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ON THE POLYCONVOLUTION WITH THE WEIGHT FUNCTION $\gamma(y) = \cos y$ OF HARTLEY INTEGRAL TRANSFORMS $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1$ AND INTEGRAL EQUATIONS

ПРО ПОЛІЗГОРТКУ З ВАГОВОЮ ФУНКЦІЄЮ $\gamma(y) = \cos y$ ДЛЯ ІНТЕГРАЛЬНИХ ПЕРЕТВОРЕНЬ ХАРТЛІ $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1$ ТА ІНТЕГРАЛЬНИХ РІВНЯНЬ

We construct and investigate new polyconvolution with the weight function $\gamma(y) = \cos y$ of Hartley integral transforms $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1$ and apply it to solve integral equations and a system of integral equations of polyconvolution type.

Побудовано та досліджено нову полізгортку з ваговою функцією $\gamma(y) = \cos y$ для інтегральних перетворень Хартлі $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1$, яку застосовано для розв'язку інтегральних рівнянь та системи інтегральних рівнянь полізгорткового типу.

1. Introduction. In 1997, Kakichev [7] proposed the polyconvolution for $n+1$ arbitrary integral transforms $K, K_1, K_2, \ldots, K_n$ with the weight function $\gamma(y)$ of functions $f_1, f_2, \ldots, f_n$ which satisfies the following factorization identity:

$$K\Big[\overset{\gamma}{*}(f_1, f_2, \ldots, f_n)\Big](y) = \gamma(y)(K_1 f_1)(y)(K_2 f_2)(y)\ldots(K_n f_n)(y).$$

In recently time, there were some polyconvolutions [13, 14] related to the Hartley integral transforms and some differential integral transforms. At the same time, there were some polyconvolutions [10, 11] related only to the Hartley integral transforms and some differential integral transforms.

In this article, we expand to construct and study the new polyconvolution with the weight function $\gamma(y) = \cos y$ related to Hartley integral transforms $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1$. We will apply this polyconvolution to solve some nonstandard integral equations and system of integral equations. We realize that for such integral equation, a representation of their solution in a closed form is interesting and open problem [4, 8, 9].

In this section, we recall some known convolution, generalized convolutions. The Hartley integral transform $\mathcal{H}_1, \mathcal{H}_2$ was introduced in [3]

$$(\mathcal{H}f)_{\begin{Bmatrix}1\\2\end{Bmatrix}}(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} f(x)\,\mathrm{cas}\,(\pm xy)dy, \quad y \in \mathbb{R}.$$

Here $\mathrm{cas}\,(\pm\theta) = \cos\theta \pm \sin\theta$. The convolution for the Hartley integral transform $\mathcal{H}_1$ [5, 6, 12]

$$\Big(f \underset{\mathcal{H}_1}{*} g\Big)(x) = \frac{1}{2\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(u)[g(x+u) + g(x-u) + g(u-x) - f(-x-u)]du, \tag{1.1}$$

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satisfies the factorization identity

$$\mathcal{H}_1\Big(f \underset{\mathcal{H}_1}{*} g\Big)(y) = (\mathcal{H}_1 f)(y)(\mathcal{H}_1 g)(y).$$

The Hartley integral transform $\mathcal{H}_1,\ \mathcal{H}_1,\ \mathcal{H}_2$ was introduced in [3]

$$\Big(f \underset{\mathcal{H}_1,\mathcal{H}_1,\mathcal{H}_2}{*} g\Big)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} [f(x+y) + f(x-y) - f(-x+y) + f(-x-y)]g(y)dy$$

satisfies the factorization identity

$$\mathcal{H}_1\Big(f \underset{\mathcal{H}_1,\mathcal{H}_1,\mathcal{H}_2}{*} g\Big)(y) = (\mathcal{H}_1 f)(y)(\mathcal{H}_2 g)(y).$$

2. Polyconvolution with the weight function $\gamma(y) = \cos y$ for Hartley integral transforms $\mathcal{H}_1,\ \mathcal{H}_2,\ \mathcal{H}_1$.

