DOI: 10.37863/umzh.v75i4.6924

UDC 512.5

M. Ashraf, M. A. Ansari[1] (Department of Mathematics, Aligarh Muslim University, India)

σ-CENTRALIZERS OF TRIANGULAR ALGEBRAS

σ-ЦЕНТРАЛІЗАТОРИ ТРИКУТНИХ АЛГЕБР

In this paper, we characterize Lie (Jordan) σ-centralizers of triangular algebras. More precisely, we prove that, under certain conditions, every Lie σ-centralizer of a triangular algebra can be represented as the sum of a σ-centralizer and a central-valued mapping. Further, it is shown that every Jordan σ-centralizer of a triangular algebra is a σ-centralizer.

Охарактеризовано σ-централізатори Лі (Джордана) трикутних алгебр. Більш точно, доведено, що за певних умов кожен σ-централізатор Лі трикутної алгебри можна записати як суму σ-централізатора та центральнозначного відображення. Крім того, показано, що кожен σ-централізатор Джордана трикутної алгебри є σ-централізатором.

1. Introduction. Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ be a unital $\mathcal{R}$-algebra and $\mathcal{Z}(\mathcal{A})$ be the centre of $\mathcal{A}$. For any $a,b \in \mathcal{A}$, $[a,b] = ab - ba$ (resp., $a \circ b = ab + ba$) will denote the Lie product (resp., Jordan product). Let σ be an automorphism of $\mathcal{A}$. An $\mathcal{R}$-linear mapping $L\colon \mathcal{A} \to \mathcal{A}$ is called a left σ-centralizer (resp., right σ-centralizer) if $L(ab) = L(a)\sigma(b)$ (resp., $L(ab) = \sigma(a)L(b)$) for all $a,b \in \mathcal{A}$. It is called a σ-centralizer if it is both a left σ-centralizer as well as a right σ-centralizer. An $\mathcal{R}$-linear mapping $L\colon \mathcal{A} \to \mathcal{A}$ is called a Lie σ-centralizer if $L([a,b]) = [L(a), \sigma(b)]$ (or $L([a,b]) = [\sigma(a), L(b)]$) for all $a,b \in \mathcal{A}$. An $\mathcal{R}$-linear mapping $L\colon \mathcal{A} \to \mathcal{A}$ is called a Jordan σ-centralizer if $L(a \circ b) = L(a) \circ \sigma(b)$ (or $L(a \circ b) = \sigma(a) \circ L(b)$) for all $a,b \in \mathcal{A}$. One can easily see that the conditions $L([a,b]) = [L(a), \sigma(b)]$ (resp., $L(a \circ b) = L(a) \circ \sigma(b)$) and $L([a,b]) = [\sigma(a), L(b)]$ (resp., $L(a \circ b) = \sigma(a) \circ L(b)$) are equivalent. Obviously, every σ-centralizer is a Lie σ-centralizer as well as a Jordan σ-centralizer but the converse statements are not true in general. If $d\colon \mathcal{A} \to \mathcal{A}$ is a σ-centralizer and $\ell\colon \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ is a linear mapping, then $d + \ell$ is a Lie σ-centralizer on $\mathcal{A}$ if and only if $\ell([a,b]) = 0$ for all $a,b \in \mathcal{A}$. A Lie σ-centralizer which can be written as the sum of a σ-centralizer and a central valued mapping is called proper.

Over the past decades, a lot of work concerning characterizations of Lie (Jordan) maps on different rings and algebras have been done (see [1, 3, 4, 6, 15] and references therein). In the year 1957, Herstein [16] proved that every Jordan derivation on a 2-torsion free prime ring is a derivation. Brešar [7] extended Herstein's result for 2-torsion free semiprime rings. In the year 1964, Martindale [23] obtained the first characterization of Lie derivations on a primitive ring and he proved that every Lie derivation on a primitive ring is proper, that is, it can be written as the sum of a derivation and a central mapping. In [9], Cheung initiated the study of Lie derivations of triangular algebras and gave a sufficient condition under which every Lie derivation on a triangular algebra is proper. In [26], Zhang an Yu showed that every Jordan derivation on a triangular algebra is a derivation. Han and Wei [15] studied Jordan (α,β)-derivations on triangular algebras. Yang and Zhu [24] characterized additive

[1] Corresponding author, e-mail: afzalgh1786@gmail.com.

ξ-Lie (α, β)-derivations on triangular algebras. Further, González et al. considered σ-biderivations and σ-commuting mappings of triangular algebras in [14]. Zalar [25] introduced the notion of Jordan centralizers and proved that every Jordan centralizer on a 2-torsion free semiprime ring is centralizer. Fošner and Jing [10] introduced the notion of Lie centralizer and investigated the additivity of Lie centralizers on triangular rings. In addition, centralizers on different rings and algebras have been broadly examined by many algebraists (see [5, 10–13, 17, 21, 22]).

In this article, we characterize Lie (Jordan) σ-centralizers of triangular algebras. In fact, we prove that under certain restrictions every Lie σ-centralizer of a triangular algebra is proper (Theorem 4.2); every Jordan σ-centralizer of a triangular algebra is a σ-centralizer (Theorem 5.1).

2. Triangular algebras. Let $\mathcal{R}$ be a commutative ring with identity. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital algebras over $\mathcal{R}$ and $\mathcal{M}$ is a nonzero $(\mathcal{A}, \mathcal{B})$-bimodule. An $\mathcal{R}$-algebra

$$\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \;\middle|\; a \in \mathcal{A},\ m \in \mathcal{M},\ b \in \mathcal{B} \right\}$$

under the usual matrix operations is called a triangular algebra consisting of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$. Basic examples of triangular algebras are upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras. In view of [8, Theorem 1.4.1], the center of $\mathfrak{A}$ is given by

$$\mathcal{Z}(\mathfrak{A}) = \left\{ a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \;\middle|\; a \in \mathcal{Z}(\mathcal{A}),\ b \in \mathcal{Z}(\mathcal{B}),\ am = mb \ \text{ for all } \ m \in \mathcal{M} \right\}.$$

Define two natural projections $\pi_{\mathcal{A}} : \mathfrak{A} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathfrak{A} \to \mathcal{B}$ by

$$\pi_{\mathcal{A}} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = a \quad \text{and} \quad \pi_{\mathcal{B}} \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = b, \quad \text{respectively.}$$

Recall that an $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$ is said to be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule if for $a \in \mathcal{A}$, $a\mathcal{M} = \{0\}$ implies $a = 0$ and for $b \in \mathcal{B}$, $\mathcal{M}b = \{0\}$ implies $b = 0$.

