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# SOME COMMUTATIVITY CRITERIA FOR PRIME RINGS WITH INVOLUTION INVOLVING SYMMETRIC AND SKEW SYMMETRIC ELEMENTS

# ДЕЯКІ КРИТЕРІЇ КОМУТАТИВНОСТІ ПРОСТИХ КІЛЕЦЬ З ІНВОЛЮЦІЄЮ СИМЕТРИЧНИХ ТА КОСОСИМЕТРИЧНИХ ЕЛЕМЕНТІВ

We study the Posner second theorem [Proc. Amer. Math. Soc., **8**, 1093 – 1100 (1957)] and strong commutativity preserving problem for symmetric and skew symmetric elements involving generalized derivations on prime rings with involution. The obtained results cover numerous known theorems. We also provide examples showing that the obtained results hold neither in the case of involution of the first kind, nor in the case where the ring is not prime.

Вивчається друга теорема Познера [Proc. Amer. Math. Soc., **8**, 1093 – 1100 (1957)] та проблема збереження сильної комутативності для симетричних і кососиметричних елементів, що включає узагальнені похідні на простих кільцях з інволюцією. Отримані результати охоплюють багато відомих теорем. Крім того, наведено приклади, які показують, що отримані результати несправедливі ні у випадку інволюції першого роду, ні у випадку, коли кільце не є простим.

**1. Introduction.** Throughout this paper, $R$ will denote an associative ring with a center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. An additive mapping $d\colon R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. If $d$ is a derivation on $R$, then an additive mapping $F\colon R \to R$ is called a generalized derivation of $R$ (with an associated derivation $d$) if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$ (see [8]). Basic examples of generalized derivations are usual derivations and generalized inner derivations (i.e., maps of the type $x \mapsto ax + xb$, $x \in R$, for fixed $a, b \in R$). A derivation $d\colon R \to R$ is said to be centralizing on $R$ if $[d(x), x] \in Z(R)$ for all $x \in R$. Centralizing derivations were first considered by Posner [20], who proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Over the years, this result was extended in various directions (see, for example, [5, 6] and references therein).

A mapping $\phi\colon R \to R$ preserves commutativity if $[\phi(x), \phi(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in R$. Commutativity preserving maps have been studied intensively in matrix theory, operator theory and ring theory (see, for example, [9, 22]). Following [7], a map $\phi\colon R \to R$ is said to be strong commutativity preserving (SCP) on a subset $S \subseteq R$ if $[\phi(x), \phi(y)] = [x, y]$ for all $x, y \in S$. In the course of time several techniques have been developed to investigate the behavior of strong commutativity preserving maps using restrictions on polynomials invoking derivations, generalized derivations etcetera.

In [4], Bell and Daif investigated the commutativity of rings admitting a derivation which is strong commutativity preserving on a nonzero right ideal. More precisely, they proved that if a

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semiprime ring $R$ admits a derivation $d$ satisfying $[d(x),d(y)] = [x,y]$ for all $x,y \in I$, where $I$ is a right ideal of $R$, then $I \subseteq Z(R)$. In particular, $R$ is commutative if $I = R$. Later, Deng and Ashraf [12] proved that if there exists a derivation $d$ of a semiprime ring $R$ and a map $f\colon I \to R$ defined on a nonzero ideal $I$ of $R$ such that $[f(x),d(y)] = [x,y]$ for all $x,y \in I$, then $R$ contains a nonzero central ideal. Thus, $R$ is commutative in the special case when $I = R$. Recently, this result was extended to Lie ideals and symmetric elements of prime rings by Lin and Liu in [15] and [16], respectively, and to the case of generalized derivations by Ma et al. [18]. For related generalizations of these results we refer the reader to [3, 10, 13, 14, 17, 21], where further references can be found.

The above mentioned problems were also studied in the setting of rings with involution (see, for example, [11] and [19] and references therein). Recall that a ring $R$ is called a $*$-ring or a ring with involution $*$ if there is an additive map $*\colon R \to R$ satisfying $(xy)^* = y^*x^*$ and $\left(x^*\right)^* = x$ for all $x,y \in R$. An element $x$ in a $*$-ring is said to be symmetric if $x^* = x$ and skew-symmetric if $x^* = -x$. The sets of all symmetric and skew-symmetric elements will be denoted by $H(R)$ and $S(R)$, respectively. If $\mathrm{char}(R) = 2$, then, obviously, $H(R) = S(R)$. Thus, we will consider only $*$-rings $R$ with $\mathrm{char}(R) \neq 2$. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case $S(R) \cap Z(R) \neq (0)$ (e.g., involution in the case of ring of quaternions).

The aim of this paper is to generalize the results proved by Alahmadi et al. [1] and Dar and Khan [11] for the symmetric and skew symmetric elements of $R$. The paper is organized as follows. In Section 2, we discuss identities $F(x)x - xF(x) \in Z(R)$ and $F(x)x + xF(x) \in Z(R)$ for symmetric (skew-symmetric) elements $x$ in $R$. In Section 3, we consider the problem of strong commutativity preserving generalized derivations on rings with involution involving symmetric (skew-symmetric) elements. In Section 4, various examples are provided to show that our results hold neither in the case when the involution is of the first kind nor in the case when the ring is not prime. In fact, our results extend and unify the results proved in [1, 11, 19, 23].