Definition 2.1. *The polyconvolution with the weight function $\gamma(y) = \cos y$ for Hartley integral transforms of the functions f, g and h is defined as follows*:

$$[\overset{\gamma}{*}(f,g,h)](x) = \frac{1}{8\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} [f(x+v+w+1) + f(x+v+w-1)$$

$$+ f(x-v-w+1) + f(x-v-w-1)]g(v)h(w)dvdw, \quad x \in \mathbb{R}. \qquad (2.1)$$

Theorem 2.1. *Let f, g and h be functions in $L(\mathbb{R})$. Then the polyconvolution with the weight function $\gamma(y) = \cos y$ (2.1) for the Hartley integral transforms of the functions $f,\ g$ and h belongs to $L(\mathbb{R})$ and the factorization identity holds*

$$\mathcal{H}_1\Big[\overset{\gamma}{*}(f,g,h)\Big](y) = \cos y(\mathcal{H}_1 f)(y)(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y) \quad \forall y \in \mathbb{R}. \qquad (2.2)$$

Proof. First of all, we prove that $[\overset{\gamma}{*}(f,g,h)](x) \in L(\mathbb{R})$. Indeed, we have

$$\int\limits_{-\infty}^{\infty} \Big|\overset{\gamma}{*}(f,g,h)(x)\Big| dx$$

$$\leq \frac{1}{8\pi} \int\limits_{-\infty}^{\infty} |g(v)|dv \int\limits_{-\infty}^{\infty} |h(w)|dw \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} [|f(x+v+w+1)|$$

$$+ |f(x+v+w-1)| + |f(x-v-w+1)| + |f(x-v-w-1)|]dx.$$

It is easy to see that

$$\int\limits_{-\infty}^{\infty} [|f(x+v+w+1)| + |f(x+v+w-1)|$$

$$+\,|f(x-v-w+1)|+|f(x-v-w-1)|]dx = 4\int\limits_{-\infty}^{\infty}|f(u)|du.$$

For this reason, we obtain

$$\int\limits_{-\infty}^{\infty}\Big|\overset{\gamma}{*}(f,g,h)(x)\Big|dx \le \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}|g(v)|dv\int\limits_{-\infty}^{\infty}|h(w)|dw\int\limits_{-\infty}^{\infty}|f(u)|du < +\infty.$$

So, $\overset{\gamma}{*}(f,g,h)(x)$ belong to $L(\mathbb{R})$. Now, we prove the factorization identity (2.2). Since

$$2\pi\sqrt{2\pi}\cos y(\mathcal{H}_1 f)(y)(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)$$
$$=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\cos y\,\mathrm{cas}\,(yu)\,\mathrm{cas}\,(-yv)\,\mathrm{cas}\,(yw)f(u)g(v)h(w)dudvdw,$$

and using the trigonometric identity, we get

$$\cos y\,\mathrm{cas}\,(yu)\,\mathrm{cas}\,(yv)\,\mathrm{cas}\,(yw)$$
$$=\frac{1}{4}[\,\mathrm{cas}\,y(u+v+w+1)+\,\mathrm{cas}\,y(u+v+w-1)+\,\mathrm{cas}\,y(u-v-w+1)$$
$$+\,\mathrm{cas}\,y(u-v-w-1)].$$

Thus,

$$\cos y(\mathcal{H}_1 f)(y)(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)$$
$$=\frac{1}{8\pi\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}[\,\mathrm{cas}\,y(u+v+w+1)+\,\mathrm{cas}\,y(u+v+w-1)$$
$$+\,\mathrm{cas}\,y(u-v-w+1)+\,\mathrm{cas}\,y(u-v-w-1)]f(u)g(v)h(w)dudvdw$$
$$=\frac{1}{8\pi\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mathrm{cas}\,(yt)[f(t-v-w-1)+f(t-v-w+1)$$
$$+\,f(t+v+w-1)+f(t+v+w+1)]g(v)h(w)dtdvdw$$
$$=\mathcal{H}_1\Big[\overset{\gamma}{*}(f,g,h)\Big](y)\quad \forall y\in\mathbb{R}.$$

Theorem 2.1 is proved.