By [8, Theorem 1.4.4], if $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, then the centre of $\mathfrak{A}$ coincides with

$$\mathcal{Z}(\mathfrak{A}) = \left\{ a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \;\middle|\; a \in \mathcal{A},\ b \in \mathcal{B},\ am = mb \ \text{ for all } \ m \in \mathcal{M} \right\}.$$

Moreover, $\pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{A})) \subseteq \mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathfrak{A})) \subseteq \mathcal{Z}(\mathfrak{A})$, and there exists a unique algebra isomorphism $\xi : \pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{A})) \to \pi_{\mathcal{B}}(\mathcal{Z}(\mathfrak{A}))$ such that $am = m\xi(a)$ for all $a \in \pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{A}))$, $m \in \mathcal{M}$.

The study of group of automorphisms is an important key for understanding the underlying algebraic structure and hence it has been extensively investigated. Automorphisms of triangular algebras were studied in [2, 8, 18–20]. In the year 2003, Khazal et al. [20] obtained the following structure of automorphisms of triangular algebras.

Lemma 2.1 [20, Theorem 1]. *Let $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$, and $\sigma : \mathfrak{A} \to \mathfrak{A}$ be an $\mathcal{R}$-linear mapping. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents. Then σ is an automorphism of $\mathfrak{A}$ if and only if it is of the form*

$$\sigma \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} \gamma(a) & \gamma(a)m_0 - m_0\delta(b) + u(m) \\ 0 & \delta(b) \end{pmatrix},$$

where $m_0 \in \mathcal{M}$ *is a fixed element;* γ, δ *are automorphisms of* $\mathcal{A}$, $\mathcal{B}$, *respectively; and* u *is an* $\mathcal{R}$ *linear bijective mapping from* $\mathcal{M}$ *into itself such that* $u(am) = \gamma(a)u(m)$, $u(mb) = u(m)\delta(b)$ *for all* $a \in \mathcal{A}$, $b \in \mathcal{B}$ *and* $m \in \mathcal{M}$.

3. σ-Centralizers on triangular algebras. In this section, we give the structure of σ-centralizers on triangular algebras with associated automorphism given in Lemma 2.1.

Proposition 3.1. *Let* $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *be a triangular algebra consisting of* $\mathcal{A}$, $\mathcal{B}$ *and* $\mathcal{M}$. *Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *have only trivial idempotents. An* $\mathcal{R}$*-linear mapping* $\Theta\colon \mathfrak{A} \to \mathfrak{A}$ *is a* σ*-centralizer on* $\mathfrak{A}$ *if and only if* Θ *has the following form:*

$$\Theta\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} A_{11}(a) & A_{11}(a)m_0 - m_0B_{22}(b) + C_{12}(m) \\ 0 & B_{22}(b) \end{pmatrix},$$

where $A_{11}\colon \mathcal{A} \to \mathcal{A}$, $C_{12}\colon \mathcal{M} \to \mathcal{M}$, *and* $B_{22}\colon \mathcal{B} \to \mathcal{B}$ *are* $\mathcal{R}$*-linear mappings satisfying the following conditions:*

(1) A_{11} *is a* γ*-centralizer on* $\mathcal{A}$, $C_{12}(am) = A_{11}(a)u(m) = \gamma(a)C_{12}(m)$;

(2) B_{22} *is a* δ*-centralizer on* $\mathcal{B}$, $C_{12}(mb) = C_{12}(m)\delta(b) = u(m)B_{22}(b)$.

Proof. Suppose that σ-centralizer Θ on $\mathfrak{A}$ has the form

$$\Theta\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} A_{11}(a) + B_{11}(b) + C_{11}(m) & A_{12}(a) + B_{12}(b) + C_{12}(m) \\ 0 & A_{22}(a) + B_{22}(b) + C_{22}(m) \end{pmatrix}$$

for all $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in \mathfrak{A}$, where A_{11}, B_{11}, C_{11} are $\mathcal{R}$-linear mappings from $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$ to $\mathcal{A}$, respectively; A_{12}, B_{12}, C_{12} are $\mathcal{R}$-linear mappings from $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$ to $\mathcal{M}$, respectively; A_{22}, B_{22}, C_{22} are $\mathcal{R}$-linear mappings from $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$ to $\mathcal{B}$, respectively.

Since Θ is σ-centralizer on $\mathfrak{A}$, we have

$$\Theta(xy) = \Theta(x)\sigma(y) = \sigma(x)\Theta(y) \quad \text{for all} \quad x, y \in \mathfrak{A}. \tag{3.1}$$

Let us choose $x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ in (3.1). Then, we obtain

$$\begin{pmatrix} C_{11}(am) & C_{12}(am) \\ 0 & C_{22}(am) \end{pmatrix} = \begin{pmatrix} 0 & A_{11}(a)u(m) \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \gamma(a)C_{11}(m) & \gamma(a)C_{12}(m) + \gamma(a)m_0C_{22}(m) \\ 0 & 0 \end{pmatrix}.$$

This implies that $C_{11}(am) = 0 = \gamma(a)C_{11}(m)$, $C_{12}(am) = A_{11}(a)u(m) = \gamma(a)C_{12}(m) + \gamma(a)m_0C_{22}(m)$ and $C_{22}(am) = 0$. Putting $a = 1_{\mathcal{A}}$, we get $C_{11}(m) = 0$, $C_{22}(m) = 0$. Thus, $C_{12}(am) = A_{11}(a)u(m) = \gamma(a)C_{12}(m)$. Similarly, if we choose $x = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ in (3.1), we can obtain $C_{12}(mb) = C_{12}(m)\delta(b) = u(m)B_{22}(b)$.

If we consider $x = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}$ in (3.1). Then

$$\begin{pmatrix} A_{11}(a_1a_2) & A_{12}(a_1a_2) \\ 0 & A_{22}(a_1a_2) \end{pmatrix} = \begin{pmatrix} A_{11}(a_1)\gamma(a_2) & A_{11}(a_1)\gamma(a_2)m_0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma(a_1)A_{11}(a_2) & \gamma(a_1)A_{12}(a_2) + \gamma(a_1)m_0A_{22}(a_2) \\ 0 & 0 \end{pmatrix}.$$

From the above relation, we find that $A_{11}(a_1a_2) = A_{11}(a_1)\gamma(a_2) = \gamma(a_1)A_{11}(a_2)$, $A_{12}(a_1a_2) = A_{11}(a_1)\gamma(a_2)m_0 = \gamma(a_1)A_{12}(a_2) + \gamma(a_1)m_0A_{22}(a_2)$ and $A_{22}(a_1a_2) = 0$. Putting $a_2 = 1_{\mathcal{A}}$, we have $A_{12}(a_1) = A_{11}(a_1)m_0$ and $A_{22}(a_1) = 0$. Thus, the above equations give $A_{11}(a_1a_2) = A_{11}(a_1)\gamma(a_2) = \gamma(a_1)A_{11}(a_2)$, i.e., A_{11} is a γ-centralizer on $\mathcal{A}$, $A_{12}(a) = A_{11}(a)m_0$ and $A_{22}(a) = 0$. Similarly, considering $x = \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}$ in (3.1) to obtain $B_{11}(b) = 0$, $B_{12}(b) = -m_0B_{22}(b)$ and $B_{22}(b_1b_2) = B_{22}(b_1)\delta(b_2) = \delta(b_1)B_{22}(b_2)$, i.e., B_{22} is a δ-centralizer on $\mathcal{B}$.