**2. When $\boldsymbol{F}$ is centralizing on $\boldsymbol{H(R)}$ and $\boldsymbol{S(R)}$.** Throughout this section, $(R,*)$ is a 2-torsion free prime ring with involution of the second kind. In [20], Posner proved that, if $d$ is a nonzero derivation of a prime ring $R$ such that $[d(x),x] \in Z(R)$ for all $x \in R$, then $R$ is commutative. Inspired by Posner's result, Ali and Dar [2] proved a $*$-version of Posner's result as follows: Let $R$ be a prime ring with involution $*$ such that $\mathrm{char}(R) \neq 2$. Let $d$ be a nonzero derivation of $R$ such that $\left[d(x),x^*\right] \in Z(R)$ for all $x \in R$ and $d(S(R) \cap Z(R)) \neq (0)$. Then $R$ is commutative. Later, Nejjar et al. [19] generalized the above mentioned result by relaxing the condition $(d(S(R) \cap Z(R)) \neq (0))$. Recently, Alahmadi et al. [1] extended this theorem for generalized derivation as follows: Let $R$ be a prime ring with involution of the second kind such that $\mathrm{char}(R) \neq 2$. If $R$ admits a nonzero generalized derivation $F\colon R \to R$ such that $\left[F(x),x^*\right] \in Z(R)$ for all $x \in R$, then $R$ is commutative. Here, we shall handle the aforementioned result for both the symmetric and skew symmetric cases.

***Remark* 2.1.** If $[h,k] \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$, then $R$ is a commutative integral domain.

***Remark* 2.2.** If $[h,h'] \in Z(R)$ for all $h,h' \in H(R)$, then $R$ is a commutative integral domain.

***Remark* 2.3.** If $[k,k'] \in Z(R)$ for all $k,k' \in S(R)$, then $R$ is a commutative integral domain.

**Lemma 2.1.** *If $(R, *)$ admits a nonzero derivation $d\colon R \to R$ such that $d(h)h - hd(h) \in Z(R)$ for all $h \in H(R)$, then $R$ is a commutative integral domain.*

***Proof.*** Suppose that

$$[d(h), h] \in Z(R) \tag{2.1}$$

for all $h \in H(R)$. Linearizing (2.1), we get $[d(h), h'] + [d(h'), h] \in Z(R)$ for all $h, h' \in H(R)$. Thus $[d(h), h^2] + [d(h^2), h] \in Z(R)$ for all $h \in H(R)$. Expanding this and using (2.1), we get $4h[d(h), h] \in Z(R)$ for all $h \in H(R)$. Since $[d(h), h] \in Z(R)$, we have $h \in Z(R)$ or $[d(h), h] = 0$ for all $h \in H(R)$. Now $h \in Z(R)$ implies $R$ is a commutative integral domain in view of Remark 2.1. Therefore, the remaining possibility we consider

$$[d(h), h] = 0 \tag{2.2}$$

for all $h \in H(R)$. A linearization of this expression yields that

$$\big[d(h), h'\big] + \big[d\big(h'\big), h\big] = 0 \tag{2.3}$$

for all $h, h' \in H(R)$. Rearranging (2.3), we get $[d(h'), h] = [h', d(h)]$ for all $h, h' \in H(R)$. Substituting $h^2$ for $h$ in above expression, we obtain

$$\big[d\big(h'\big), h^2\big] = \big[h', d(h)\big]h + h\big[h', d(h)\big] + d(h)\big[h', h\big] + \big[h', h\big]d(h) \tag{2.4}$$

for all $h, h' \in H(R)$. Moreover, we have

$$\big[d\big(h'\big), h^2\big] = \big[d\big(h'\big), h\big]h + h\big[d\big(h'\big), h\big] = \big[h', d(h)\big]h + h\big[h', d(h)\big] \tag{2.5}$$

for all $h, h' \in H(R)$. Combining (2.4) and (2.5), we have

$$d(h)\big[h', h\big] + \big[h', h\big]d(h) = 0 \tag{2.6}$$

for all $h, h' \in H(R)$. Taking $h' = kk_0$ in (2.6), where $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$, and using the fact that $S(R) \cap Z(R) \neq (0)$, we arrive at

$$d(h)[k, h] + [k, h]d(h) = 0 \tag{2.7}$$

for all $h \in H(R)$ and $k \in S(R)$. Since every $x \in R$ can be represented as $2x = h + k$, $h \in H(R)$ and $k \in S(R)$. Therefore, in view of (2.6) and (2.7), we finally arrive at $d(h)[x, h] + [x, h]d(h) = 0$ for all $h \in H(R)$ and $x \in R$. Substituting $h + h_0$ in place of $h$, where $h_0 \in H(R) \cap Z(R)$ and $h \in H(R)$, we get $d(h_0)[x, h] + [x, h]d(h_0) = 0$. That is, $d(h_0)[x, h] = 0$ for all $x \in R$ because $R$ is 2-torsion free. Thus, in view of Remark 2.1 either $R$ is a commutative integral domain or $d(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. Taking $h_0 = k_0^2$, we have $d(k_0) = 0$ for all $k_0 \in S(R) \cap Z(R)$. Hence

$$d(Z(R)) = (0).$$

Following (2.2), we have $0 = [d(kk_0), kk_0] = [d(k), k]k_0^2$ for all $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$. Using the primeness of $R$ and the fact that $S(R) \cap Z(R) \neq (0)$, we obtain

$$[d(k), k] = 0 \tag{2.8}$$

for all $k \in S(R)$. Similarly taking $h' = kk_0$ in (2.3) and using the fact that $S(R) \cap Z(R) \neq (0)$, we arrive at

$$[d(h), k] + [d(k), h] = 0 \tag{2.9}$$

for all $h \in H(R)$ and $k \in S(R)$. Again since every $x \in R$ can be represented as $2x = h + k$, $h \in H(R),\ k \in S(R)$. Therefore, using (2.2), (2.8) and (2.9), we get

$$4[d(x), x] = [d(2x), 2x] = [d(h+k), h+k] =$$

$$= [d(h), h] + [d(h), k] + [d(k), h] + [d(k), k] = 0.$$

Thus $[d(x), x] = 0$ for all $x \in R$, because $R$ is 2-torsion free. Hence, $R$ is a commutative integral domain in view of [20, Theorem 2].