***Corollary* 2.1.** *In the space* $L(\mathbb{R})$ *the polyconvulution* (2.1) *has following equality*:

$$[\overset{\gamma}{*}(f,g,h)](x) = [\overset{\gamma}{*}(h,g,f)(x)].$$

Proof. From factorization identity (2.2), we have

$$\mathcal{H}\Big[\overset{\gamma}{*}(f,g,h)\Big](y) = (\mathcal{H}_1 f)(y)(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)$$

$$= (\mathcal{H}_1 h)(y)(\mathcal{H}_2 g)(y)(\mathcal{H}_1 f)(y) = \mathcal{H}\Big[\overset{\gamma}{*}(h,g,f)\Big](y).$$

Thus, $[\overset{\gamma}{*}(f,g,h)](x) = [\overset{\gamma}{*}(h,g,f)(x)]$.

Theorem 2.2. *If f, g, h belong to $L(\mathbb{R})$, then the following inequality holds*:

$$\Big\|\overset{\gamma}{*}(f,g,h)\Big\| \leq \|f\|\|g\|\|h\|.$$

Proof. From the proof of Theorem 2.1, we get

$$\int\limits_{-\infty}^{\infty} \Big|\overset{\gamma}{*}(f,g,h)(x)\Big| dx \leq \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |f(t)|dt \int\limits_{-\infty}^{\infty} |g(v)|dv \int\limits_{-\infty}^{\infty} |h(w)|dw$$

$$= \frac{1}{\sqrt[3]{2\pi}} \int\limits_{-\infty}^{\infty} |f(t)|dt \frac{1}{\sqrt[3]{2\pi}} \int\limits_{-\infty}^{\infty} |g(v)|dv \frac{1}{\sqrt[3]{2\pi}} \int\limits_{-\infty}^{\infty} |h(w)|dw.$$

Hence,

$$\Big\|\overset{\gamma}{*}(f,g,h)\Big\| \leq \|f\|\|g\|\|h\|.$$

Theorem 2.2 is proved.

Theorem 2.3. *Let $g \in L_p(\mathbb{R})$, $h \in L_q(\mathbb{R})$ and $f \in L_r(\mathbb{R})$ such that p, q, $r > 1$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 2$. Then the following inequality holds*:

$$\Big\|\overset{\gamma}{*}(f,g,h)\Big\| \leq \frac{1}{2\pi} \|g\|^p_{L_p(\mathbb{R}} \|h\|^q_{L_q(\mathbb{R}} \|f\|^r_{L_r(\mathbb{R}}.$$

Proof. From (2.1), we have the following estimation

$$\Big|\overset{\gamma}{*}(f,g,h)(x)\Big| \leq \frac{1}{8\pi} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} |g(v)||h(w)||f(x+v+w+1)|dvdw$$

$$+ \frac{1}{8\pi} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} |g(v)||h(w)||f(x+v+w-1)|dvdw$$

$$+ \frac{1}{8\pi} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} |g(v)||h(w)||f(x-v-w+1)|dvdw$$

$$+ \frac{1}{8\pi} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} |g(v)||h(w)||f(x-v-w-1)|dvdw. \tag{2.3}$$

Let $I_1, I_2, \ldots, I_4$ be the corresponding integral terms in the above expression. Without loss of generality we consider

$$I_1 = \frac{1}{8\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} |g(v)||h(w)||f(x+v+w+1)|dvdw, \quad x \in \mathbb{R}.$$