Conversely, suppose that Θ is a linear mapping on $\mathfrak{A}$ of the form

$$\Theta\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} A_{11}(a) & A_{11}(a)m_0 - m_0B_{22}(b) + C_{12}(m) \\ 0 & B_{22}(b) \end{pmatrix}$$

satisfying the assumptions (1) and (2). Then it is easy to check that Θ satisfies the relation $\Theta(xy) = \Theta(x)\sigma(y) = \sigma(x)\Theta(y)$ for all $x, y \in \mathfrak{A}$, that is, Θ is a σ-centralizer on $\mathfrak{A}$.

Proposition 3.1 is proved.

If $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, then conditions "A_{11} is a γ-centralizer on $\mathcal{A}$" and "B_{22} is a δ-centralizer on $\mathcal{B}$" in Proposition 3.1 become redundant.

***Corollary* 3.1.** *Let $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$. Suppose that $\mathcal{A}$, $\mathcal{B}$ have only trivial idempotents and $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule. An $\mathcal{R}$-linear mapping $\Theta : \mathfrak{A} \to \mathfrak{A}$ is a σ-centralizer on $\mathfrak{A}$ if and only if Θ has the following form:*

$$\Theta\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} A_{11}(a) & A_{11}(a)m_0 - m_0B_{22}(b) + C_{12}(m) \\ 0 & B_{22}(b) \end{pmatrix},$$

where $A_{11} : \mathcal{A} \to \mathcal{A}$, $C_{12} : \mathcal{M} \to \mathcal{M}$, and $B_{22} : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(1) $C_{12}(am) = A_{11}(a)u(m) = \gamma(a)C_{12}(m)$;

(2) $C_{12}(mb) = C_{12}(m)\delta(b) = u(m)B_{22}(b)$.

Proof. In view of Proposition 3.1, it suffices to show that if $\mathcal{M}$ is faithful, then A_{11} is a γ-centralizer on $\mathcal{A}$ and B_{22} is a δ-centralizer on $\mathcal{B}$. For all $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$, we have

$$A_{11}(a_1a_2)u(m) = C_{12}(a_1a_2m) = A_{11}(a_1)u(a_2m) = A_{11}(a_1)\gamma(a_2)u(m).$$

Thus, $\{A_{11}(a_1a_2) - A_{11}(a_1)\gamma(a_2)\}\mathcal{M} = \{0\}$. Since $\mathcal{M}$ is faithful left $\mathcal{A}$-module, we have $A_{11}(a_1a_2) = A_{11}(a_1)\gamma(a_2)$. Again,

$$A_{11}(a_1a_2)u(m) = C_{12}(a_1a_2m) = \gamma(a_1)C_{12}(a_2m) = \gamma(a_1)A_{11}(a_2)u(m)$$

for all $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$. Hence, $\{A_{11}(a_1a_2) - \gamma(a_1)A_{11}(a_2)\}\mathcal{M} = \{0\}$ which implies that $A_{11}(a_1a_2) = \gamma(a_1)A_{11}(a_2)$. Therefore, A_{11} is a γ-centralizer on $\mathcal{A}$. In a similar manner, one can prove that B_{22} is a δ-centralizer on $\mathcal{B}$.

Corollary 3.1 is proved.

4. Lie σ-centralizers on triangular algebras. Fošner and Jing [10] introduced the notion of Lie centralizer and investigated the additivity of Lie centralizers on triangular rings. Yang and Zhu [24] studied Lie (α,β)-derivations on triangular algebras and related mappings. Motivated by this work, in this section, we characterize Lie σ-centralizers on triangular algebras.

Proposition 4.1. *Let $\mathfrak{A} = \mathrm{Tri}\,(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents. An $\mathcal{R}$-linear mapping $\mathcal{L}\colon \mathfrak{A}\to\mathfrak{A}$ is a Lie σ-centralizer on $\mathfrak{A}$ if and only if $\mathcal{L}$ has the following form:*

$$\mathcal{L}\begin{pmatrix} a & m \\ 0 & b\end{pmatrix} = \begin{pmatrix} R_{11}(a)+S_{11}(b) & (R_{11}(a)+S_{11}(b))m_0 - m_0(R_{22}(a)+S_{22}(b)) + T_{12}(m) \\ 0 & R_{22}(a)+S_{22}(b)\end{pmatrix},$$

where $R_{11}:\mathcal{A}\to\mathcal{A}$, $S_{11}:\mathcal{B}\to\mathcal{Z}(\mathcal{A})$, $T_{12}:\mathcal{M}\to\mathcal{M}$, $R_{22}:\mathcal{A}\to\mathcal{Z}(\mathcal{B})$ and $S_{22}:\mathcal{B}\to\mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(1) *R_{11} is a Lie γ-centralizer on $\mathcal{A}$, $T_{12}(am) = R_{11}(a)u(m) - u(m)R_{22}(a) = \gamma(a)T_{12}(m)$;*

(2) *S_{22} is a Lie δ-centralizer on $\mathcal{B}$, $T_{12}(mb) = T_{12}(m)\delta(b) = u(m)S_{22}(b) - S_{11}(b)u(m)$;*

(3) *$R_{22}([a_1,a_2]) = 0$ and $S_{11}([b_1,b_2]) = 0$.*

Proof. Suppose that Lie σ-centralizer $\mathcal{L}$ on $\mathfrak{A}$ has the form

$$\mathcal{L}\begin{pmatrix} a & m \\ 0 & b\end{pmatrix} = \begin{pmatrix} R_{11}(a)+S_{11}(b)+T_{11}(m) & R_{12}(a)+S_{12}(b)+T_{12}(m) \\ 0 & R_{22}(a)+S_{22}(b)+T_{22}(m)\end{pmatrix}$$

for all $\begin{pmatrix} a & m \\ 0 & b\end{pmatrix}\in\mathcal{G}$, where $R_{11},\ S_{11},\ T_{11}$ are $\mathcal{R}$-linear mappings from $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$ to $\mathcal{A}$, respectively; $R_{12},\ S_{12},\ T_{12}$ are $\mathcal{R}$-linear mappings from $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$ to $\mathcal{M}$, respectively; $R_{22},\ S_{22},\ T_{22}$ are $\mathcal{R}$-linear mappings from $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$ to $\mathcal{B}$, respectively.