Lemma 2.1 is proved.

Now we are in a position to prove our first main result of this paper.

**Theorem 2.1.** *If $(R, *)$ admits a generalized derivation $F : R \to R$ associated with a nonzero derivation $d \colon R \to R$ such that $F(h)h - hF(h) \in Z(R)$ for all $h \in H(R)$, then $R$ is a commutative integral domain.*

***Proof.*** By the given assumption, we have

$$[F(h), h] \in Z(R) \tag{2.10}$$

for all $h \in H(R)$. Linearization of (2.10) gives

$$\big[F(h), h'\big] + \big[F\big(h'\big), h\big] \in Z(R) \tag{2.11}$$

for all $h, h' \in H(R)$. In particular, we obtain

$$[F(h_1), h] \in Z(R) \tag{2.12}$$

for all $h_1 \in H(R) \cap Z(R)$ and $h \in H(R)$. Thus, $\big[F(h_1), h^2\big] \in Z(R)$. This gives $2h[F(h_1), h] \in Z(R)$. Since $R$ is 2-torsion free, we finally get

$$h[F(h_1), h] \in Z(R) \tag{2.13}$$

for all $h_1 \in H(R) \cap Z(R)$ and $h \in H(R)$. Taking $h^2$ for $h'$ in (2.11) where $h \in H(R)$, we have $\big[F(h), h^2\big] + \big[F\big(h^2\big), h\big] \in Z(R)$ for all $h \in H(R)$. Expanding this and using (2.10), we get $3h[F(h), h] + h[d(h), h] \in Z(R)$. Replacing $h$ by $h + h_1$, where $h_1 \in H(R) \cap Z(R)$, and making use of (2.12) and (2.13), we arrive at $3h_1[F(h), h] + h_1[d(h), h] \in Z(R)$. Primeness of $R$ forces $3[F(h), h] + [d(h), h] \in Z(R)$ for all $h \in H(R)$. Since $[F(h), h] \in Z(R)$, we conclude

$$[d(h), h] \in Z(R)$$

for all $h \in H(R)$. Hence, $R$ is a commutative integral domain in view of Lemma 2.1.

Theorem 2.1 is proved.

Next we turn to a corresponding result in the skew symmetric case.

**Theorem 2.2.** *If $(R, *)$ admits a generalized derivation $F : R \to R$ associated with a nonzero derivation $d \colon R \to R$ such that $F(k)k - kF(k) \in Z(R)$ for all $k \in S(R)$, then $R$ is a commutative integral domain.*

***Proof.*** Taking $hk_0$ for $k$, where $h \in H(R)$ and $k_0 \in S(R) \cap Z(R)$ in $[F(k), k] \in Z(R)$, we get $[F(h), h]k_0^2 \in Z(R)$ for all $h \in H(R)$ and $k_0 \in S(R) \cap Z(R)$. Since $S(R) \cap Z(R) \neq (0)$, we obtain $[F(h), h] \in Z(R)$ for all $h \in H(R)$. Hence, $R$ is a commutative integral domain in view of Theorem 2.1.

**Theorem 2.3.** *If $(R, *)$ admits a generalized derivation $F : R \to R$ associated with a nonzero derivation $d : R \to R$ such that $F(h)h + hF(h) \in Z(R)$ for all $h \in H(R)$, then $R$ is a commutative integral domain.*

***Proof.*** For $h_0 \in H(R) \cap Z(R)$, $F(h)h + hF(h) \in Z(R)$ implies that $F(h_0)h_0 \in Z(R)$ for all $h_0 \in H(R) \cap Z(R)$, since $R$ is 2-torsion free. Using the primeness of $R$, we have $F(h_0) \in Z(R)$ for all $h_0 \in H(R) \cap Z(R)$. Now $F(h)h + hF(h) \in Z(R)$ for all $h \in H(R)$ implies that $[F(h)h + hF(h), h] = 0$. That is, $[F(h), h]h + h[F(h), h] = 0$ for all $h \in H(R)$. Taking $h = h + h_0$, where $h_0 \in H(R) \cap Z(R)$, we arrive at $[F(h), h]h_0 = 0$. By the primeness and the fact that $S(R) \cap Z(R) \neq (0)$, we get $[F(h), h] = 0$ for all $h \in H(R)$. Hence, $R$ is a commutative integral domain in view of Theorem 2.1.

On similar lines, we can also prove the following result.

**Theorem 2.4.** *If $(R, *)$ admits a generalized derivation $F : R \to R$ associated with a nonzero derivation $d : R \to R$ such that $F(k)k + kF(k) \in Z(R)$ for all $k \in S(R)$, then $R$ is a commutative integral domain.*

As the applications of the aforementioned results, we obtain the following corollaries.