Let $p_1,\ q_1,\ r_1$ be the conjugate exponentials of $p,\ q,\ r$ and

$$A_1(u,v) = |h(w)|^{q/p_1}|f(x+v+w+1)|^{q/p_1} \in L_{p_1}(\mathbb{R}^2),$$

$$A_2(u,v) = |f(x+v+w+1)|^{r/q_1}|g(v)|^{p/q_1} \in L_{q_1}(\mathbb{R}^2),$$

$$A_3(u,v) = |g(v)|^{p/r_1}|h(w)|^{q/r_1} \in L_{r_1}(\mathbb{R}^2).$$

We see that

$$A_1.A_2.A_3 = |g(v)||h(w)||f(x+v+w+1)|.$$

Using the definition of the norm on the space $L_{p_1}(\mathbb{R})^2$ and with the help of the Fubini theorem, we get

$$\begin{aligned}
||A_1||^{p_1}_{L_{p_1}(\mathbb{R}^2)} &= \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \Big\{ |h(w)|^{q/p_1}|f(x+v+w+1)|^{r/p_1} \Big\}^{p_1} \\
&= \int\limits_{-\infty}^{\infty} |h(w)|^q \left(\int\limits_{-\infty}^{\infty} |f(x+v+w+1)|^r dv \right) dw \\
&= \int\limits_{-\infty}^{\infty} |h(w)|^q ||f||^r_{L_r(\mathbb{R})} dw = ||h||^q_{L_q(\mathbb{R})} ||f||^r_{L_r(\mathbb{R})}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
||A_2||^{q_1}_{L_{q_1}(\mathbb{R}^2)} &= ||f||^r_{L_r(\mathbb{R})} ||g||^p_{L_p(\mathbb{R})}, \\
||A_3||^{r_1}_{L_{r_1}(\mathbb{R}^2)} &= ||g||^p_{L_p(\mathbb{R})} ||h||^q_{L_q(\mathbb{R})}.
\end{aligned} \tag{2.4}$$

From the hypothesis $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, it follows that $\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{r_1} = 1$. Using Hölder inequality and (2.4), we obtain following estimation:

$$\begin{aligned}
I_1 &= \frac{1}{8\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} A_1 A_2 A_3 dvdw \\
&\leq \frac{1}{8\pi} \left(\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} A_1^{p_1} dvdw \right)^{\frac{1}{p_1}} \left(\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} A_2^{q_1} dvdw \right)^{\frac{1}{q_1}} \left(\int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} A_3^{r_1} dvdw \right)^{\frac{1}{r_1}}
\end{aligned}$$

$$= \frac{1}{8\pi}||A_1||^{p_1}_{L_{p_1}(\mathbb{R}^2)}||A_2||^{q_1}_{L_{q_1}(\mathbb{R}^2)}||A_3||^{r_1}_{L_{r_1}(\mathbb{R}^2)}$$

$$= \frac{1}{8\pi}||g||^{p}_{L_p(\mathbb{R})}||h||^{q}_{L_q(\mathbb{R})}||f||^{r}_{L_r(\mathbb{R})}. \tag{2.5}$$

In the same way, we get following estimations for $I_2,\ I_3,\ I_4$:

$$I_k \leq \frac{1}{8\pi}||g||^{p}_{L_p(\mathbb{R})}||h||^{q}_{L_q(\mathbb{R})}||f||^{r}_{L_r(\mathbb{R})} \tag{2.6}$$

for all $k = 2, 3, 4$. Furthermore, from (2.3)–(2.6), we obtain

$$\left\|\overset{\gamma}{*}(f,g,h)\right\| \leq \frac{1}{2\pi}\|g\|^{p}_{L_p(\mathbb{R}}\|h\|^{q}_{L_q(\mathbb{R}}\|f\|^{r}_{L_r(\mathbb{R}}.$$

Theorem 2.3 is proved.

Theorem 2.4 (Titchmarch-type theorem). *Let $f, g, h \in L(\mathbb{R})$. If, for all $x \in \mathbb{R}$, $\overset{\gamma}{*}(f,g,h)(x) \equiv 0$, then either $f(x) = 0$, or $g(x) = 0$, or $h(x) = 0$ for all $x \in \mathbb{R}$.*

Proof. The hypothesis $\overset{\gamma}{*}(f,g,h)(x) \equiv 0$ implies that $\mathcal{H}_1[\overset{\gamma}{*}(f,g,h)](y) = 0\ \forall y \in \mathbb{R}$. Due to Theorem 2.1, we get

$$\cos y(\mathcal{H}_1 f)(y)(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y) = 0 \quad \forall y \in \mathbb{R}. \tag{2.7}$$

As $(\mathcal{H}_1 f)(y)$, $(\mathcal{H}_2 g)(y)$, $(\mathcal{H}_1 h)(y)$ are analytic for all $y \in \mathbb{R}$, (2.7) implies that $(\mathcal{H}_1 f) = 0\ \forall y \in \mathbb{R}$, or $(\mathcal{H}_2 g) = 0\ \forall y \in \mathbb{R}$, or $(\mathcal{H}_1 h) = 0\ \forall y \in \mathbb{R}$.