Since $\mathcal{L}$ is a Lie σ-centralizer, we have

$$\mathcal{L}([x,y]) = [\mathcal{L}(x),\sigma(y)] = [\sigma(x),\mathcal{L}(y)] \quad \text{for all} \quad x,y\in\mathfrak{A}. \tag{4.1}$$

Let us choose $x = \begin{pmatrix} a & 0 \\ 0 & 0\end{pmatrix}$ and $y = \begin{pmatrix} 0 & m \\ 0 & 0\end{pmatrix}$ in (4.1). Then we find that

$$\begin{pmatrix} T_{11}(am) & T_{12}(am) \\ 0 & T_{22}(am)\end{pmatrix} = \begin{pmatrix} 0 & R_{11}(a)u(m) - u(m)R_{22}(a) \\ 0 & 0\end{pmatrix}$$

$$= \begin{pmatrix} [\gamma(a),T_{11}(m)] & \gamma(a)T_{12}(m) + \gamma(a)m_0T_{22}(m) - T_{11}(m)\gamma(a)m_0 \\ 0 & 0\end{pmatrix}.$$

From the above equation, we get $T_{11}(am) = 0 = [\gamma(a),T_{11}(m)]$, $T_{12}(am) = R_{11}(a)u(m) - u(m)R_{22}(a) = \gamma(a)T_{12}(m) + \gamma(a)m_0T_{22}(m) - T_{11}(m)\gamma(a)m_0$ and $T_{22}(am) = 0$. Putting $a = 1_{\mathcal{A}}$, we get $T_{11}(m) = 0$ and $T_{22}(m) = 0$. Thus, $T_{12}(am) = R_{11}(a)u(m) - u(m)R_{22}(a) = \gamma(a)T_{12}(m)$. Similarly, if we choose $x = \begin{pmatrix} 0 & m \\ 0 & 0\end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b\end{pmatrix}$ in (4.1), we can obtain $T_{12}(mb) = u(m)S_{22}(b) - S_{11}(b)u(m) = T_{12}(m)\delta(b)$.

If we consider $x = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}$ in (4.1). Then

$$\begin{pmatrix} R_{11}([a_1,a_2]) & R_{12}([a_1,a_2]) \\ 0 & R_{22}([a_1,a_2]) \end{pmatrix}$$
$$= \begin{pmatrix} [R_{11}(a_1),\gamma(a_2)] & R_{11}(a_1)\gamma(a_2)m_0 - \gamma(a_2)R_{12}(a_1) - \gamma(a_2)m_0R_{22}(a_1) \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} [\gamma(a_1),R_{11}(a_2)] & \gamma(a_1)R_{12}(a_2) + \gamma(a_1)m_0R_{22}(a_2) - R_{11}(a_2)\gamma(a_1)m_0 \\ 0 & 0 \end{pmatrix}.$$

From the above relation, we find that $R_{11}([a_1,a_2]) = [R_{11}(a_1),\gamma(a_2)] = [\gamma(a_1),R_{11}(a_2)]$, i.e., R_{11} is a Lie γ-centralizer on $\mathcal{A}$, $R_{12}(a) = R_{11}(a)m_0 - m_0R_{22}(a)$ and $R_{22}([a_1,a_2]) = 0$.

Similarly, by considering $x = \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}$ in (4.1) we obtain $S_{11}([b_1,b_2]) = 0$, $S_{12}(b) = S_{11}(b)m_0 - m_0S_{22}(b)$ and $S_{22}([b_1,b_2]) = [S_{22}(b_1),\delta(b_2)] = [\delta(b_1),S_{22}(b_2)]$, i.e., S_{22} is a Lie δ-centralizer on $\mathcal{B}$.

Taking $x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ in (4.1), we have

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -R_{11}(a)m_0\delta(b) + R_{12}(a)\delta(b) + m_0\delta(b)R_{22}(b) \\ 0 & [R_{22}(a),\delta(b)] \end{pmatrix}$$
$$= \begin{pmatrix} [\gamma(a),S_{11}(b)] & \gamma(a)S_{12}(b) + \gamma(a)m_0S_{22}(b) - S_{11}(b)\gamma(a)m_0 \\ 0 & 0 \end{pmatrix}.$$

From the above equation, we obtain $[R_{22}(a),\delta(b)] = 0$ and $[\gamma(a),S_{11}(b)] = 0$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Since γ and δ are automorphisms on $\mathcal{A}$ and $\mathcal{B}$, respectively, we conclude $R_{22}(a) \in \mathcal{Z}(\mathcal{B})$ and $S_{11}(b) \in \mathcal{Z}(\mathcal{A})$.

Conversely, suppose that $\mathcal{L}$ is a linear mapping on $\mathfrak{A}$ of the form

$$\mathcal{L}\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} R_{11}(a) + S_{11}(b) & (R_{11}(a) + S_{11}(b))m_0 - m_0(R_{22}(a) + S_{22}(b)) + T_{12}(m) \\ 0 & R_{22}(a) + S_{22}(b) \end{pmatrix}$$

satisfying the assumptions (1)–(3). Then it is easy to check that $\mathcal{L}$ satisfies the relation

$$\mathcal{L}([x,y]) = [\mathcal{L}(x),\sigma(y)] = [\sigma(x),\mathcal{L}(y)] \quad \text{for all} \quad x,y \in \mathfrak{A},$$

that is, $\mathcal{L}$ is a Lie σ-centralizer on $\mathfrak{A}$.

Proposition 4.1 is proved.

If $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule, then the conditions $R_{22}([a_1,a_2]) = 0$ and $S_{11}([b_1,b_2]) = 0$ in the above theorem become superfluous.

