DOI: 10.37863/umzh.v75i4.6838

UDC 517.5

İ. Gençtürk[1] (Kırıkkale University, Turkey)

ROBIN BOUNDARY-VALUE PROBLEM FOR THE BELTRAMI EQUATION
КРАЙОВА ЗАДАЧА РОБІНА ДЛЯ РІВНЯННЯ БЕЛЬТРАМІ

We investigate the unique solution of the Robin boundary-value problem for the Beltrami equation with constant coefficients in the unit disc by using a technique based on a singular integral operator defined on $L_p(\mathbb{D})$ for all $p > 2$.

Досліджено єдиний розв'язок крайової задачі Робіна для рівняння Бельтрамі зі сталими коефіцієнтами в одиничному крузі за допомогою процедури, яка ґрунтується на сингулярному інтегральному операторі, визначеному на $L_p(\mathbb{D})$ для всіх $p > 2$.

1. Introduction. So far some boundary-value problems for the Beltrami equation were studied in the literature. For instance Schwarz and Dirichlet, Neumann problems for the Beltrami equation are solved in [5, 7], respectively. On the other hand, the solution of the Robin boundary-value problem for the Poisson equation is given in [4]. However, the Robin boundary-value problem for the Beltrami equation has not been investigated using Begehr's approach in the literature. In Begehr's approach, solutions of the boundary-value problems for certain types of complex partial differential equations under appropriate conditions are put forward as elementary. In other methods, by transforming boundary-value problems for complex partial differential equations into singular integral equations, existence and uniqueness of solutions of boundary-value problems are investigated using functional analytic tools, such as fixed point theorems.

As one of the main examples of complex partial differential equations, the Cauchy – Riemann system, written in complex form as $w_{\overline{z}} = 0$ is a special form of an elliptic system of two real first order partial differential equations. The Beltrami system is a more general system of the same type, and in a regular domain $D \subset \mathbb{C}$, in complex notation $w = u + iv$, $z = x + iy$, $w_{\overline{z}} = \partial_{\overline{z}} w = \dfrac{1}{2}(w_x + iw_y)$ and $w_z = \partial_z w = \dfrac{1}{2}(w_x - iw_y)$, it has the form $w_{\overline{z}} + q(z)w_z = f(z)$, where $q\colon D \to \mathbb{C}$ is a measurable function satisfying $|q(z)| \leq q_0 < 1$.

This condition, guaranteeing the strong ellipticity of the system, is called the ellipticity condition.

The solutions of Schwarz and Dirichlet boundary-value problems for the Beltrami equation are investigated in [7] by using a technique that relies on a singular integral operator defined on $L_p(D)$ for all $p > 2$. In [6], by considering some special cases ($q \equiv 0$ and q is a constant) for the Beltrami equation $Bw = w_{\overline{z}} + qw_z = 0$, the Riemann – Hilbert boundary-value problems are solved explicitly for the half-plane and for an ellipse. In addition in [10], Vekua investigated the Beltrami equation in the theory of quasiconformal mappings. In [8, 9], authors study homeomorphic solutions of the Dirichlet problem for the Beltrami equations in arbitrary Jordan domains.

In this paper, we obtain the solvability conditions and solution of the Robin boundary-value problem for the Beltrami equation with constant coefficients for $q(z) \equiv c$ and $|c| < 1$, in the unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$,

[1] E-mail: ilkergencturk@gmail.com.

$$w_{\bar{z}} + c w_z = f, \quad f \in L_p(\mathbb{D}, \mathbb{C}) \cap C(\overline{\mathbb{D}}, \mathbb{C}), \quad p > 2, \tag{1.1}$$

$$(w + \partial_v w)|_{\partial\mathbb{D}} = \gamma, \quad \gamma \in C(\partial\mathbb{D}, \mathbb{C}). \tag{1.2}$$

Here, ∂_v is directional derivative in the direction of the outward normal vector to the boundary $\partial\mathbb{D}$.