***Corollary* 2.1** [19, Theorem 3.7]**.** *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind and let $d$ be a nonzero derivation of $R$. Then the following assertions are equivalent:*

(i) $\big[d(x), x*\big] \in Z(R)$ *for all* $x \in R$;

(ii) $d(x) \circ x* \in Z(R)$ *for all* $x \in R$;

(iii) $R$ *is commutative.*

***Corollary* 2.2** [1, Theorem 4.1]**.** *Let $R$ be a prime ring with involution of the second kind such that* $\operatorname{char}(R) \neq 2$. *If $R$ admits a nonzero generalized derivation $F : R \to R$ such that $\big[F(x), x^*\big] \in Z(R)$ for all $x \in R$, then $R$ is commutative.*

***Corollary* 2.3.** *Let $R$ be a prime ring with involution of the second kind such that* $\operatorname{char}(R) \neq 2$. *If $R$ admits a nonzero generalized derivation $F : R \to R$ such that $F(x) \circ x^* \in Z(R)$ for all $x \in R$, then $R$ is commutative.*

***Corollary* 2.4.** *If $(R, *)$ admits a generalized derivation $F : R \to R$ associated with a nonzero derivation $d : R \to R$ such that $F(x)x - xF(x) \in Z(R)$ for all $x \in R$, then $R$ is a commutative integral domain.*

***Corollary* 2.5.** *If $(R, *)$ admits a generalized derivation $F : R \to R$ associated with a nonzero derivation $d : R \to R$ such that $F(x)x + xF(x) \in Z(R)$ for all $x \in R$, then $R$ is a commutative integral domain.*

**3. When $F$ is SCP on $H(R)$ and $S(R)$.** In [11], Dar and Khan discussed the strong commutativity problem in the setting of rings with involution and proved that if $R$ is a noncommutative prime ring with involution of the second kind such that $\operatorname{char}(R) \neq 2$ and $F : R \to R$ is generalized derivation of $R$ associated with a derivation $d : R \to R$ such that $\big[F(x), F\big(x^*\big)\big] - \big[x, x^*\big] \in Z(R)$ for all $x \in R$, then $F(x) = x$ or $F(x) = -x$ for all $x \in R$. We extend this result for symmetric elements. More precise, we have the following result.

**Theorem 3.1.** *Let $(R,*)$ be a 2-torsion free noncommutative prime ring with involution of the second kind. If $R$ admits a generalized derivation $F: R \to R$ associated with a nonzero derivation $d: R \to R$ such that $[F(h), F(k)] - [h,k] \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$, then $F(x) = x$ for all $x \in R$ or $F(x) = -x$ for all $x \in R$.*

***Proof.*** By the given assumption, we have

$$[F(h), F(k)] - [h,k] \in Z(R) \tag{3.1}$$

for all $h \in H(R)$ and $k \in S(R)$. Replacing $h$ by $kk_0$ in (3.1) where $k_0 \in S(R) \cap Z(R)$, we get $[k, F(k)]d(k_0) \in Z(R)$ for all $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$. Using the primeness of $R$, we have $[F(k), k] \in Z(R)$ for all $k \in S(R)$ or $d(k_0) = 0$ for all $k_0 \in S(R) \cap Z(R)$. In the first case we get a contradiction. Therefore, we may assume that $d(Z(R)) = (0)$. Taking $k'k_0$ for $h$ in (3.1), where $k' \in S(R)$ and $k_0 \in S(R) \cap Z(R)$, we arrive at $\left(\left[F\left(k'\right), F(k)\right] - \left[k', k\right]\right)k_0 \in Z(R)$. Using the primeness of $R$ and the fact that $S(R) \cap Z(R) \neq (0)$, we obtain

$$\left[F\left(k'\right), F(k)\right] - \left[k', k\right] \in Z(R) \tag{3.2}$$

for all $k, k' \in S(R)$. Since $R$ is 2-torsion free prime ring, every $x \in R$ can be represented as $2x = h + k$, $h \in H(R)$ and $k \in S(R)$. Thus, in view of (3.1) and (3.2), we obtain

$$2[F(x), F(k)] - 2[x,k] = [F(2x), F(k)] - [2x, k] = \left[F\left(h + k'\right), F(k)\right] - \left[h + k', k\right] =$$

$$= [F(h), F(k)] + \left[F\left(k'\right), F(k)\right] - [h,k] - \left[k', k\right].$$

This gives $[F(x), F(k)] - [x,k] \in Z(R)$ for all $x \in R$ and $k \in S(R)$. Again replacing $k$ by $hk_0$, where $h \in H(R)$ and $k_0 \in S(R) \cap Z(R)$, we get $[F(x), F(h)] - [x,h] \in Z(R)$ for all $x \in R$ and $h \in H(R)$. Thus, proceeding as above, we finally arrive at $[F(x), F(y)] - [x,y] \in Z(R)$ for all $x, y \in R$. Thus, in view of [18, Theorem 4], we get $F(x) = x$ for all $x \in R$ or $F(x) = -x$ for all $x \in R$.

Theorem 3.1 proved.