***Corollary* 4.1.** *Let* $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *be a triangular algebra consisting of* $\mathcal{A}$, $\mathcal{B}$ *and* $\mathcal{M}$. *Suppose that* $\mathcal{A}$, $\mathcal{B}$ *have only trivial idempotents and* $\mathcal{M}$ *is a faithful* $(\mathcal{A}, \mathcal{B})$*-bimodule. An* $\mathcal{R}$*-linear mapping* $\mathcal{L}\colon \mathfrak{A} \to \mathfrak{A}$ *is a Lie* σ*-centralizer on* $\mathfrak{A}$ *if and only if* $\mathcal{L}$ *has the following form:*

$$\mathcal{L}\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} R_{11}(a) + S_{11}(b) & (R_{11}(a) + S_{11}(b))m_0 - m_0(R_{22}(a) + S_{22}(b)) + T_{12}(m) \\ 0 & R_{22}(a) + S_{22}(b) \end{pmatrix},$$

where $R_{11}\colon \mathcal{A} \to \mathcal{A}$, $S_{11}\colon \mathcal{A} \to \mathcal{Z}(\mathcal{A})$, $T_{12}\colon \mathcal{M} \to \mathcal{M}$, $R_{22}\colon \mathcal{A} \to \mathcal{Z}(\mathcal{B})$ *and* $S_{22}\colon \mathcal{B} \to \mathcal{B}$ *are* $\mathcal{R}$*-linear mappings satisfying the following conditions:*

(1) R_{11} *is a Lie* γ*-centralizer on* $\mathcal{A}$, $T_{12}(am) = R_{11}(a)u(m) - u(m)R_{22}(a) = \gamma(a)T_{12}(m)$;

(2) S_{22} *is a Lie* δ*-centralizer on* $\mathcal{B}$, $T_{12}(mb) = T_{12}(m)\delta(b) = u(m)S_{22}(b) - S_{11}(b)u(m)$.

Proof. In view of Proposition 4.1, it is sufficient to show that if $\mathcal{M}$ is faithful, then $R_{22}([a_1, a_2]) = 0$ and $S_{11}([b_1, b_2]) = 0$. For all $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$, we have

$$\begin{aligned} &R_{11}([a_1, a_2])u(m) - u(m)R_{22}([a_1, a_2]) = \\ &\quad = T_{12}([a_1, a_2]m) = T_{12}(a_1a_2m) - T_{12}(a_2a_1m) \\ &\quad = R_{11}(a_1)u(a_2m) - u(a_2m)R_{22}(a_1) - \gamma(a_2)T_{12}(a_1m) \\ &\quad = R_{11}(a_1)\gamma(a_2)u(m) - \gamma(a_2)u(m)R_{22}(a_1) - \gamma(a_2)(R_{11}(a_1)u(m) - u(m)R_{22}(a_1)) \\ &\quad = [R_{11}(a_1), \gamma(a_2)]u(m). \end{aligned}$$

This implies that $\mathcal{M}R_{22}([a_1, a_2]) = \{0\}$. Since $\mathcal{M}$ is a faithful right $\mathcal{B}$-module, we obtain $R_{22}([a_1, a_2]) = 0$. Similarly, one can prove that $S_{11}([b_1, b_2]) = 0$.

Corollary 4.1 is proved.

The following theorem gives a necessary and sufficient for a Lie σ-centralizer on a triangular algebras to be proper.

Theorem 4.1. *Let* $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *be a triangular algebra consisting of* $\mathcal{A}$, $\mathcal{B}$ *and* $\mathcal{M}$. *Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *have only trivial idempotents. A Lie* σ*-centralizer* $\mathcal{L}$ *on* $\mathfrak{A}$ *of the form presented in Proposition* 4.1 *is proper if and only if there exist linear mappings* $\ell_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ *and* $\ell_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ *satisfying the following conditions:*

(1) $R_{11} - \ell_{\mathcal{A}}$ *is a* γ*-centralizer on* $\mathcal{A}$ *and* $S_{22} - \ell_{\mathcal{B}}$ *is a* δ*-centralizer on* $\mathcal{B}$;

(2) $\ell_{\mathcal{A}}(a) \oplus R_{22}(a) \in \mathcal{Z}(\mathfrak{A})$ *and* $S_{11}(b) \oplus \ell_{\mathcal{B}}(b) \in \mathcal{Z}(\mathfrak{A})$ *for all* $a \in \mathcal{A}$, $b \in \mathcal{B}$.

Proof. Assume that $\mathcal{L}$ is a Lie σ-centralizer on $\mathfrak{A}$ of the form presented in Proposition 4.1 and there exist linear mappings $\ell_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ and $\ell_{\mathcal{B}}\colon \mathcal{B} \to \mathcal{Z}(\mathcal{B})$ satisfying conditions (1) and (2). Define two mappings Θ and τ as follows:

$$\Theta\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} (R_{11} - \ell_{\mathcal{A}})(a) & (R_{11} - \ell_{\mathcal{A}})(a)m_0 - m_0(S_{22} - \ell_{\mathcal{B}})(b) + T_{12}(m) \\ 0 & (S_{22} - \ell_{\mathcal{B}})(b) \end{pmatrix}$$

and

$$\tau\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} \ell_{\mathcal{A}}(a) + S_{11}(b) & 0 \\ 0 & R_{22}(a) + \ell_{\mathcal{B}}(b) \end{pmatrix}.$$

It is easy to see that Θ and τ are $\mathcal{R}$-linear mappings and $\mathcal{L} = \Theta + \tau$. Moreover, it follows from Proposition 3.1 that Θ is a σ-centralizer on $\mathfrak{A}$. It only remains to show that $\tau(\mathfrak{A}) \subseteq \mathcal{Z}(\mathfrak{A})$. Using assumption (2), we have

$$(\ell_{\mathcal{A}}(a)+S_{11}(b))m=\ell_{\mathcal{A}}(a)m+S_{11}(b)m=mR_{22}(a)+m\ell_{\mathcal{B}}(b)=m(R_{22}(a)+\ell_{\mathcal{B}}(b))$$

for all $m\in\mathcal{M}$. Hence, $\tau(\mathfrak{A})\subseteq\mathcal{Z}(\mathfrak{A})$.

Conversely, suppose that $\mathcal{L}$ is proper, that is, $\mathcal{L}=\Theta+\tau$, where Θ is a σ-centralizer and τ is a central valued mapping. In view of the representations of $\mathcal{L}$ and Θ, the mapping $\tau=\mathcal{L}-\Theta$ has the following form:

$$\tau\begin{pmatrix} a & m\\ 0 & b\end{pmatrix}=\begin{pmatrix}(R_{11}-A_{11})(a)+S_{11}(b) & 0\\ 0 & R_{22}(a)+(S_{22}-B_{22})(b)\end{pmatrix}.$$

Set $\ell_{\mathcal{A}}=R_{11}-A_{11}$ and $\ell_{\mathcal{B}}=S_{22}-B_{22}$. Since τ is a central valued mapping, it follows that $\ell_{\mathcal{A}}$ and $\ell_{\mathcal{B}}$ are the desired mappings satisfying assumptions (1) and (2).

Theorem 4.1 is proved.

Note that if $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule, then the condition (1) of the above theorem becomes superfluous. As a consequence of Theorem 4.1, we have the following corollary.