We need the following formulas in order to calculate some integrals: Let $D \subseteq \mathbb{C}$ be a regular domain and $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Then, for $z \in D$, we have

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial D} w(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int\limits_{D} w_{\overline{\zeta}}(\zeta) \frac{d\xi d\eta}{\zeta - z}, \tag{1.3}$$

$$w(z) = -\frac{1}{2\pi i} \int\limits_{\partial D} w(\zeta) \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}} - \frac{1}{\pi} \int\limits_{D} w_{\zeta}(\zeta) \frac{d\xi d\eta}{\overline{\zeta} - \overline{z}}, \tag{1.4}$$

$$\int\limits_{D} w_{\overline{\zeta}}(\zeta) d\xi d\eta = \frac{1}{2i} \int\limits_{\partial D} w(\zeta) d\zeta, \tag{1.5}$$

$$\int\limits_{D} w_{\zeta}(\zeta) d\xi d\eta = -\frac{1}{2i} \int\limits_{\partial D} w(\zeta) d\overline{\zeta}, \tag{1.6}$$

where $\zeta = \xi + i\eta$. For the proofs of these representations cf. [2].

2. The general solution of the Beltrami equation. In [10], the general solution of the Beltrami equation

$$w_{\bar{z}}(z) + q(z) w_z(z) = f(z), \quad z \in D, \quad f \in L_p(D, \mathbb{C}), \quad p > 2,$$

is constructed as in the form

$$w(z) = \varphi(z) + T\rho(z),$$

where φ is an arbitrary analytic function in the considered domain D, T is the well-known Pompeiu operator defined by

$$Tf(z) = -\frac{1}{\pi} \int\limits_{D} f(\zeta) \frac{d\xi d\eta}{\zeta - z},$$

and $\rho(z)$ is a solution to the related singular integral equation

$$\rho(z) + q\Pi[\rho(z)] = f(z) - q\varphi'(z),$$

where Π denotes the well-known, strongly singular integral operator, the Ahlfors – Beurling operator defined by

$$\Pi\rho(z) = -\frac{1}{\pi} \int\limits_{D} \rho(\zeta) \frac{d\xi d\eta}{(\zeta - z)^2}.$$

We note that (generalized) derivative of $T\rho(z)$ with respect to z $(\bar{z})$ is

$$\frac{\partial}{\partial z}T\rho(z) = \Pi\rho(z) \quad \left(\frac{\partial}{\partial \bar{z}}T\rho(z) = \rho(z)\right).$$

With boundness of the operator Π, we refer to [1, Theorem 32], [10, p. 337] and by using condition on f and q, the solution of the singular integral equation can be given the following converge Neumann series:

$$\rho(z) = \sum_{k=0}^{\infty}(-1)^k(q\Pi)^k[f - q\varphi'](z).$$

If we consider the simple particular case where $q(z) = \text{constant} := c, z \in D$, with some calculations, we have

$$\rho(z) = f(z) - c\varphi'(z) + \sum_{k=1}^{\infty} kc^k \frac{1}{\pi} \int\limits_{|\zeta|<1} \Big(f(\zeta) - c\varphi'(\zeta)\Big) \frac{(\overline{\zeta - z})^{k-1}}{(\zeta - z)^{k+1}} d\xi d\eta. \tag{2.1}$$

For the proof of this equation, we refer the reader to [7].

3. Robin boundary-value problem. In this section, we consider the Robin boundary-value problem (1.1), (1.2) for the Beltrami equation in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

After a simple calculation, the Robin boundary condition (1.2) can be written as

$$(w + zw_z + \bar{z}w_{\bar{z}})(z) = \gamma(z), \quad z \in \partial\mathbb{D}.$$

Hence,

$$w_z(z) = \varphi'(z) + \Pi\rho(z) \quad \text{and} \quad w_{\bar{z}}(z) = \rho(z).$$

By defining $\hat{\gamma}(z)$ for $z \in \partial\mathbb{D}$ as

$$(\varphi + z\varphi_z)(z)|_{\partial\mathbb{D}} = \gamma(z) + \frac{1}{\pi}\int\limits_D \frac{\rho(\zeta)}{\zeta - z} d\xi d\eta + \frac{z}{\pi}\int\limits_D \frac{\rho(\zeta)}{(\zeta - z)^2} d\xi d\eta - \bar{z}\rho(z) := \hat{\gamma}(z), \tag{3.1}$$

we get the Robin boundary-value problem for the analytic function φ

$$(\varphi + z\varphi_z)(z)|_{\partial\mathbb{D}} = \hat{\gamma}(z). \tag{3.2}$$

According to [3, Theorem 1.1], this Robin boundary-value problem for analytic functions is solvable if and only if $\hat{\gamma}$ satisfies the condition

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \hat{\gamma}(\zeta) \frac{\bar{z}}{1 - \bar{z}\zeta} d\zeta = 0 \tag{3.3}$$

for all $|z| < 1$.