It follows that either $f(x) = 0\ \forall x \in \mathbb{R}$, or $g(x) = 0\ \forall x \in \mathbb{R}$, or $h(x) = 0\ \forall x \in \mathbb{R}$.

Theorem 2.4 is proved.

3. Application to solving an integral equation and system of integral equations of polyconvolution type. ***3.1. A single integral equation.*** In this subsection, we apply the obtained result in solving an integral equation of polyconvolution type. To deal with this equation, we prove the existence of a solution as well a present it in a closed form. We examine the following integral equation:

$$f(x) + \frac{1}{8\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}[f(x+v+w+1) + f(x+v+w-1)$$

$$+ f(x-v-w+1) + f(x-v-w-1)]g(v)h(w)dvdw = k(x) \quad \forall y \in \mathbb{R}. \tag{3.1}$$

Here, g, h and k are functions of $L(\mathbb{R})$, f is an unknown function.

Theorem 3.1. *Let k, g, $h \in L(\mathbb{R})$ be given. Equation* (3.1) *has a unique solution $f(x) = k(x) - \left(k \underset{\mathcal{H}}{*} l\right)(x)$ in $L(\mathbb{R})$ if $1 + \cos y(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y) \neq 0\ \forall y \in \mathbb{R}$. Here, $l \in L(\mathbb{R})$ and it is determined by the equation*

$$(\mathcal{H}l)(y) = \frac{\cos y(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)}{1 + \cos y(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)}.$$

Proof. The equation (3.1) can be rewriten in the form

$$f(x) + \left[\overset{\gamma}{*}g(f,g,h)\right](x) = k(x).$$

Due to Theorem 2.1, we have

$$(\mathcal{H}_1 f)(y) + \cos y(\mathcal{H}_1 f)(y)(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y) = (\mathcal{H}_1 k)(y) \quad \forall y \in \mathbb{R}.$$

It follows that

$$(\mathcal{H}_1 f)(y)[1 + \cos y(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)] = (\mathcal{H}_1 k)(y).$$

With the condition $1 + \cos y(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y) \neq 0\ \forall y \in \mathbb{R}$, we get

$$(\mathcal{H}_1 f)(y) = (\mathcal{H}_1 k)(y)\left[1 - \frac{\cos y(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)}{1 + \cos y(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)}\right].$$

Therefore, by Wiener–Levy's theorem [1, 12], there exists a function $l \in L(\mathbb{R})$ such that

$$(\mathcal{H}_1 l)(y) = \frac{\cos y(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)}{1 + \cos y(\mathcal{H}_2 g)(y)(\mathcal{H}_1 h)(y)}.$$

Hence,

$$\begin{aligned}(\mathcal{H}_1 f)(y) &= (\mathcal{H}_1 k)(y) - (\mathcal{H}_1 k)(y)(\mathcal{H}_1 l)(y)\\ &= (\mathcal{H}_1 k)(y) - \mathcal{H}_1\left(k \underset{\mathcal{H}}{*} l\right)(y).\end{aligned}$$

Thus,

$$f(x) = k(x) - \left(k \underset{\mathcal{H}_1}{*} l\right)(x).$$

Theorem 3.1 is proved.