***Corollary* 3.1** [11, Theorem 2.3]. *Let $(R,*)$ be a 2-torsion free noncommutative prime ring with involution of the second kind. If $R$ admits a generalized derivation $F: R \to R$ associated with a nonzero derivation $d: R \to R$ such that $\left[F(x), F\left(x^*\right)\right] - \left[x, x^*\right] \in Z(R)$ for all $x \in R$, then $F(x) = x$ for all $x \in R$ or $F(x) = -x$ for all $x \in R$.*

***Proof.*** We have

$$[F(x), F(x^*)] - [x, x^*] \in Z(R) \quad \text{for all} \quad x \in R. \tag{3.3}$$

Replacing $x$ by $h + k$ in (3.3), we obtain

$$[F(h+k), F(h-k)] - [h+k, h-k] \in Z(R) \quad \text{for all} \quad h \in H(R) \quad \text{and} \quad k \in S(R).$$

This implies that

$$2([F(h), F(k)] - [h,k]) \in Z(R) \quad \text{for all} \quad h \in H(R) \quad \text{and} \quad k \in S(R).$$

Since $\operatorname{char}(R) \neq 2$, we have

$$[F(h), F(k)] - [h,k] \in Z(R) \quad \text{for all} \quad h \in H(R) \quad \text{and} \quad k \in S(R).$$

Making use of Theorem 3.1, we obtain the required result.

***Corollary* 3.2.** *Let $(R, *)$ be a 2-torsion free noncommutative prime ring with involution of the second kind. If $R$ admits a generalized derivation $F : R \to R$ associated with a nonzero derivation $d : R \to R$ such that $[F(x), F(y)] - [x, y] \in Z(R)$ for all $x, y \in R$, then $F(x) = x$ for all $x \in R$ or $F(x) = -x$ for all $x \in R$.*

**Theorem 3.2.** *Let $(R, *)$ be a 2-torsion free noncommutative prime ring with involution of the second kind. If $R$ admits a generalized derivation $F : R \to R$ associated with a nonzero derivation $d : R \to R$ such that $\big[F(h), F\big(h'\big)\big] - \big[h, h'\big] \in Z(R)$ for all $h' \in H(R)$, then $F(x) = x$ for all $x \in R$ or $F(x) = -x$ for all $x \in R$.*

***Proof.*** By the given assumption, we have

$$\big[F(h), F\big(h'\big)\big] - \big[h, h'\big] \in Z(R) \tag{3.4}$$

for all $h, h' \in H(R)$. Replacing $h'$ by $hh_0$, $h_0 \in H(R) \cap Z(R)$ in (3.4), we get $[F(h), h]d(h_0) \in Z(R)$ for all $h \in H(R)$ and $h_0 \in H(R) \cap Z(R)$. Using the primeness of $R$, we have $[F(h), h] \in Z(R)$ for all $h \in H(R)$ or $d(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. In the first case $R$ is a commutative integral domain in view of Theorem 2.1, a contradiction. Thus, we must have $d(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. This further implies that $d(k_0) = 0$ for all $k_0 \in S(R) \cap Z(R)$ and hence $d(Z(R)) = (0)$. Taking $kk_0$ for $h$ in (3.4), where $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$, we arrive at $\big(\big[F(k), F\big(h'\big)\big] - \big[k, h'\big]\big)k_0 \in Z(R)$. Using the primeness of $R$ and the fact that $S(R) \cap Z(R) \neq (0)$, we obtain

$$\big[F(k), F\big(h'\big)\big] - \big[k, h'\big] \in Z(R) \tag{3.5}$$

for all $h' \in H(R)$ and $k \in S(R)$. Proceeding on similar lines as in Theorem 3.1 and using (3.4), (3.5), we obtain $\big[F(x), F\big(h'\big)\big] - \big[x, h'\big] \in Z(R)$ for all $x \in R$ and $h' \in H(R)$. Replacing $h'$ by $kk_0$, where $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$, we finally arrive at $[F(x), F(y)] - [x, y] \in Z(R)$ for all $x, y \in R$. By [18, Theorem 4] we get $F(x) = x$ for all $x \in R$ or $F(x) = -x$ for all $x \in R$.

Theorem 3.2 proved.

**Theorem 3.3.** *Let $(R, *)$ be a 2-torsion free noncommutative prime ring with involution of the second kind. If $R$ admits a generalized derivation $F : R \to R$ associated with a nonzero derivation $d : R \to R$ such that $\big[F(k), F\big(k'\big)\big] - \big[k, k'\big] \in Z(R)$ for all $k, k' \in S(R)$, then $F(x) = x$ for all $x \in R$ or $F(x) = -x$ for all $x \in R$.*

***Proof.*** By the given assumption, we have

$$\big[F(k), F\big(k'\big)\big] - \big[k, k'\big] \in Z(R) \tag{3.6}$$

for all $k, k' \in S(R)$. Replacing $k'$ by $kh_0$, where $h_0 \in H(R) \cap Z(R)$ in (3.6), we get $[F(k), k]d(h_0) \in Z(R)$ for all $k \in S(R)$ and $h_0 \in H(R) \cap Z(R)$. Using the primeness of $R$, we have $[F(k), k] \in Z(R)$ for all $k \in S(R)$ or $d(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. In the first case, $R$ is a commutative integral domain in view of Theorem 2.2, a contradiction. Hence $d(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. This further implies that $d(Z(R)) = (0)$. Taking $hk_0$ for $k$ in (3.6), where $h \in H(R)$ and $k_0 \in S(R) \cap Z(R)$, we arrive at $\big([F(h), F\big(k'\big)] - \big[h, k'\big]\big)k_0 \in Z(R)$. Using the primeness of $R$ and the fact that $S(R) \cap Z(R) \neq (0)$, we obtain

$$\big[F(h), F\big(k'\big)\big] - \big[h, k'\big] \in Z(R) \tag{3.7}$$

for all $h \in H(R)$ and $k' \in S(R)$. Since every $x \in R$ can be represented as $2x = h + k$, $h \in H(R)$ and $k \in S(R)$. Therefore, proceeding on similar lines as in Theorem 3.2 and making use of (3.6) and

