(s, \bar{u}_\varepsilon(s)), v(s) - \bar{u}_\varepsilon(s)) d\bar{W}_\varepsilon(s) \\
&\qquad\quad + 2\int_0^t (\tilde{v}(s), v(s) - \bar{u}_\varepsilon(s)) dW(s) + \langle\langle v - \bar{u}_\varepsilon\rangle\rangle_t^H,
\end{aligned}$$

where $\langle\langle\cdot\rangle\rangle_t^H$ denotes the quadratic variation of $\cdot$ with respect to the space H at time t, namely, given an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of H

$$\langle\langle v - \bar{u}_\varepsilon\rangle\rangle_t^H = \sum_{i=1}^\infty \left(\int_0^t \tilde{v}(s)dW(s) - \int_0^t g(s, \bar{u}_\varepsilon(s))d\bar{W}_\varepsilon(s), e_i\right)_H^2 \quad P\text{-a.s.} \tag{53}$$

Taking account of the L^2 integrability, with respect to time of the integrands in the stochastic integrals and their F_t adaptive nature, leading to the vanishing of their expectations, we have

$$\begin{aligned}
&E\big\|v(t) - \bar{u}_\varepsilon(t)\big\|_H^2 \\
&\quad = E\|v_0 - u_0\|_H^2 + 2E\int_0^t \langle v^*(s) - f(s, \bar{u}_\varepsilon(s)), v(s) - \bar{u}_\varepsilon(s)\rangle ds \\
&\qquad + 2E\int_0^t \Big\langle v^*(s) + A(\bar{u}_\varepsilon(s)) + B(\bar{u}_\varepsilon(s)) + \frac{1}{\varepsilon}\nabla\varphi_\varepsilon\big(\bar{u}_\varepsilon(s)\big), v(s) - \bar{u}_\varepsilon(s)\Big\rangle ds \\
&\qquad + E\langle\langle v - \bar{u}_\varepsilon\rangle\rangle_t^H.
\end{aligned}$$

Let us show that

$$\lim_{\varepsilon\to 0} E\langle\langle v - \bar{u}_\varepsilon\rangle\rangle_t^H = E\int_0^t \big\|\tilde{v}(s) - g(s, \bar{u}_\varepsilon(s))\big\|_H^2 ds. \tag{54}$$

Expanding the square in (53), we have

$$\langle\langle v-\bar u_\varepsilon\rangle\rangle_t=\sum_{i=1}^{\infty}\left(\int_0^t(\tilde v(s)-g(s,\bar u_\varepsilon(s)))dW(s)-\int_0^t g(s,\bar u_\varepsilon(s))d\big(\bar W_\varepsilon-W\big)(s),e_i\right)_H^2$$

$$=\left\|\int_0^t\Big(\tilde v(s)-g\big(s,\bar u_\varepsilon(s)\big)\Big)dW(s)\right\|_H^2+\left\|\int_0^t g\big(s,\bar u_\varepsilon(s)\big)d\big(\bar W_\varepsilon-W\big)(s)\right\|_H^2$$

$$-2\left(\int_0^t\big(\tilde v(s)-g\big(s,\bar u_\varepsilon(s)\big)\big)dW(s),\int_0^t g\big(s,\bar u_\varepsilon(s)\big)d\big(\bar W_\varepsilon-W\big)(s)\right)_H. \qquad (55)$$

It is clear that

$$E\left\|\int_0^t\Big(\tilde v(s)-g\big(s,\bar u_\varepsilon(s)\big)\Big)dW(s)\right\|_H^2=E\int_0^t\left\|\tilde v(s)-g\big(s,\bar u_\varepsilon(s)\big)\right\|_H^2ds,$$

thanks to Fubini's theorem and Itô's isometry.

Using Vitali's convergence theorem and proceeding as in [25] (Section 5), we show that the last two terms in (55) converge to zero in the mean as $\varepsilon\to 0$.

Using the property (21) of φ_ε, namely, $\dfrac{1}{\varepsilon}\langle\nabla\varphi_\varepsilon(\bar u_\varepsilon),v-\bar u_\varepsilon\rangle\le\varphi(v)-\varphi\big(J_\varepsilon(\bar u_\varepsilon)\big)$ and the vanishing property (9) of B, we get

$$\begin{aligned}&2E\int_0^t\langle v^*(s)-f(s,\bar u_\varepsilon(s)),v(s)-\bar u_\varepsilon(s)\rangle ds\\&\quad+2E\int_0^t\langle A(\bar u_\varepsilon(s))+B(\bar u_\varepsilon(s)),v(s)\rangle ds+2E\int_0^t\varphi(v(s))ds\\&\quad+2E\int_0^t\langle v^*(s),v(s)-\bar u_\varepsilon(s)\rangle ds+E\langle\langle v-\bar u_\varepsilon\rangle\rangle_t^H\\&\ge 2E\int_0^t\varphi(J_\varepsilon(\bar u_\varepsilon(s)))ds+2E\int_0^t\langle A(\bar u_\varepsilon(s)),\bar u_\varepsilon(s)\rangle ds-E\|v_0-u_0\|_H^2. \qquad (56)\end{aligned}$$

We are now left with the convergence of the expressions involving the sequence $\bar u_\varepsilon$.

By (49), (51) we have

$$E\int_0^t\langle f(s,\bar u_\varepsilon(s)),v(s)-\bar u_\varepsilon(s)\rangle ds\to E\int_0^t\langle f(s,u(s)),v(s)-\bar u_\varepsilon(s)u(s)\rangle ds. \qquad (57)$$

Similarly, owing to (52) and (54), we infer that

$$E\langle\langle v-\bar{u}_\varepsilon\rangle\rangle_t^H \to E\int_0^t \left\|\tilde{v}(s)-g(s,u(s))\right\|_H^2 ds. \tag{58}$$

Since φ is convex lower semicontinuous and $J_\varepsilon\big(\bar{u}_\varepsilon(s)\big)$ converges to $u(s)$ in $L^2(0,T,H)$ P-a.s., thanks to (47), we have by Fatou lemma that

$$\lim_{\varepsilon\to 0}\inf E\int_0^t \varphi\big(J_\varepsilon(\bar{u}_\varepsilon(s))\big)ds \geq E\int_0^t \varphi(u(s))ds. \tag{59}$$

The function

$$L^2\big(\Omega, L^2(0,T;V)\big) \ni \phi \to E\int_0^t \langle A(\phi(s)),\phi(s)\rangle ds \in [0,\infty)$$

involving the Stokes operator A is weakly lower semicontinuous. Thus, since $\bar{u}_\varepsilon$ converges to u weakly in $L^2\big(\Omega; L^2(0,T;V)\big)$, we have

$$2\lim_{\varepsilon\to 0}\inf E\int_0^t \langle A(\bar{u}_\varepsilon(s)),\bar{u}_\varepsilon(s)\rangle ds \geq 2E\int_0^t \langle A(u(s)),u(s)\rangle ds. \tag{60}$$

The combination of the convergences (57) – (60) and the relation (56) finally settles the estimate (14).

Theorem 1 is proved.

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