***Corollary* 4.2.** *Let $\mathfrak{A}=\mathrm{Tri}\,(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents. If $\mathcal{L}$ is proper, then $R_{22}(\mathcal{A})\subseteq\pi_{\mathcal{B}}(\mathcal{Z}(\mathfrak{A}))$ and $S_{11}(\mathcal{B})\subseteq\pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{A}))$. The converse also holds provided $\mathcal{M}$ is faithful.*

Proof. Assume that $\mathcal{L}$ is a proper Lie σ-centralizer on $\mathfrak{A}$. Then the required conditions directly follow from Theorem 4.1. Conversely, suppose that $\mathcal{L}$ is a Lie σ-centralizer on $\mathfrak{A}$ satisfying $R_{22}(\mathcal{A})\subseteq\pi_{\mathcal{B}}(\mathcal{Z}(\mathfrak{A}))$ and $S_{11}(\mathcal{B})\subseteq\pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{A}))$. Since $\mathcal{M}$ is faithful $(\mathcal{A},\mathcal{B})$-bimodule, there exists a unique algebra isomorphism $\xi\colon\pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{A}))\to\pi_{\mathcal{B}}(\mathcal{Z}(\mathfrak{A}))$ such that $a\oplus\xi(a)\in\mathcal{Z}(\mathfrak{A})$ for all $a\in\pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{A}))$. Define $\ell_{\mathcal{A}}\colon\mathcal{A}\to\mathcal{Z}(\mathcal{A})$ and $\ell_{\mathcal{B}}\colon\mathcal{B}\to\mathcal{Z}(\mathcal{B})$ by $\ell_{\mathcal{A}}=\xi^{-1}\circ R_{22}$ and $\ell_{\mathcal{B}}=\xi\circ S_{11}$, respectively. It is easy to verify that $\ell_{\mathcal{A}}$ and $\ell_{\mathcal{B}}$ satisfy $\ell_{\mathcal{A}}(a)\oplus R_{22}(a)\in\mathcal{Z}(\mathfrak{A})$ and $S_{11}(b)\oplus\ell_{\mathcal{B}}(b)\in\mathcal{Z}(\mathfrak{A})$ for all $a\in\mathcal{A}$, $b\in\mathcal{B}$. Therefore, by Theorem 4.1, $\mathcal{L}$ is proper.

Corollary 4.2 is proved.

Now, we are in a position to state and prove our first main result of this paper which provides a sufficient condition for a Lie σ-centralizer on a triangular algebra to be proper.

Theorem 4.2. *Let $\mathfrak{A}=\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular algebra consisting of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$. Suppose that $\mathcal{A}$, $\mathcal{B}$ have only trivial idempotents and $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule. A Lie σ-centralizer on $\mathfrak{A}$ is proper if $\pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{A}))=\mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathfrak{A}))=\mathcal{Z}(\mathcal{B})$.*

Proof. Assume that $\mathcal{L}$ is a Lie σ-centralizer on $\mathfrak{A}$ of the form presented in Proposition 4.1. Then $R_{22}(\mathcal{A})\subseteq\mathcal{Z}(\mathcal{B})=\pi_{\mathcal{B}}(\mathcal{Z}(\mathfrak{A}))$ and $S_{11}(\mathcal{B})\subseteq\mathcal{Z}(\mathcal{A})=\pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{A}))$. Hence, the theorem follows from Corollary 4.2.

If σ is the identity automorphism of the triangular algebra $\mathfrak{A}=\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$, then Theorem 4.2 gives the following result.

***Corollary* 4.3** [10, Corollary 3.3]**.** *Let $\mathfrak{A}=\mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ be a triangular algebra with faithful $\mathcal{M}$. A Lie centralizer on $\mathfrak{A}$ is proper if $\pi_{\mathcal{A}}(\mathcal{Z}(\mathfrak{A}))=\mathcal{Z}(\mathcal{A})$ and $\pi_{\mathcal{B}}(\mathcal{Z}(\mathfrak{A}))=\mathcal{Z}(\mathcal{B})$.*

5. Jordan σ-centralizer on triangular algebras. Zalar [25] introduced the notion of Jordan centralizers and proved that every Jordan centralizer on a 2-torsion free semiprime ring is centralizer. Han and Wei [15] studied Jordan (α,β)-derivations on triangular algebras. In this section, we show that every Jordan σ-centralizer on a 2-torsion free triangular algebra is a σ-centralizer.

Proposition 5.1. *Let $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a 2-torsion free triangular algebra consisting of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents. An $\mathcal{R}$-linear mapping $J\colon \mathfrak{A} \to \mathfrak{A}$ is a Jordan σ-centralizer on $\mathfrak{A}$ if and only if J has the following form:*

$$J\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} R_{11}(a) & R_{11}(a)m_0 - m_0S_{22}(b) + T_{12}(m) \\ 0 & S_{22}(b) \end{pmatrix},$$

where $R_{11}\colon \mathcal{A} \to \mathcal{A}$, $T_{12}\colon \mathcal{M} \to \mathcal{M}$ and $S_{22}\colon \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(1) *R_{11} is a Jordan γ-centralizer on $\mathcal{A}$, $T_{12}(am) = R_{11}(a)u(m) = \gamma(a)T_{12}(m)$;*

(2) *S_{22} is a Jordan δ-centralizer on $\mathcal{B}$, $T_{12}(mb) = T_{12}(m)\delta(b) = u(m)S_{22}(b)$.*

Proof. Suppose that Jordan σ-centralizer J on $\mathfrak{A}$ has the form

$$J\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} R_{11}(a) + S_{11}(b) + T_{11}(m) & R_{12}(a) + S_{12}(b) + T_{12}(m) \\ 0 & R_{22}(a) + S_{22}(b) + T_{22}(m) \end{pmatrix}$$

for all $\begin{pmatrix} a & m \\ n & b \end{pmatrix} \in \mathfrak{A}$, where R_{11}, S_{11}, T_{11} are $\mathcal{R}$-linear mappings from $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$ to $\mathcal{A}$, respectively; R_{12}, S_{12}, T_{12} are $\mathcal{R}$-linear mappings from $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$ to $\mathcal{M}$, respectively; R_{22}, S_{22}, T_{22} are $\mathcal{R}$-linear mappings from $\mathcal{A}$, $\mathcal{B}$, $\mathcal{M}$ to $\mathcal{B}$, respectively.