(3.7), we have that $\big[F(x), F\big(k'\big)\big] - \big[x, k'\big] \in Z(R)$ for all $x \in R$ and $k' \in S(R)$. Again replacing $k'$ by $hk_0$, where $h \in H(R)$ and $k_0 \in S(R) \cap Z(R)$, we get $[F(x), F(h)] - [x, h] \in Z(R)$ for all $x \in R$ and $h \in H(R)$. Thus, following the same argument, we finally arrive at $[F(x), F(y)] - [x, y] \in Z(R)$ for all $x, y \in R$. Thus, in view of [18, Theorem 4], we get $F(x) = x$ for all $x \in R$ or $F(x) = -x$ for all $x \in R$.

Theorem 3.3 is proved.

**Theorem 3.4.** *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If $R$ admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$ such that $\big[F(h), d\big(h'\big)\big] - \big[h, h'\big] \in Z(R)$ for all $h, h' \in H(R)$, then $R$ is a commutative integral domain.*

***Proof.*** By the given assumption, we have

$$\big[F(h), d\big(h'\big)\big] - \big[h, h'\big] \in Z(R) \tag{3.8}$$

for all $h, h' \in H(R)$. If $d = 0$, then the result follows from Remark 2.2. Thus, we may assume $d \neq 0$. Taking $h = h'$ in (3.8), we get

$$[F(h), d(h)] \in Z(R) \tag{3.9}$$

for all $h \in H(R)$. Replacing $h'$ by $hh_0$, $h_0 \in H(R) \cap Z(R)$ in (3.8), we have $[F(h), d(h)]h_0 + [F(h), h]d(h_0) \in Z(R)$ for all $h \in H(R)$ and $h_0 \in H(R) \cap Z(R)$. In view of (3.9), we obtain $[F(h), h]d(h_0) \in Z(R)$ for all $h \in H(R)$ and $h_0 \in H(R) \cap Z(R)$. Thus, using the primeness of $R$, we have $[F(h), h] \in Z(R)$ for all $h \in H(R)$ or $d(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. In the first case $R$ is a commutative integral domain in view of Theorem 2.1. Therefore, we must have $d(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. This further implies that $d(k_0) = 0$ for all $k_0 \in S(R) \cap Z(R)$. Substituting $kk_0$ for $h$ in (3.8), where $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$ and proceeding as in Theorem 3.2, we finally arrive at $\big[F(x), d\big(h'\big)\big] - \big[x, h'\big] \in Z(R)$ for all $x \in R$ and $h' \in H(R)$. Again replacing $h'$ by $kk_0$, where $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$, we get $[F(x), d(k)] - [x, k] \in Z(R)$ for all $x \in R$ and $k \in S(R)$. Hence one can find that $[F(x), d(y)] - [x, y] \in Z(R)$ for all $x, y \in R$. By [12, Theorem 1] ring $R$ is a commutative integral domain.

Theorem 3.4 is proved.

The skew symmetric version of Theorem 3.4 is the following result.

**Theorem 3.5.** *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If $R$ admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$ such that $\big[F(k), d\big(k'\big)\big] - \big[k, k'\big] \in Z(R)$ for all $k, k' \in S(R)$, then $R$ is a commutative integral domain.*

***Proof.*** By the given assumption, we have

$$\big[F(k), d\big(k'\big)\big] - \big[k, k'\big] \in Z(R) \tag{3.10}$$

for all $k, k' \in S(R)$. If $d = 0$, then the result follows from Remark 2.3. Thus, we may assume $d \neq 0$. From (3.10), we have, for $k = k'$,

$$[F(k), d(k)] \in Z(R) \tag{3.11}$$

for all $k \in S(R)$. Replacing $k'$ by $kh_0$, where $h_0 \in H(R) \cap Z(R)$ in (3.10), we get $[F(k), d(k)]h_0 + [F(k), k]d(h_0) \in Z(R)$ for all $k \in S(R)$ and $h_0 \in H(R) \cap Z(R)$. In view of (3.11), we obtain $[F(k), k]d(h_0) \in Z(R)$ for all $k \in S(R)$ and $h_0 \in H(R) \cap Z(R)$. Thus, using the primeness of $R$, we have $[F(k), k] \in Z(R)$ for all $k \in S(R)$ or $d(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. In the first case

$R$ is a commutative integral domain in view of Theorem 2.2. Therefore, we must have $d(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. This further implies that $d(k_0) = 0$ for all $k_0 \in S(R) \cap Z(R)$. Using $hk_0$ for $k$, where $h \in H(R)$ and $k_0 \in S(R) \cap Z(R)$ in (3.10), one can obtain that $\big[F(h), d\big(k'\big)\big] - \big[h, k'\big] \in Z(R)$ for all $h \in H(R)$ and $k' \in S(R)$. Hence $\big[F(x), d\big(k'\big)\big] - \big[x, k'\big] \in Z(R)$ for all $x \in R$ and $k' \in S(R)$. Again replacing $k'$ by $hk_0$, where $h \in H(R)$ and $k_0 \in S(R) \cap Z(R)$, we get $[F(x), d(h)] - [x, h] \in Z(R)$ for all $x \in R$ and $h \in H(R)$. Hence $[F(x), d(y)] - [x, y] \in Z(R)$ for all $x, y \in R$. By [12, Theorem 1], ring $R$ is a commutative integral domain.