In this case, the solution has the form

$$\varphi(z) = -\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \hat{\gamma}(\zeta) \frac{\log(1 - z\bar{\zeta})}{z} d\zeta. \tag{3.4}$$

It is noted that the point $z = 0$ is a removable singular point of the function $\dfrac{\log(1 - z\bar{\zeta})}{z}$ and it is clear that

$$\frac{\log(1 - z\bar{\zeta})}{z} = -\sum_{k=0}^{\infty} \frac{1}{k+1} \frac{z^k}{\zeta^{k+1}}.$$

If $\hat{\gamma}$ in (3.1) is substituted in (3.3), we get, for $t = t_1 + it_2$,

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \hat{\gamma}(\zeta) \frac{\bar{z}}{1 - \bar{z}\zeta} d\zeta = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \big(\gamma(\zeta) - \bar{\zeta}\rho(\zeta)\big) \frac{\bar{z}}{1 - \bar{z}\zeta} d\zeta$$

$$+ \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \left[\frac{1}{\pi} \int\limits_{|t|<1} \rho(t) \frac{dt_1 dt_2}{t - \zeta} \right] \frac{\bar{z}}{1 - \bar{z}\zeta} d\zeta$$

$$+ \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \left[\frac{\zeta}{\pi} \int\limits_{|t|<1} \rho(t) \frac{dt_1 dt_2}{(t - \zeta)^2} \right] \frac{\bar{z}}{1 - \bar{z}\zeta} d\zeta = 0. \tag{3.5}$$

Then, by changing the order of integrations the right-hand side of (3.5), we have

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \big[\gamma(\zeta) - \bar{\zeta}\rho(\zeta)\big] \frac{\bar{z}}{1 - \bar{z}\zeta} d\zeta + \frac{1}{\pi} \int\limits_{|t|<1} \rho(t) \frac{\bar{z}}{2\pi i} \int\limits_{|\zeta|=1} \frac{d\zeta}{(t - \zeta)(1 - \bar{z}\zeta)} dt_1 dt_2$$

$$+ \frac{1}{\pi} \int\limits_{|t|<1} \rho(t) \frac{\bar{z}}{2\pi i} \int\limits_{|\zeta|=1} \frac{\zeta d\zeta}{(t - \zeta)^2 (1 - \bar{z}\zeta)} dt_1 dt_2 = 0, \tag{3.6}$$

and using the Cauchy integral formula, we see that

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{d\zeta}{(t - \zeta)(1 - \bar{z}\zeta)} = -\frac{1}{1 - \bar{z}t} \quad \text{and} \quad \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\zeta d\zeta}{(t - \zeta)^2 (1 - \bar{z}\zeta)} = \frac{1}{(1 - \bar{z}t)^2}.$$

Hence, from (3.6), for $|z| < 1$, the solvability condition can be found as

$$\bar{z} \left\{ \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{1 - \bar{z}\zeta} - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \bar{\zeta}\rho(\zeta) \frac{d\zeta}{1 - \bar{z}\zeta} + \frac{\bar{z}}{\pi} \int\limits_{|\zeta|<1} \rho(\zeta) \frac{\zeta}{(1 - \bar{z}\zeta)^2} d\xi d\eta \right\} = 0. \tag{3.7}$$