3.2. A system of two integral equations of polyconvolution type

$$\begin{aligned} &f(x) + \frac{1}{8\pi}\int\limits_{-\infty}^{\infty}[f(x+v+w+1) + f(x+v+w-1)\\ &\qquad + f(x-v-w+1) + f(x-v-w-1)]\varphi(v)\psi(w)dvdw = h(x),\\ &\frac{1}{2\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(v)[p(x+v) + p(x-v) + p(-x+v) - p(-x-v)]dv\\ &\qquad + g(x) = k(x), \quad x \in \mathbb{R}. \end{aligned} \tag{3.2}$$

Here, φ, ψ, p, h and k are given functions in $L(\mathbb{R})$, f and g are the unknown functions.

Theorem 3.2. *Assume that* $1 - \mathcal{H}_1\left[\overset{\gamma}{*}(p,\varphi,\psi)\right](y) \neq 0\ \forall y \in \mathbb{R}$, *there exists a unique solution in* $L(\mathbb{R})$ *of* (3.2), *which is defined by*

$$f(x) = h(x) + \left(h \underset{\mathcal{H}_1}{*} l\right)(x) - \left[\overset{\gamma}{*}(k,\varphi,\psi)\right](x) - \left\{\left[\overset{\gamma}{*}(k,\varphi,\psi)\right] \underset{\mathcal{H}_1}{*} l\right\}(x),$$

$$g(x) = k(x) + \left(k \underset{\mathcal{H}_1}{*} l\right)(x) - \left(h \underset{\mathcal{H}_1}{*} p\right)(x) - \left[\left(h \underset{\mathcal{H}_1}{*} p\right) \underset{\mathcal{H}_1}{*} l\right](x).$$

Here, $l \in L(\mathbb{R})$ *and defined by the equations*

$$(\mathcal{H}_1 l)(y) = \frac{\mathcal{H}_1\Big[\overset{\gamma}{*}(p,\varphi,\psi)\Big](y)}{1 - \mathcal{H}_1\Big[\overset{\gamma}{*}(p,\varphi,\psi)\Big](y)}.$$

Proof. System (3.2) can be written in the form

$$f(x) + \Big[\overset{\gamma}{*}(g,\varphi,\psi)\Big](x) = h(x),$$

$$\Big(f \underset{\mathcal{H}_1}{*} p\Big)(x) + g(x) = k(x), \quad x \in \mathbb{R}.$$

Using the factorization property of the polyconvolution (2.1) and the convolution (1.1), we obtain the linear system of algebraic equations with, respectively, to $(\mathcal{H}_1 f)(y)$ and $(\mathcal{H}_1 g)(y)$:

$$(\mathcal{H}_1 f)(y) + \cos y (\mathcal{H}_1 g)(y)(\mathcal{H}_2 \varphi)(y)(\mathcal{H}_1 \psi)(y) = (\mathcal{H}_1 h)(y),$$

$$(\mathcal{H}_1 f)(y)(\mathcal{H}_1 p)(y) + (\mathcal{H}_1 g)(y) = (\mathcal{H}_1 k)(y), \quad y \in \mathbb{R}.$$

We calculate the determinants of the system

$$\Delta = \begin{vmatrix} 1 & \cos y(\mathcal{H}_2\varphi)(y)(\mathcal{H}_1\psi)(y) \\ (\mathcal{H}_1 p)(y) & 1 \end{vmatrix}$$

$$= 1 - \cos y(\mathcal{H}_1 p)(y)(\mathcal{H}_2\varphi)(y)(\mathcal{H}_1\psi)(y) = 1 - \mathcal{H}_1\Big[\overset{\gamma}{*}(p,\varphi,\psi)\Big](y),$$

$$\Delta_1 = \begin{vmatrix} (\mathcal{H}_1 h)(y) & \cos y(\mathcal{H}_2\varphi)(y)(\mathcal{H}_1\psi)(y) \\ (\mathcal{H}_1 k)(y) & 1 \end{vmatrix}$$

$$= (\mathcal{H}_1 h)(y) - \mathcal{H}_1\Big[\overset{\gamma}{*}(k,\varphi,\psi)\Big](y),$$

$$\Delta_2 = \begin{vmatrix} 1 & (\mathcal{H}_1 h)(y) \\ (\mathcal{H}_1 p)(y) & (\mathcal{H}_1 k)(y) \end{vmatrix} = (\mathcal{H}_1 k)(y) - \mathcal{H}_1\Big(h \underset{\mathcal{H}_1}{*} p\Big)(y).$$