Theorem 3.5 is proved.

**Theorem 3.6.** *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If $R$ admits a generalized derivation $F\colon R \to R$ associated with a derivation $d\colon R \to R$ such that $[F(h), d(k)] - [h, k] \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$, then $R$ is a commutative integral domain.*

***Proof.*** By the given assumption, we have

$$[F(h), d(k)] - [h, k] \in Z(R) \tag{3.12}$$

for all $h \in H(R)$ and $k \in S(R)$. If $d = 0$, then result follows by Remark 2.1. Henceforward, we assume that $d \neq 0$. Replacing $h$ by $k_0^2$, $k_0 \in S(R) \cap Z(R)$, we get

$$[F(k_0), d(k)]k_0 \in Z(R)$$

for all $k_0 \in S(R) \cap Z(R)$ and $k \in S(R)$. Using the primeness and the fact that $S(R) \cap Z(R) \neq (0)$, we obtain

$$[F(k_0), d(k)] \in Z(R) \tag{3.13}$$

for all $k_0 \in S(R) \cap Z(R)$ and $k \in S(R)$. Replacing $k$ by $hk_0$ in (3.13), we have

$$[F(k_0), d(h)]k_0 + [F(k_0), h]d(k_0) \in Z(R) \tag{3.14}$$

for all $k_0 \in S(R) \cap Z(R)$ and $h \in H(R)$. Now taking $hh_0$ for $h$, where $h_0 \in H(R) \cap Z(R)$ in (3.14), we arrive at

$$[F(k_0), d(h)]h_0k_0 + [F(k_0), h]d(h_0)k_0 + [F(k_0), h]d(k_0)h_0 \in Z(R)$$

for all $k_0 \in S(R) \cap Z(R)$, $h_0 \in H(R) \cap Z(R)$ and $h \in H(R)$. In view of (3.14), we finally get $[F(k_0), h]d(h_0)k_0 \in Z(R)$. Since $S(R) \cap Z(R) \neq (0)$, we have $[F(k_0), h]d(h_0) \in Z(R)$. Again using the primeness of $R$, we obtain $[F(k_0), h] \in Z(R)$ or $d(h_0) = 0$. Suppose that $d(h_0) = 0$ for all $h_0 \in H(R) \cap Z(R)$. Substituting $kk_0$ for $h$ in (3.12), where $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$, we get $[F(k), d(k)]k_0 \in Z(R)$ and, hence, $[F(k), d(k)] \in Z(R)$, since $S(R) \cap Z(R) \neq (0)$. Thus, by (3.12), we get $[h, k] \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$. Hence, $R$ is a commutative integral domain in view of Remark 2.1. On the other hand, suppose that $[F(k_0), h] \in Z(R)$ for all $k_0 \in S(R) \cap Z(R)$ and $h \in H(R)$. Taking $h^2$ for $h$, we obtain that $[F(k_0), h]h \in Z(R)$. Using the primeness of $R$, we have $[F(k_0), h] = 0$ or $h \in Z(R)$. By Remark 2.1 and since $h \in Z(R)$ for all $h \in H(R)$, we get that $R$ is a commutative integral domain. At the end, we have to consider the case

$$[F(k_0), h] = 0 \tag{3.15}$$

for all $k_0 \in S(R) \cap Z(R)$ and $h \in H(R)$. Putting $kk_0$ for $h$, where $k_0 \in S(R) \cap Z(R)$, we obtain

$$[F(k_0), k] = 0 \tag{3.16}$$

for all $k_0 \in S(R) \cap Z(R)$ and $k \in S(R)$. Let us write $2x = h + k$ for some $h \in H(R)$, $k \in S(R)$. By (3.15) and (3.16), we get

$$2[F(k_0), x] = [F(k_0), 2x] = [F(k_0), h + k] = [F(k_0), h] + [F(k_0), k] = 0.$$

Since $R$ is 2-torsion free, we finally have $F(k_0) \in Z(R)$ for all $k_0 \in S(R) \cap Z(R)$.

Using $kk_0$ for $h$, where $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$ in (3.12), we have that

$$[F(k), d(k)]k_0 + [k, d(k)]d(k_0) \in Z(R) \tag{3.17}$$

for all $h \in H(R)$, $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$. Similarly, substituting $k^2$ for $h$ in (3.12), we obtain that $[F(k), d(k)]k + F(k)[k, d(k)] + [k, d(k)]d(k) \in Z(R)$ for all $k \in S(R)$. On substituting $k + k_0$, $k_0 \in S(R) \cap Z(R)$ for $k$ in above equation, we arrive at