By substituting the value of $\rho(z)$ which is (2.1) in (3.7), we get

$$\bar{z} \left\{ \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{d\zeta}{1 - \bar{z}\zeta} - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \bar{\zeta} \left(f(\zeta) - c\varphi'(\zeta) \right. \right.$$

$$\left. + \sum_{k=1}^{\infty} k c^k \frac{1}{\pi} \int\limits_{|t|<1} \big(f(t) - c\varphi'(t)\big) \frac{\overline{(t - \zeta)}^{k-1}}{(t - \zeta)^{k+1}} dt_1 dt_2 \right) \frac{d\zeta}{1 - \bar{z}\zeta}$$

$$+\frac{\bar z}{\pi}\int\limits_{|\zeta|<1}\left(f(\zeta)-c\varphi'(\zeta)+\sum_{k=1}^{\infty}kc^k\frac{1}{\pi}\int\limits_{|t|<1}\big(f(t)-c\varphi'(t)\big)\frac{(\overline{t-\zeta})^{k-1}}{(t-\zeta)^{k+1}}dt_1dt_2\right)\frac{\zeta d\xi d\eta}{(1-\bar z\zeta)^2}\Bigg\}=0.$$

Again by changing the order of integrations, the solvability condition can be written as

$$\bar z\Bigg\{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{1-\bar z\zeta}-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\big(f(\zeta)-c\varphi'(\zeta)\big)\frac{\overline{\zeta}d\zeta}{1-\bar z\zeta}$$

$$-\frac{1}{\pi}\int\limits_{|t|<1}\sum_{k=1}^{\infty}kc^k\big(f(t)-c\varphi'(t)\big)\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{t-\zeta})^{k-1}}{(t-\zeta)^{k+1}}\frac{\overline{\zeta}d\zeta}{1-\bar z\zeta}\right)dt_1dt_2$$

$$+\frac{\bar z}{\pi}\int\limits_{|\zeta|<1}\big(f(\zeta)-c\varphi'(\zeta)\big)\frac{\zeta d\xi d\eta}{(1-\bar z\zeta)^2}$$

$$+\frac{\bar z}{\pi}\int\limits_{|t|<1}\sum_{k=1}^{\infty}kc^k\big(f(t)-c\varphi'(t)\big)\left(\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{t-\zeta})^{k-1}}{(t-\zeta)^{k+1}}\frac{\zeta d\xi d\eta}{(1-\bar z\zeta)^2}\right)dt_1dt_2\Bigg\}=0. \tag{3.8}$$

In order to evaluate (3.8) we need the following lemma.

Lemma 3.1. *For* $|z|<1,\ |t|<1$ *and* $k\in\mathbb{N}_0$,

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{t-\zeta})^{k-1}}{t-\zeta}\frac{\overline{\zeta}d\zeta}{1-\bar z\zeta}=-\frac{\overline{z}(\overline{t-z})^{k-1}}{1-\bar z t}$$

and

$$\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{t-\zeta})^{k-1}}{t-\zeta}\frac{\zeta d\xi d\eta}{(1-\bar z\zeta)^2}=-\frac{(1-t\bar t)(\overline{t-z})^{k-1}}{(1-\bar z t)^2}$$

hold.

Proof. For $|z|<1,\ |t|<1$ and $k\in\mathbb{N}_0$, by using Cauchy integral, Cauchy's differentiation formula and from (1.3), we get

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{t-\zeta})^{k-1}}{t-\zeta}\frac{\overline{\zeta}d\zeta}{1-\bar z\zeta}=\overline{-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(t-\zeta)^{k-1}\zeta}{1-\bar t\zeta}\frac{d\zeta}{\zeta-z}}=-\frac{\overline{z}(\overline{t-z})^{k-1}}{1-\overline{z}t}$$

and similarly

$$\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{t-\zeta})^{k-1}}{t-\zeta}\frac{\zeta d\xi d\eta}{(1-\bar z\zeta)^2}=\frac{1}{k}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{t-\zeta})^{k}}{(1-\bar z\zeta)^2}\frac{d\zeta}{\zeta-t}+w(t)$$

$$=\overline{\frac{1}{k}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(t-\zeta)^{k}}{1-\zeta\bar t}\frac{d\zeta}{(\zeta-z)^2}}=-\frac{(1-t\bar t)(\overline{t-z})^{k-1}}{(1-\bar z t)^2},$$

where $w(t)=0$, because of $w(\zeta)=\dfrac{(\overline{t-\zeta})^k}{k(1-\bar z\zeta)^2}$.

Lemma 3.1 is proved.