Since J is a Jordan σ-centralizer, we have

$$J(x \circ y) = J(x) \circ \sigma(y) = \sigma(x) \circ J(y) \quad \text{for all} \quad x, y \in \mathfrak{A}. \tag{5.1}$$

If we consider $x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ in (5.1), we have

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -R_{11}(a)m_0\delta(b) + R_{12}(a)\delta(b) - m_0\delta(b)R_{22}(b) \\ 0 & R_{22}(a) \circ \delta(b) \end{pmatrix}$$

$$= \begin{pmatrix} \gamma(a) \circ S_{11}(b) & \gamma(a)S_{12}(b) + \gamma(a)m_0S_{22}(b) + S_{11}(b)\gamma(a)m_0 \\ 0 & 0 \end{pmatrix}.$$

From the above equation, we obtain $R_{22}(a) \circ \delta(b) = 0$ and $\gamma(a) \circ S_{11}(b) = 0$ for all $a \in \mathcal{A}$, $b \in \mathcal{B}$. Putting $b = 1_{\mathcal{B}}$ and $a = 1_{\mathcal{A}}$ in the above equations, respectively, we get $2R_{22}(a) = 0$ and $2S_{11}(b) = 0$. Since $\mathfrak{A}$ is 2-torsion free, we obtain $R_{22}(a) = 0$ and $S_{11}(b) = 0$.

Let us choose $x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ in (5.1). Then we find that

$$\begin{pmatrix} T_{11}(am) & T_{12}(am) \\ 0 & T_{22}(am) \end{pmatrix} = \begin{pmatrix} 0 & R_{11}(a)u(m) \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma(a) \circ T_{11}(m) & \gamma(a)T_{12}(m) + \gamma(a)m_0T_{22}(m) + T_{11}(m)\gamma(a)m_0 \\ 0 & 0 \end{pmatrix}.$$

This implies that $T_{11}(am) = 0 = \gamma(a) \circ T_{11}(m)$, $T_{12}(am) = R_{11}(a)u(m) = \gamma(a)T_{12}(m) + \gamma(a)m_0T_{22}(m) + T_{11}(m)\gamma(a)m_0$, and $T_{22}(am) = 0$. Putting $a = 1_{\mathcal{A}}$, we get $T_{11}(m) = 0$ and $T_{22}(m) = 0$. Thus, $T_{12}(am) = R_{11}(a)u(m) = \gamma(a)T_{12}(m)$. Similarly, if we choose $x = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$ in (5.1), we can obtain $T_{12}(mb) = u(m)S_{22}(b) = T_{12}(m)\delta(b)$.

If we consider $x = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}$ in (5.1). Then

$$\begin{pmatrix} R_{11}(a_1 \circ a_2) & R_{12}(a_1 \circ a_2) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R_{11}(a_1) \circ \gamma(a_2) & R_{11}(a_1)\gamma(a_2)m_0 + \gamma(a_2)R_{12}(a_1) \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma(a_1) \circ R_{11}(a_2) & \gamma(a_1)R_{12}(a_2) + R_{11}(a_2)\gamma(a_1)m_0 \\ 0 & 0 \end{pmatrix}.$$

From the above relation, we find that $R_{11}(a_1 \circ a_2) = R_{11}(a_1) \circ \gamma(a_2) = \gamma(a_1) \circ R_{11}(a_2)$, i.e., R_{11} is a Jordan γ-centralizer on $\mathcal{A}$ and $R_{12}(a) = R_{11}(a)m_0$. Similarly, by considering $x = \begin{pmatrix} 0 & 0 \\ 0 & b_1 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}$ in (5.1) to get $S_{12}(b) = -m_0S_{22}(b)$ and $S_{22}(b_1 \circ b_2) = S_{22}(b_1) \circ \gamma(b_2) = \gamma(b_1) \circ S_{22}(b_2)$, i.e., S_{22} is a Jordan δ-centralizer on $\mathcal{B}$.

Conversely, suppose that J is a linear mapping on $\mathfrak{A}$ of the form

$$J\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} R_{11}(a) & R_{11}(a)m_0 - m_0S_{22}(b) + T_{12}(m) \\ 0 & S_{22}(b) \end{pmatrix}$$

satisfying the assumptions (1) and (2). Then it is easy to check that J satisfies the relation $J(x \circ y) = J(x) \circ \sigma(y) = \sigma(x) \circ J(y)$ for all $x, y \in \mathfrak{A}$, that is, J is Jordan σ-centralizer on $\mathfrak{A}$.

Proposition 5.1 is proved.

If $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule, then the conditions "R_{11} is a Jordan γ-centralizer of $\mathcal{A}$" and "S_{22} is a Jordan δ-centralizer of $\mathcal{B}$" in Proposition 5.1 become superfluous.

***Corollary* 5.1.** *Let $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be a 2-torsion free triangular algebra consisting of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$. Suppose that $\mathcal{A}$, $\mathcal{B}$ have only trivial idempotents and $\mathcal{M}$ is a faithful $(\mathcal{A}, \mathcal{B})$-bimodule. An $\mathcal{R}$-linear mapping $J : \mathfrak{A} \to \mathfrak{A}$ is a Jordan σ-centralizer on $\mathfrak{A}$ if and only if J has the following form:*

$$J\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} R_{11}(a) & R_{11}(a)m_0 - m_0S_{22}(b) + T_{12}(m) \\ 0 & S_{22}(b) \end{pmatrix},$$

where $R_{11} : \mathcal{A} \to \mathcal{A}$, $T_{12} : \mathcal{M} \to \mathcal{M}$ and $S_{22} : \mathcal{B} \to \mathcal{B}$ are $\mathcal{R}$-linear mappings satisfying the following conditions:

(1) $T_{12}(am) = R_{11}(a)u(m) = \gamma(a)T_{12}(m)$;

(2) $T_{12}(mb) = T_{12}(m)\delta(b) = u(m)S_{22}(b)$.

Proof. In view of Proposition 5.1, it suffices to show that if $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule, then R_{11} is a Jordan γ-centralizer of $\mathcal{A}$ and S_{22} is a Jordan δ-centralizer of $\mathcal{B}$. For any $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$, we have

$$\begin{aligned} R_{11}(a_1 \circ a_2)u(m) &= T_{12}((a_1 \circ a_2)m) = T_{12}(a_1a_2m + a_2a_1m) \\ &= R_{11}(a_1)u(a_2m) + \gamma(a_2)T_{12}(a_1m) \\ &= R_{11}(a_1)\gamma(a_2)u(m) + \gamma(a_2)R_{11}(a_1)u(m) \\ &= (R_{11}(a_1) \circ \gamma(a_2))u(m). \end{aligned}$$

This implies that $\{R_{11}(a_1 \circ a_2) - R_{11}(a_1) \circ \gamma(a_2)\}\mathcal{M} = \{0\}$. Since $\mathcal{M}$ is faithful as a left $\mathcal{A}$-module, we conclude $R_{11}(a_1 \circ a_2) = R_{11}(a_1) \circ \gamma(a_2)$. In a similar manner, one can obtain $R_{11}(a_1 \circ a_2) = \gamma(a_1) \circ R_{11}(a_2)$ for all $a_1, a_2 \in \mathcal{A}$. Thus, R_{11} is a Jordan γ-centralizer of $\mathcal{A}$. Similarly, we can show that S_{22} is a Jordan δ-centralizer of $\mathcal{B}$.