Since $1 - \mathcal{H}_1\Big[\overset{\gamma}{*}(p,\varphi,\psi)\Big](y) \neq 0 \ \forall y \in \mathbb{R}$, we have

$$(\mathcal{H}_1 f)(y) = \Big\{(\mathcal{H}_1 h)(y) - \mathcal{H}_1\Big[\overset{\gamma}{*}(k,\varphi,\psi)\Big](y)\Big\} \frac{1}{1 - \mathcal{H}_1\Big[\overset{\gamma}{*}(p,\varphi,\psi)\Big](y)}$$

$$= \Big\{(\mathcal{H}_1 h)(y) - \mathcal{H}_1\Big[\overset{\gamma}{*}(k,\varphi,\psi)\Big](y)\Big\} \left\{1 + \frac{\mathcal{H}_1\Big[\overset{\gamma}{*}(p,\varphi,\psi)\Big](y)}{1 - \mathcal{H}_1\Big[\overset{\gamma}{*}(p,\varphi,\psi)\Big](y)}\right\}.$$

Furthermore, according to Wiener – Levy's theorem [1, 12], there exists a function $l \in L(\mathbb{R})$ such that

$$(\mathcal{H}_1 l)(y) = \frac{\mathcal{H}_1\Big[\overset{\gamma}{*}(p,\varphi,\psi)\Big](y)}{1 - \mathcal{H}_1\Big[\overset{\gamma}{*}(p,\varphi,\psi)\Big](y)}.$$

It follows that

$$(\mathcal{H}_1 f)(y) = \Big\{(\mathcal{H}_1 h)(y) - \mathcal{H}_1\Big[\overset{\gamma}{*}(k,\varphi,\psi)\Big](y)\Big\}\{1 + (\mathcal{H}_1 l)(y)\}$$

$$= (\mathcal{H}_1 h)(y) + \mathcal{H}_1\Big(h \underset{\mathcal{H}_1}{*} l\Big)(y) - \mathcal{H}_1\Big[\overset{\gamma}{*}(k,\varphi,\psi)\Big](y)$$

$$- \mathcal{H}_1\Big\{\Big[\overset{\gamma}{*}(k,\varphi,\psi)\Big] \underset{\mathcal{H}_1}{*} l\Big\}(y).$$

So,

$$f(x) = h(x) + \Big(h \underset{\mathcal{H}_1}{*} l\Big)(x) - \Big[\overset{\gamma}{*}(k,\varphi,\psi)\Big](x) - \Big\{\Big[\overset{\gamma}{*}(k,\varphi,\psi)\Big] \underset{\mathcal{H}_1}{*} l\Big\}(x) \in L(\mathbb{R}).$$

in the same way, we obtain

$$(\mathcal{H}_1 g)(y) = \Big\{(\mathcal{H}_1 k)(y) - \mathcal{H}_1\Big(h \underset{\mathcal{H}_1}{*} p\Big)(y)\Big\}\{1 + (\mathcal{H}_1 l)(y)\}$$

$$= (\mathcal{H}_1 k)(y) + \mathcal{H}_1\Big(k \underset{\mathcal{H}_1}{*} l\Big)(y) - \mathcal{H}_1\Big(h \underset{\mathcal{H}_1}{*} p\Big)(y) - \mathcal{H}_1\Big[\Big(h \underset{\mathcal{H}_1}{*} p\Big) \underset{\mathcal{H}_1}{*} l\Big](y).$$

It follows that

$$g(x) = k(x) + \Big(k \underset{\mathcal{H}_1}{*} l\Big)(x) - \Big(h \underset{\mathcal{H}_1}{*} p\Big)(x) - \Big[\Big(h \underset{\mathcal{H}_1}{*} p\Big) \underset{\mathcal{H}_1}{*} l\Big] \quad (x) \in L(\mathbb{R}).$$

Theorem 3.2 is proved.

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