By Lemma 3.1, we get

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{t-\zeta})^{k-1}}{(t-\zeta)^{k+1}}\frac{\overline{\zeta}d\zeta}{1-\bar z\zeta}=-\frac{\bar z^{k+1}(\overline{t-z})^{k-1}}{(1-\bar z t)^{k+1}},$$

$$\frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{(\overline{t-\zeta})^{k-1}}{(t-\zeta)^{k+1}}\frac{\zeta d\xi d\eta}{(1-\bar z\zeta)^2}=-\frac{\bar z^{k-1}(k(\overline{t-z})-\overline{z}(1-|t|^2))}{(1-\bar z t)^{k+2}}.$$

If these expressions are written in (3.8), we have solvability condition for the Robin boundary-value problem (1.1), (1.2) as

$$\bar z\Bigg\{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Big(\gamma(\zeta)-\overline{\zeta}f(\zeta)-c\overline{\zeta}\varphi'(\zeta)\Big)\frac{d\zeta}{1-\bar z\zeta}+\frac{1}{\pi}\int\limits_{|\zeta|<1}\big(f(\zeta)-c\varphi'(\zeta)\big)\frac{\zeta d\xi d\eta}{(1-\bar z\zeta)^2}$$
$$+\frac{1}{\pi}\int\limits_{|\zeta|<1}\sum_{k=1}^{\infty}(-1)^k k c^k\big(f(\zeta)-c\varphi'(\zeta)\big)$$
$$\times\left(\frac{\bar z^{k+1}(\overline{t-z})^{k-1}}{(1-\bar z t)^{k+1}}-\frac{\bar z^{k-1}\big(k(\overline{t-z})-\overline{z}(1-|t|^2)\big)}{(1-\bar z t)^{k+2}}\right)d\xi d\eta\Bigg\}=0. \tag{3.9}$$

Similarly, if the value of $\hat\gamma$ is plugged in (3.4), we have

$$\varphi(z)=-\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\hat\gamma(\zeta)\frac{\log(1-z\bar\zeta)}{z}d\zeta$$
$$=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\big(\bar\zeta\rho(\zeta)-\gamma(\zeta)\big)\frac{\log(1-z\bar\zeta)}{z}d\zeta-\frac{1}{\pi}\int\limits_{|t|<1}\rho(t)\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-z\bar\zeta)}{t-\zeta}\frac{d\zeta}{z}\right)dt_1dt_2$$
$$-\frac{1}{\pi}\int\limits_{|t|<1}\rho(t)\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-z\bar\zeta)}{(t-\zeta)^2}\frac{\zeta}{z}d\zeta\right)dt_1dt_2.$$

On the other hand, because of

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-z\bar\zeta)}{t-\zeta}\frac{\zeta}{z}d\zeta=-\frac{1}{z}\overline{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-\bar z\zeta)}{1-\zeta\bar t}\frac{d\zeta}{\zeta}}=0,$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-z\bar\zeta)}{(t-\zeta)^2}\frac{\zeta}{z}d\zeta=\frac{1}{z}\overline{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-\bar z\zeta)}{(1-\zeta\bar t)^2}\frac{d\zeta}{\zeta}}=0,$$

the solution $\varphi(z)$ can be found as

$$\varphi(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\big(\overline{\zeta}\rho(\zeta)-\gamma(\zeta)\big)\frac{\log(1-z\bar{\zeta})}{z}d\zeta. \tag{3.10}$$

Again, if the value of $\rho(z)$ in (2.1) is inserted in (3.10), we obtain

$$\varphi(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\left(\bar{\zeta}f(\zeta)-c\bar{\zeta}\varphi'(\zeta)-\gamma(\zeta)\right)\frac{\log(1-z\bar{\zeta})}{z}d\zeta$$

$$+\sum_{k=1}^{\infty}kc^k\frac{1}{\pi}\int\limits_{|t|<1}\big(f(t)-c\varphi'(t)\big)\left(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{t-\zeta})^{k-1}}{(t-\zeta)^{k+1}}\frac{\log(1-z\bar{\zeta})}{z}\bar{\zeta}d\zeta\right)dt_1dt_2.$$