Corollary 5.1 is proved.

In view of Corollaries 3.1 and 5.1, we obtain the second main result of this paper.

Theorem 5.1. *Let $\mathfrak{A} = \mathrm{Tri}\,(\mathcal{A},\mathcal{M},\mathcal{B})$ be a 2-torsion free triangular algebra consisting of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$. Suppose that $\mathcal{A}$, $\mathcal{B}$ have only trivial idempotents and $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule. Then every Jordan σ-centralizer on $\mathfrak{A}$ is a σ-centralizer.*

Bahmani et al. [5, Corollary 2.12] proved that every Jordan centralizer on a 2-torsion free triangular algebra $\mathfrak{A} = \mathrm{Tri}\,(\mathcal{A},\mathcal{M},\mathcal{B})$ is a centralizer provided $\mathrm{l.Ann}_{\mathcal{A}}(\mathcal{M}) = \{0\} = \mathrm{r.Ann}_{\mathcal{B}}(\mathcal{M})$. If $\mathcal{M}$ is a faithful $(\mathcal{A},\mathcal{B})$-bimodule, then the condition $\mathrm{l.Ann}_{\mathcal{A}}(\mathcal{M}) = \{0\} = \mathrm{r.Ann}_{\mathcal{B}}(\mathcal{M})$ holds trivially. Thus, as a direct consequence of Theorem 5.1, we obtain the following result.

***Corollary* 5.2** [5, Corollary 2.12]. *Let $\mathfrak{A} = \mathrm{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$ be a 2-torsion free triangular algebra with faithful $\mathcal{M}$. Then every Jordan centralizer on $\mathfrak{A}$ is a centralizer.*

Acknowledgements. The authors are indebted to the referee for his/her helpful comments and suggestions which have improved the article. The second author is partially supported by a research grant from DST (No. DST/INSPIRE/03/2017/IF170834).

References

1. A. N. Alkenani, M. Ashraf, A. Jabeen, *Nonlinear generalized Jordan (σ,τ)-derivations on triangular algebras*, Spec. Matrices, **6**, 216 – 228 (2018).
2. P. N. Anh, L. Van Wyk, *Automorphism groups of generalized triangular matrix rings*, Linear Algebra and Appl., **434**, 1018 – 1026 (2011).
3. M. A. Ansari, M. Ashraf, M. S. Akhtar, *Lie triple derivations on trivial extension algebras*, Bull. Iranian Math. Soc. (2021); DOI: https://doi.org/10.1007/s41980-021-00618-3.
4. M. Ashraf, M. S. Akhtar, M. A. Ansari, *Generalized Lie (Jordan) triple derivations on arbitrary triangular algebras*, Bull. Malays. Math. Sci. Soc. (2021); DOI: https://doi.org/10.1007/s40840-021-01148-1.
5. M. A. Bahmani, F. Ghomanjani, S. Shateyi, *Jordan centralizer maps on trivial extension algebras*, Demonstr. Math., **53**, 58 – 66 (2020).
6. D. Benkovič, *Jordan σ-derivations of triangular algebras,* Linear and Multilinear Algebra, **64**, № 2, 143 – 155 (2016).
7. M. Brešar, *Jordan derivations of semiprime rings*, Proc. Amer. Math. Soc., **104**, 1003 – 1006 (1988).
8. W. S. Cheung, *Maps on triangular algebras*, Ph. D. Diss., Univ. Victoria (2000).
9. W. S. Cheung, *Lie derivation of triangular algebras*, Linear and Multilinear Algebra, **51**, 299 – 310 (2003).
10. A. Fošner, W. Jing, *Lie centralizers on triangular rings and nest algebras*, Adv. Oper. Theory, **4**, № 2, 342 – 350 (2019).

11. H. Ghahramani, *On centralizers of Banach algebras*, Bull. Malays. Math. Sci. Soc., **38**, № 1, 155 – 164 (2015).
12. H. Ghahramani, *Characterizing Jordan maps on triangular rings through commutative zero products*, Mediterr. J. Math.; https://doi.org/10.1007/s00009-018-1082-3.
13. F. Ghomanjani, M. A. Bahmani, *A note on Lie centralizer maps*, Palest. J. Math., **7**, № 2, 468 – 471 (2018).
14. C. R. González, J. Repka, J. Sánchez-Ortega, *Automorphisms, σ-biderivations and σ-commuting maps of triangular algebras,* Mediterr. J. Math.; https://doi.org/10.1007/s00009-016-0809-2.
15. D. Han, F. Wei, *Jordan (α, β)-derivations on triangular algebras and related mappings*, Linear Algebra and Appl., **434**, 259 – 284 (2011).
16. I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc., **8**, 1104 – 1110 (1957).
17. A. Jabeen, *Lie (Jordan) centralizers on generalized matrix algebras*, Comm. Algebra (2021); DOI: https://doi.org/10.1080/00927872.2020.1797759.
18. S. Jondrup, *Automorphisms and derivations of upper triangular matrix rings*, Linear Algebra and Appl., **221**, 205 – 218 (1995).
19. T. P. Kezlan, *A note on algebra automorphisms of triangular matrices over commutative rings*, Linear Algebra and Appl., **135**, 181 – 184 (1990).
20. R. Khazal, S. Dăscălescu, L. Van Wyk, *Isomorphism of generalized triangular matrix-rings and recovery of tiles*, Int. J. Math. and Math. Sci., **9**, 533 – 538 (2003).
21. L. Liu, *On Jordan centralizers of triangular algebras*, Banach J. Math. Anal., **10**, № 2, 223 – 234 (2016).
22. L. Liu, *On nonlinear Lie centralizers of generalized matrix algebras*, Linear and Multilinear Algebra; https://doi.org/10.1080/03081087.2020.1810605.
23. W. S. Martindale, *Lie derivations of primitive rings, III*, Michigan Math. J., **11**, 183 – 187 (1964).
24. W. Yang, J. Zhu, *Characterizations of additive (generalized) ξ-Lie (α, β)-derivations on triangular algebras*, Linear and Multilinear Algebra, **61**, 811 – 830 (2013).
25. B. Zalar, *On centralizers of semiprime rings*, Comment. Math. Univ. Carolin., **32**, 609 – 614 (1991).
26. J.-H. Zhang, W.-Y. Yu, *Jordan derivations of triangular algebras,* Linear Algebra and Appl., **419**, 251 – 255 (2006).

Received 21.09.21