$$[F(k), d(k)]k_0 + F(k_0)[k, d(k)] + [k, d(k)]d(k_0) \in Z(R) \tag{3.18}$$

for all $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$. Comparing equations (3.17) and (3.18), we get that $F(k_0)[k, d(k)] \in Z(R)$ for all $k \in S(R)$ and $k_0 \in S(R) \cap Z(R)$. Thus, using the primeness of $R$, we have that $[d(k), k] \in Z(R)$ for all $k \in S(R)$ or $F(k_0) = 0$ for all $k_0 \in S(R) \cap Z(R)$. In the first case $R$ is a commutative integral domain in view of Theorem 2.2. Therefore, we must have $F(k_0) = 0$ for all $k_0 \in S(R) \cap Z(R)$. Substituting $k_0k'$ for $h$ in (3.12), where $k' \in S(R)$ and $k_0 \in S(R) \cap Z(R)$, we get that $k_0\big([d(k'), d(k)] - [k', k]\big) \in Z(R)$ for all $k \in S(R)$ and $k' \in S(R)$. Thus, using the primeness of $R$, we obtain that

$$\big[d\big(k'\big), d(k)\big] + \big[k', k\big] \in Z(R)$$

for all $k \in S(R)$ and $k' \in S(R) \cap Z(R)$. Proceeding in the same way as in Theorem 3.3, we finally arrive at $[d(x), d(y)] - [x, y] \in Z(R)$ for all $x, y \in R$. Hence, $R$ is a commutative integral domain in view of [4, Theorem 1].

Theorem 3.6 is proved.

***Corollary* 3.3.** *Let $(R, *)$ be a 2-torsion free prime ring with involution of the second kind. If $R$ admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$ such that $[F(x), d(y)] - [x, y] \in Z(R)$ for all $x, y \in R$, then $R$ is a commutative integral domain.*

**4. Examples.** We begin this section with certain examples showing that our results do not hold in case when the involution is of the first kind.

***Example* 4.1.** Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z} \right\}$. Of course, $R$ with matrix addition and matrix multiplication is a prime ring and $Z(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| a \in \mathbb{Z} \right\}$. Let $* : R \longrightarrow R$ be a mapping defined by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then $x^* = x$ for all $x \in Z(R)$, and, hence, $Z(R) \subseteq H(R)$, which shows that the involution $*$ is of the first kind. Let us define mappings $F : R \longrightarrow R$ and $d : R \longrightarrow R$ by

$$F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}, \quad d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}.$$

Then $F$ is a generalized derivation on $R$ associated with a nonzero derivation $d$ and the following conditions hold:

(i) $F(h)h + hF(h) \in Z(R)$ for all $h \in h(R)$,

(ii) $F(h)h - hF(h) \in Z(R)$ for all $h \in H(R)$,

(iii) $\big[F(h), F\big(h'\big)\big] - \big[h, h'\big] \in Z(R)$ for all $h, h' \in H(R)$,

(iv) $[F(h), F(k)] - [h, k] \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$,

(v) $\big[F(h), d\big(h'\big)\big] - \big[h, h'\big] \in Z(R)$ for all $h, h' \in H(R)$,

(vi) $[F(h), d(k)] - [h, k] \in Z(R)$ for all $h \in H(R)$ and $k \in S(R)$.

However, $R$ is not commutative and neither $F(x) = x$ nor $F(x) = -x$ for all $x \in R$. If we consider $* : R \longrightarrow R$ as usual transpose mapping, then the condition $F(k)k + kF(k) \in Z(R)$ for all $k \in S(R)$ is satisfied, but $R$ is not commutative.

***Example* 4.2.** Let $R$ be the ring of real quaternions. If we define $* : R \longrightarrow R$ by $(\alpha + \beta i + \gamma j + \delta k)^* = \alpha - \beta i + \gamma j + \delta k$, then $*$ is an involution of the first kind and all skew symmetric elements commute. Thus, if $F$ is a generalized inner derivation induced by some skew symmetric elements $a, b \in R$ (associated with the inner derivation induced by $b$), then the following conditions hold:

(i) $F(k)k - kF(k) \in Z(R)$ for all $k \in S(R)$,

(ii) $\big[F(k), F\big(k'\big)\big] - \big[k, k'\big] \in Z(R)$ for all $k, k' \in S(R)$,

(iii) $[F(k), d(k')] - [k, k'] \in Z(R)$ for all $k, k' \in S(R)$.

However, $R$ is not commutative and neither $F(x) = x$ nor $F(x) = -x$ for all $x \in R$.

We end our paper with following example showing that the primeness hypothesis in our results is necessary. In particular, our results cannot be extended to semiprime rings.

***Example* 4.3.** Let $R_1$ be the ring as in Example 4.1 and $\mathbb{C}$ be the field of complex numbers. Consider $R = R_1 \times \mathbb{C}$. Then $R$ is a non prime ring provided with the involution $\sigma : R \to R$ of the second kind defined by $\sigma(x, z) = \big(x^*, \bar{z}\big)$. Let $G$ be the derivation of $R$ defined by $G(x, z) = (F(x), 0)$. Then one can see that $G(h)h - hG(h) = 0$ for all $h \in H(R)$ and $\big[G(h), G\big(h'\big)\big] - \big[h, h'\big] = 0$ for all $h, h' \in H(R)$. But $R$ is not commutative and neither $F(x) = x$ nor $F(x) = -x$ for all $x \in R$.

On the other hand, if we consider $R_1$ to be the ring as in Example 4.2, then one can easily find that $G(k)k - kG(k) = 0$ for all $k \in S(R)$ and $\big[G(k), G\big(k'\big)\big] - \big[k, k'\big] = 0$ for all $k, k' \in S(R)$. But again $R$ is not commutative and neither $F(x) = x$ nor $F(x) = -x$ for all $x \in R$.

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