As a next step, by using the Cauchy integral formula, we see that

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(\overline{t-\zeta})^{k-1}}{(t-\zeta)^{k+1}}\frac{\log(1-z\bar{\zeta})}{z}\bar{\zeta}d\zeta=\frac{1}{z}\overline{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{(t-\zeta)^{k-1}}{(\bar{t}\zeta-1)^{k+1}}\log(1-\overline{z}\zeta)\zeta^k d\zeta}=0.$$

So, the solution of the Robin boundary-value problem for the analytic function (3.2) is

$$\varphi(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\big(\bar{\zeta}f(\zeta)-c\bar{\zeta}\varphi'(\zeta)-\gamma(\zeta)\big)\frac{\log(1-z\bar{\zeta})}{z}d\zeta. \tag{3.11}$$

Hence, by considering (2.1) and (3.11) we conclude that the solution of the Robin boundary-value problem for the Beltrami equation with constant coefficients is

$$w(z)=\varphi(z)+T\rho(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\big(\bar{\zeta}f(\zeta)-\bar{\zeta}c\varphi'(\zeta)-\gamma(\zeta)\big)\frac{\log(1-z\bar{\zeta})}{z}d\zeta$$

$$-\sum_{k=0}^{\infty}c^k\frac{1}{\pi}\int\limits_{|\zeta|<1}\big(f(\zeta)-c\varphi'(\zeta)\big)\frac{(\overline{\zeta-z})^k}{(\zeta-z)^{k+1}}d\xi d\eta.$$

Since $|c|<1$, the solution of the boundary-value problem (1.1), (1.2) can be shown to have the simple form

$$w(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\big(\bar{\zeta}f(\zeta)-c\bar{\zeta}\varphi'(\zeta)-\gamma(\zeta)\big)\frac{\log(1-z\bar{\zeta})}{z}d\zeta$$

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1}\big(f(\zeta)-c\varphi'(\zeta)\big)\frac{1}{\zeta-z-c(\overline{\zeta-z})}d\xi d\eta. \tag{3.12}$$

Finally we get following theorem.

Theorem 3.1. *In the unit disc, the Robin boundary-value problem* (1.1), (1.2) *for the Beltrami equation with constant coefficients can be solvable if and only if the condition* (3.9) *is satisfied. In this case, the unique solution of this problem has the form* (3.12).

References

1. H. Begehr, *Complex analytic methods for partial differential equations: an introductory text*, World Sci. (1994).
2. H. Begehr, *Boundary value problems in complex analysis, I*, Bol. Asoc. Mat. Venez., **12**, 65 – 85 (2005).
3. H. Begehr, G. Harutjunjan, *Robin boundary value problem for the Cauchy – Riemann operator*, Complex Var. and Elliptic Equat., **50**, № 15, 1125 – 1136 (2005).
4. H. Begehr, G. Harutjunjan, *Robin boundary value problem for the Poisson equation*, J. Anal. and Appl., **4**, № 3, 201 – 213 (2006).
5. H. Begehr, G. Harutjunjan, *Neumann problem for the Beltrami operator and for second order operators with Poisson – Bitsadze operator as main part*, Complex Var. and Elliptic Equat., **54**, № 12, 1129 – 1150 (2009).
6. H. Begehr, E. Obolashvili, *Some boundary value problems for a Beltrami equation*, Complex Var. and Elliptic Equat., **26**, № 1-2, 113 – 122 (1994).
7. G. Harutyunyan, *Boundary value problems for the Beltrami operator*, Complex Var. and Elliptic Equat., **52**, № 6, 475 – 484 (2007).
8. D. A. Kovtonyuk, I. V. Petkov, V. I. Ryazanov, R. Salimov, *Boundary behavior and the Dirichlet problem for Beltrami equations*, St. Petersburg Math. J., **25**, № 4, 587 – 603 (2014).
9. D. A. Kovtonyuk, I. V. Petkov, V. I. Ryazanov, R. Salimov, *On the Dirichlet problem for the Beltrami equation*, J. Anal. Math., **122**, № 1, 113 – 141 (2014).
10. I. N. Vekua, *Generalized analytic functions*, Pergamon, Oxford (1962).

Received 18.07.21